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MINIMAL ENERGY OF BICYCLIC GRAPHS OF A GIVEN DIAMETER

Yong Yang and Bo Zhou*

Department of Mathematics, South China Normal University, Guangzhou 510631, P. R. China

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Abstract

The energy of a graph is defined as the sum of the absolute values of its eigenvalues. Let $\mathcal{B}(n, d)$ be the class of bicyclic graphs with n vertices, diameter d and containing no vertex-disjoint odd cycles of lengths s and l with $s + l \equiv 2 \pmod 4$. In this paper, we characterize the graphs with minimal energy in $\mathcal{B}(n, d)$ for $3 \leq d \leq n - 3$. We also discuss the case $d = n - 2$.

INTRODUCTION

Let G be a simple graph with n vertices. The characteristic polynomial of G, denoted by $\phi(G, \lambda)$, is the characteristic polynomial of its adjacency matrix. The eigenvalues of G, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$, are the roots of the equation $\phi(G, \lambda) = 0$ [1]. The energy of G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

*Corresponding author. E-mail: zhoubo@scnu.edu.cn

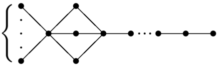

Figure 1: Graph $B_{n,d}$ with $3 \leq d \leq n - 3$.

In theoretical chemistry, the energy of a graph has been extensively studied since it can be used to approximate the total π-electron energy of the molecule [2–5].

A graph whose components are cycles and/or complete graphs with two vertices is called a Sachs graph. Let $\phi(G, \lambda) = \sum_{i=0}^{n} a_i(G)\lambda^{n-i}$. Sachs theorem [1, 2] says that for $i \geq 1$,

$$a_i(G) = \sum_{S \in L_i} (-1)^{p(S)} 2^{c(S)},$$

where L_i denotes the set of Sachs graphs of G with i vertices, $p(S)$ denotes the number of components and $c(S)$ denotes the number of cycles in S. In addition, $a_0(G) = 1$. It is known that $E(G)$ can be expressed as the Coulson integral formula (see [2])

$$E(G) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dx}{x^2} \log \left[\left(\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a_{2i} x^{2i} \right)^2 + \left(\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a_{2i+1} x^{2i+1} \right)^2 \right], \tag{1}$$

where $a_i = a_i(G)$ for $i = 0, 1, \ldots, n$.

A connected graph with n vertices and n edges is called a unicyclic graph, and a connected graph with n vertices and $n + 1$ edges is called a bicyclic graph.

From a chemical point of view, it is of greatest interest to find the extremal values of the energy for significant classes of graphs. For instance, Gutman [6] determined the trees with minimal and maximal energies. Hou [7] determined the unicyclic graphs with minimal energy. Yan and Ye [8] determined the trees of a given diameter with minimal energy. Recently, Li and Zhou [9] determined the unicyclic graphs of a given diameter with minimal energy. More results in this direction can be found in Refs. [10–19].

Let $\mathcal{B}(n)$ be the class of bicyclic graphs with n vertices and containing no vertex-disjoint odd cycles of lengths s and l with $s + l \equiv 2 \pmod 4$. Let $\mathcal{B}(n, d)$ be the class of bicyclic graphs in $\mathcal{B}(n)$ with diameter d, where $2 \leq d \leq n - 2$. By a result of Zhang and Zhou [15], for $n \geq 6$, the graph obtained by attaching $n - 4$ pendent vertices to a vertex of degree three of the graph $K_4 - e$ (the complete graph on four vertices with one edge deleted) is the unique graph in $\mathcal{B}(n, 2)$ with minimal energy. In this paper, we will show that $B_{n,d}$ is the unique graph in $\mathcal{B}(n, d)$ with minimal energy for $3 \leq d \leq n - 3$, where the graph $B_{n,d}$ is shown in Figure 1. We also discuss the case $d = n - 2$.

PRELIMINARIES

For two graphs G and H, $G = H$ means G and H are isomorphic, and $G \supseteq H$ means G contains a subgraph that is isomorphic to H. For $u \in V(G)$, $\Gamma_G(u)$ denotes the set of neighbors of u in G and the degree of u in G is $\deg_G(u) = |\Gamma_G(u)|$. Let P_n, S_n and C_n be respectively the path, star and cycle on n vertices.

For a graph G with n vertices, let $b_i(G) = |a_i(G)|$ for $i = 0, 1, \ldots, n$. Obviously, $b_0(G) = 1$, $b_1(G) = 0$ and $b_2(G)$ equals the number of edges of G. Let $b_i(G) = 0$ if $i < 0$ or $i > n$.

Let $\mathcal{Q}(n)$ be the class of graphs with n vertices whose components are (i) all trees except at most one being either a unicyclic graph or a bicyclic graph in $\mathcal{B}(m)$ with $4 \le m \le n$, or (ii) all trees except two being unicyclic graphs whose union is a subgraph of some graph in $\mathcal{B}(m)$ with $7 \le m \le n$.

A quasi-order relation can be introduced in $\mathcal{Q}(n)$: Let $G_1, G_2 \in \mathcal{Q}(n)$. If $b_i(G_1) \ge b_i(G_2)$ for $i = 0, 1, \ldots, n$, then we write $G_1 \succeq G_2$. If $G_1 \succeq G_2$ and there exists a k such that $b_k(G_1) > b_k(G_2)$, then we write $G_1 \succ G_2$.

Note that for any bipartite graph G, $a_{2i+1}(G) = 0$ (see [1, 2]), and that for any $G \in \mathcal{Q}(n)$, $(-1)^i a_{2i}(G) \ge 0$ (see [1, 7, 15]), and moreover, if $G \supseteq K_4 - e$, then $(-1)^i a_{2i+1}(G) \ge 0$ (see [15]). Thus for graphs $G_1, G_2 \in \mathcal{Q}(n)$, if G_2 is bipartite or if $G_1, G_2 \supseteq K_4 - e$, then from (1), we have the following increasing property on E:

$$G_1 \succ G_2 \Rightarrow E(G_1) > E(G_2). \tag{2}$$

Lemma 1. *Let G be a graph in $\mathcal{Q}(n)$.*

(*a*) *If uv is a cut edge of G, then*

$$b_{2i}(G) = b_{2i}(G - uv) + b_{2i-2}(G - u - v).$$

(*b*) *Suppose that G contains either a unique cycle C_s or exactly two cycles C_s and C_l and that uv is an edge on C_s. If $s \not\equiv 0 \pmod 4$, then*

$$b_{2i}(G) = b_{2i}(G - uv) + b_{2i-2}(G - u - v) + 2b_{2i-s}(G - C_s),$$

and if $s \equiv 0 \pmod 4$, then

$$b_{2i}(G) = b_{2i}(G - uv) + b_{2i-2}(G - u - v) - 2b_{2i-s}(G - C_s).$$

(*c*) *Suppose that G contains three cycles C_s, C_l and C_r and that uv is a common edge of C_s and C_l. If $s, l \not\equiv 0 \pmod 4$, then*

$$b_{2i}(G) = b_{2i}(G - uv) + b_{2i-2}(G - u - v) + 2b_{2i-s}(G - C_s) + 2b_{2i-l}(G - C_l),$$

if $s \not\equiv 0 \pmod 4$ *and* $l \equiv 0 \pmod 4$, *then*

$$b_{2i}(G) = b_{2i}(G-uv) + b_{2i-2}(G-u-v) + 2b_{2i-s}(G-C_s) - 2b_{2i-l}(G-C_l),$$

and if $s, l \equiv 0 \pmod 4$, *then*

$$b_{2i}(G) = b_{2i}(G-uv) + b_{2i-2}(G-u-v) - 2b_{2i-s}(G-C_s) - 2b_{2i-l}(G-C_l).$$

(*d*) *If* u *is a vertex outside any cycle of* G, *then*

$$b_{2i}(G) = b_{2i}(G-u) + \sum_{u' \in \Gamma_G(u)} b_{2i-2}(G-u-u'),$$

and if G *contains three cycles* C_s, C_l *and* C_r *and* u *is a common vertex of them, where* $s, l, r \equiv 0 \pmod 4$, *then*

$$\begin{aligned} b_{2i}(G) &= b_{2i}(G-u) + \sum_{u' \in \Gamma_G(u)} b_{2i-2}(G-u-u') - 2b_{2i-s}(G-C_s) \\ &\quad -2b_{2i-l}(G-C_l) - 2b_{2i-r}(G-C_r). \end{aligned}$$

Proof. For a graph $G \in \mathcal{Q}(n)$, let $\mathcal{C}(uv)$ and $\mathcal{C}(u)$ denote respectively the sets of all cycles C containing the edge uv and the vertex u in G. Then [1, 2]

$$\phi(G,\lambda) = \phi(G-uv,\lambda) - \phi(G-u-v,\lambda) - 2\sum_{C \in \mathcal{C}(uv)} \phi(G-C,\lambda), \tag{3}$$

$$\phi(G,\lambda) = \lambda\phi(G-u,\lambda) - \sum_{u' \in \Gamma_G(u)} \phi(G-u-u',\lambda) - 2\sum_{C \in \mathcal{C}(u)} \phi(G-C,\lambda). \tag{4}$$

In particular, if uv is a cut edge of G, then

$$\phi(G,\lambda) = \phi(G-uv,\lambda) - \phi(G-u-v,\lambda), \tag{5}$$

and if u is a vertex outside any cycle of G, then

$$\phi(G,\lambda) = \lambda\phi(G-u,\lambda) - \sum_{u' \in \Gamma_G(u)} \phi(G-u-u',\lambda). \tag{6}$$

Bearing in mind the facts $(-1)^i a_{2i}(G) \geq 0$ if $G \in \mathcal{Q}(n)$ [1, 7, 15] and $(-1)^i a_{2i+1}(G) \geq 0$ (resp. ≤ 0) for $s \equiv 3 \pmod 4$ (resp. $s \equiv 1 \pmod 4$) if G is a unicyclic graph whose unique cycle has length s [7], and equating coefficients of λ^{n-2i} on both sides of identities on characteristic polynomials above, we have (a) from (5), (b) and (c) from (3), and (d) from (6) and (4). $\square$

By Lemma 1 (a) and Sachs theorem, Lemmas 2 and 3 follow easily.

Lemma 2. *Let* e *be a cut edge of* $G \in \mathcal{Q}(n)$. *Then* $b_{2i}(G) \geq b_{2i}(G-e)$.

Lemma 3. *Let G be a unicyclic graph with n vertices or a bicyclic graph in $\mathcal{B}(n)$. Then $G \succ S_n$.*

Lemma 4. [2] *For $n \geq 2$, $P_n \succeq P_i \cup P_{n-i} \succeq P_1 \cup P_{n-1}$.*

Let $\mathcal{T}(n,d)$ be the class of trees with $n \geq 2$ vertices and diameter d, where $1 \leq d \leq n-1$. If $T \in \mathcal{T}(n,1)$, then $T = P_2$. For $2 \leq d \leq n-1$, let $T_{n,d}$ denote the graph obtained by attaching $n-d$ pendent vertices to an end vertex of P_d. Obviously, $T_{n,2} = S_n$ is the unique tree in $\mathcal{T}(n,2)$ and $T_{n,n-1} = P_n$ is the unique tree in $\mathcal{T}(n,n-1)$.

Lemma 5. [6] *For $3 \leq d \leq n-2$, $P_n \succeq T_{n,d} \succeq S_n$.*

Lemma 6. [8] *Let $T \in \mathcal{T}(n,d)$ with $3 \leq d \leq n-2$. Then $T \succeq T_{n,d}$.*

Lemma 7. [9] *For $3 \leq d_0 < d \leq n-2$, $T_{n,d} \succeq T_{n,d_0}$.*

Lemma 8. *For $2 \leq d_1 \leq n_1 - 2$, we have $T_{n_1,d_1} \cup T \succeq T_{n_1+n_2-1,d_1+d_2}$, where $T = T_{n_2,d_2}$ if $2 \leq d_2 \leq n_2 - 1$, and P_2 if $n_2 = 2$ and $d_2 = 1$.*

Proof. For $2 \leq d_2 \leq n_2 - 1$, we have $(n_1 - d_1)(n_2 - d_2) \geq n_1 + n_2 - d_1 - d_2 - 1$, and by Lemmas 1 (a) and 4,

$$\begin{aligned}
b_{2i}(T_{n_1,d_1} \cup T_{n_2,d_2}) &= b_{2i}(T_{n_1-1,d_1} \cup T_{n_2,d_2}) + b_{2i-2}(P_{d_1-1} \cup T_{n_2,d_2}) \\
&= b_{2i}(P_{d_1} \cup T_{n_2,d_2}) + (n_1 - d_1) b_{2i-2}(P_{d_1-1} \cup T_{n_2,d_2}) \\
&= b_{2i}(P_{d_1} \cup P_{d_2}) + (n_2 - d_2) b_{2i-2}(P_{d_1} \cup P_{d_2-1}) \\
&\quad + (n_1 - d_1) b_{2i-2}(P_{d_1-1} \cup P_{d_2}) \\
&\quad + (n_1 - d_1)(n_2 - d_2) b_{2i-4}(P_{d_1-1} \cup P_{d_2-1}) \\
&\geq b_{2i}(P_{d_1+d_2-1}) + (n_1 + n_2 - d_1 - d_2) b_{2i-2}(P_{d_1+d_2-2}) \\
&\quad + (n_1 + n_2 - d_1 - d_2 - 1) b_{2i-4}(P_{d_1+d_2-3}) \\
&= b_{2i}(T_{n_1+n_2-1,d_1+d_2}),
\end{aligned}$$

which implies $T_{n_1,d_1} \cup T_{n_2,d_2} \succeq T_{n_1+n_2-1,d_1+d_2}$. It is easy to see that $T_{n_1,d_1} \cup P_2 \succeq T_{n_1+1,d_1+1}$. $\square$

Let $\mathcal{U}(n,d)$ be the class of unicyclic graphs with n vertices and diameter d, where $1 \leq d \leq n-2$. If $U \in \mathcal{U}(n,1)$, then $U = C_3$. For $3 \leq d \leq n-2$, let $U_{n,d}$ be the graph obtained respectively by attaching $n-d-1$ pendent vertices and a path P_{d-3} to two non-adjacent vertices of a quadrangle.

Lemma 9. [9] *Let* $U \in \mathcal{U}(n,d)$ *with* $3 \le d \le n-2$. *Then* $U \succeq U_{n,d}$.

Lemma 10. *For* $3 \le d \le n-2$, $U_{n,d} \succ T_{n,d}$.

Proof. By Lemmas 1 (a) and (b),

$$\begin{aligned} b_{2i}(U_{n,d}) &= b_{2i}(T_{n,d}) + b_{2i-2}(P_{d-3} \cup S_{n-d+1}) - 2b_{2i-4}(P_{d-3}) \\ &= b_{2i}(T_{n,d}) + b_{2i-2}(P_{d-3} \cup S_{n-d-1}) \ge b_{2i}(T_{n,d}). \end{aligned}$$

It is easy to see that $b_2(U_{n,d}) > b_2(T_{n,d})$. Thus $U_{n,d} \succ T_{n,d}$. $\square$

Lemma 11. *For* $3 \le d_0 < d \le n-2$, $U_{n,d} \succeq U_{n,d_0}$.

Proof. If $d = 4$, then it can be checked that $U_{n,4} \succeq U_{n,3}$ by Sachs theorem. If $d \ge 5$, then by Lemmas 1 (a), 2 and 10,

$$\begin{aligned} b_{2i}(U_{n,d}) &= b_{2i}(U_{n-1,d-1}) + b_{2i-2}(U_{n-2,d-2}) \\ &\ge b_{2i}(U_{n-1,d-1}) + b_{2i-2}(T_{n-2,d-2}) \\ &\ge b_{2i}(U_{n-1,d-1}) + b_{2i-2}(T_{d-1,d-3}) \\ &= b_{2i}(U_{n,d-1}), \end{aligned}$$

and so $U_{n,d} \succeq U_{n,d-1} \succeq \cdots \succeq U_{n,d_0}$. $\square$

Similarly, we have

Lemma 12. *For* $3 \le d_0 < d \le n-3$, $B_{n,d} \succeq B_{n,d_0}$.

MAIN RESULTS

For a graph $G \in \mathcal{B}(n)$, it has either two or three distinct cycles. If G has three cycles, then any two cycles must have at least one edge in common, and we may choose two cycles of lengths of a and b with t common edges such that $a - t \ge t$ and $b - t \ge t$. If G has exactly two cycles, suppose that the lengths of them are a and b respectively. Then, in any case, we choose two cycles C_a and C_b in G. For convenience, let $C_a = v_0 v_1 \ldots v_{a-1} v_0$ and $C_b = u_0 u_1 \ldots u_{b-1} u_0$. If C_a and C_b have no common edges, then C_a and C_b are connected by a unique path P, say from v_0 to u_0. Let $l(G)$ be the length of P. If C_a and C_b have exactly t (≥ 1) common edges, and thus have exactly $t+1$ common vertices, say, $v_0 = u_0, v_1 = u_1, \ldots, v_t = u_t$, then $C_c = u_0 u_{b-1} \ldots u_{t+1} u_t v_{t+1} v_{t+2} \ldots v_{a-1} v_0$ is the third cycle of G, where $c = b + a - 2t$. If we write $w_0 = u_0, w_1 = u_{b-1}, \ldots, w_{c-1} = v_{a-1}$, then $C_c = w_0 w_1 \ldots w_{c-1} w_0$. Let $d(G)$ be the diameter of G.

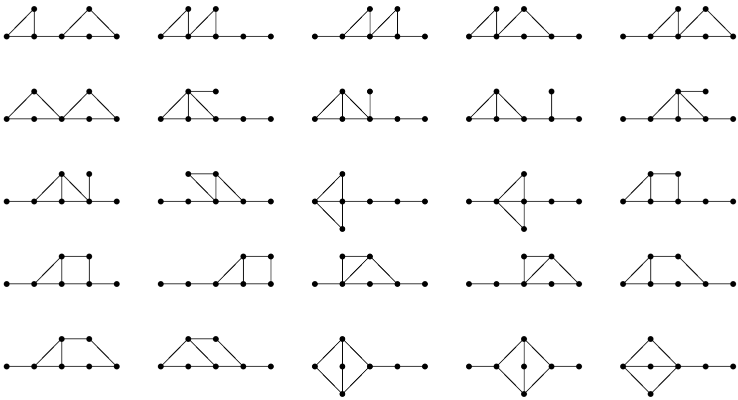

Figure 2: Graphs in $\mathcal{B}(7,4)$ except $B_{7,4}$.

Lemma 13. *Let $G \in \mathcal{B}(n, n-3)$ with $n \geq 6$ and $G \neq B_{n,n-3}$. Then $G \succ B_{n,n-3}$.*

Proof. We prove this lemma by induction on n.

If $n = 6$, then by a result of [15] or by direct check, we have $G \succ B_{6,3}$.

If $n = 7$, then G is isomorphic to one of the graphs in Figure 2. By Sachs theorem, for $i \geq 3$, we have $b_4(G) > b_4(B_{7,4}) = 7$ and $b_{2i}(B_{7,4}) = 0$. Thus $G \succ B_{7,4}$.

Suppose that $n \geq 8$ and it is true for all graphs in $\mathcal{B}(n-1, n-4)$ and $\mathcal{B}(n-2, n-5)$. Now suppose that $G \in \mathcal{B}(n, n-3)$ and $G \neq B_{n,n-3}$.

Suppose that there is a pendent vertex u in G such that the degree of its neighbor v is two. Then $G - u \in \mathcal{B}(n-1, n-4)$ and $G - u - v \in \mathcal{B}(n-2, n-5)$. Note that $G \neq B_{n,n-3}$. So $G - u \neq B_{n-1,n-4}$ or $G - u - v \neq B_{n-2,n-5}$. By the induction hypothesis, we have either $G - u \succ B_{n-1,n-4}$ and $G - u - v \succeq B_{n-2,n-5}$ or $G - u \succeq B_{n-1,n-4}$ and $G - u - v \succ B_{n-2,n-5}$. Thus $G \succ B_{n,n-3}$.

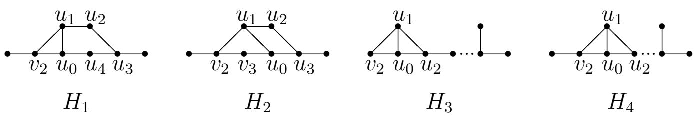

Figure 3: Graphs H_j, $j = 1, \ldots, 4$.

Suppose that the neighbor of any pendent vertex has degree at least three or there is no pendent vertex. Then G is isomorphic to some H_j in Figure 3, $j = 1, \ldots, 4$, or G contains one quadrangle which has at most one common vertex with another cycle that is a triangle or a quadrangle. For $n = 8, 9$, it can be checked by Sachs theorem that $G \succ B_{n,n-3}$. Suppose that $n \geq 10$. Choose C_a and C_b as above. Let $b \geq a$. Note that $b_4(G) > b_4(B_{n,n-3})$. If $G = H_3$ or H_4, then and by Lemmas 1 (a), (b) and (c), 2 and 6,

$$\begin{aligned} b_{2i}(G) &= b_{2i}(G - u_0u_2) + b_{2i-2}(G - u_0 - u_2) \\ &\quad +2b_{2i-3}(G - u_0 - u_1 - u_2) - 2b_{2i-4}(G - u_0 - u_1 - u_2 - v_2) \\ &\geq b_{2i}(G - u_0u_2 - u_0v_2) + b_{2i-2}(G - u_0 - v_2) \\ &\quad +b_{2i-2}(G - u_0 - u_2) - 2b_{2i-4}(G - u_0 - u_1 - u_2 - v_2) \\ &= b_{2i}(G - u_0u_2 - u_0v_2) + b_{2i-2}(G - u_0 - v_2 - u_1u_2) \\ &\quad +b_{2i-2}(G - u_0 - u_2 - u_1v_2) \\ &\geq b_{2i}(T_{n,n-3}) + 2b_{2i-2}(P_{n-6}) = b_{2i}(B_{n,n-3}), \end{aligned}$$

implying $G \succ B_{n,n-3}$. Otherwise, by Lemmas 1 (a) and (b), 2, 9 and 10,

$$\begin{aligned} b_{2i}(G) &= b_{2i}(G - u_0u_1 - u_1u_2) + b_{2i-2}(G - u_1 - u_2 - u_3u_0) \\ &\quad +b_{2i-2}(G - u_0 - u_1 - u_2u_3) \\ &\geq b_{2i}(U_{n-1,n-3}) + b_{2i-2}(U_{n-5,n-7} \cup P_2) + b_{2i-2}(U_{n-4,n-6}) \\ &\geq b_{2i}(U_{n-1,n-3}) + b_{2i-2}(T_{n-5,n-7} \cup P_2) + b_{2i-2}(T_{n-4,n-6}), \end{aligned}$$

while by Lemma 1 (a) and (c),

$$b_{2i}(B_{n,n-3}) = b_{2i}(U_{n-1,n-3}) + b_{2i-2}(P_{n-6} \cup P_2) + b_{2i-2}(P_{n-5}),$$

and so, by Lemma 2, we have $b_{2i}(G) \geq b_{2i}(B_{n,n-3})$ and then $G \succ B_{n,n-3}$. $\square$

Lemma 14. *Let $G \in \mathcal{B}(n,d)$ with $2 \leq d \leq n-4$. If G contains no pendent vertices, then $G \succ B_{n,d+1}$.*

Proof. We choose C_a, C_b in G and if there exists the third cycle, then we choose C_c and t as above. Let $b \geq a$. Since $d \leq n-4$, we have $b \geq 5$.

Case 1. C_a and C_b have no common edges. Then $d = \lfloor a/2 \rfloor + \lfloor b/2 \rfloor + l(G)$.

Subcase 1.1. $b \not\equiv 0 \pmod 4$. Then $d(G - u_1u_2) = \lfloor a/2 \rfloor + b + l(G) - 2 \geq d+1$, $d(G - u_1 - u_2) \geq d$ and $d \geq 3$. By Lemmas 1 (b), 9, 10 and 11,

$$\begin{aligned} b_{2i}(G) &\geq b_{2i}(G - u_1u_2) + b_{2i-2}(G - u_1 - u_2) \\ &\geq b_{2i}(U_{n,d+1}) + b_{2i-2}(U_{n-2,d}) \\ &\geq b_{2i}(U_{n,d+1}) + b_{2i-2}(T_{n-2,d}), \end{aligned}$$

and by Lemma 1 (c),

$$b_{2i}(B_{n,d+1}) = b_{2i}(U_{n,d+1}) + b_{2i-2}(P_{d-2} \cup S_{n-d}) - 4b_{2i-4}(P_{d-2}). \tag{7}$$

Hence, by Lemma 2, $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$.

Subcase 1.2. $b \equiv 0 \pmod 4$. Then $b \geq 8$. Hence $d(G - u_1u_2 - u_2u_3) = \lfloor a/2 \rfloor + b + l(G) - 3 \geq d+1$, $d(G - u_2 - u_3 - u_4u_5) \geq d-1$, $d(G - u_1 - u_2 - u_3u_4) \geq d$ and $d \geq 5$. By Lemmas 1 (a) and (b), 9, 10 and 11,

$$\begin{aligned} b_{2i}(G) &= b_{2i}(G - u_1u_2 - u_2u_3) + b_{2i-2}(G - u_2 - u_3) \\ &\quad + b_{2i-2}(G - u_1 - u_2) - 2b_{2i-b}(G - C_b) \\ &\geq b_{2i}(G - u_1u_2 - u_2u_3) + b_{2i-2}(G - u_2 - u_3 - u_4u_5) \\ &\quad + b_{2i-2}(G - u_1 - u_2 - u_3u_4) \\ &\geq b_{2i}(U_{n-1,d+1}) + b_{2i-2}(U_{n-3,d-1}) + b_{2i-2}(U_{n-3,d}) \\ &\geq b_{2i}(U_{n-1,d+1}) + b_{2i-2}(T_{n-3,d-1}) + b_{2i-2}(T_{n-3,d}), \end{aligned}$$

and by Lemma 1 (a) and (c),

$$b_{2i}(B_{n,d+1}) = b_{2i}(U_{n-1,d+1}) + b_{2i-2}(P_{d-2} \cup S_{n-d-2}) + b_{2i-2}(P_{d-1}). \tag{8}$$

Hence, by Lemma 2, $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$.

Case 2. C_a and C_b have at least one common edge. Note that $a - t \geq t$, $b - t \geq t$, where $t \geq 1$. Then $c \geq b$ and $d = \lfloor c/2 \rfloor = \lfloor (a+b)/2 \rfloor - t$.

Subcase 2.1. $b, c \not\equiv 0 \pmod 4$. Then $d(G - w_0w_1) = \lfloor a/2 \rfloor + b - t - 1 \geq d+1$, $d(G - w_0 - w_1) \geq c - 3 \geq d$ and $d \geq 2$. By Lemmas 1 (c), 6, 7, 9 and 11,

$$b_{2i}(G) \geq b_{2i}(G - w_0w_1) + b_{2i-2}(G - w_0 - w_1) \geq b_{2i}(U_{n,d+1}) + b_{2i-2}(T_{n-2,d}),$$

which, together with (7) and Lemma 2, implies $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$.

Subcase 2.2. $b \not\equiv 0 \pmod 4$ and $c \equiv 0 \pmod 4$. If $b = 5$, then it can be checked by Sachs theorem that $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$. Otherwise, $d(G - w_0w_1 - w_1w_2) = \lfloor a/2 \rfloor + b - t - 2 \geq d+1$, $d(G - w_1 - w_2 - w_3w_4) \geq d-1$, $d(G - w_0 - w_1 - w_2w_3) \geq c - 4 \geq d$ and $d \geq 4$. By Lemmas 1 (a), (b) and (c), 6, 7, 9, 10 and 11,

$$\begin{aligned} b_{2i}(G) &\geq b_{2i}(G - w_0w_1 - w_1w_2) + b_{2i-2}(G - w_1 - w_2 - w_3w_4) \\ &\quad + b_{2i-2}(G - w_0 - w_1 - w_2w_3) \\ &\geq b_{2i}(U_{n-1,d+1}) + b_{2i-2}(U_{n-3,d-1}) + b_{2i-2}(T_{n-3,d}) \\ &\geq b_{2i}(U_{n-1,d+1}) + b_{2i-2}(T_{n-3,d-1}) + b_{2i-2}(T_{n-3,d}), \end{aligned}$$

which, together with (8) and Lemma 2, implies $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$.

Subcase 2.3. $b \equiv 0 \pmod 4$ and $c \not\equiv 0 \pmod 4$. Then $c > b \geq 8$. By similar arguments as those in Subcase 2.2, $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$.

Subcase 2.4. $b, c \equiv 0 \pmod 4$. Then $a \equiv 0 \pmod 2$ and $c \geq b \geq 8$. If $a \equiv 2 \pmod 4$, then $a \geq 6$, and by similar arguments as those in Subcase 2.2, $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$. Suppose that $a \equiv 0 \pmod 4$. Obviously, $n = a + b - t - 1$, $d(G - u_0) = c - 2 \geq d + 2$, $d = \lfloor (a+b)/2 \rfloor - t \geq 4$, and $n - 5 \geq d$. By Lemmas 1 (a) and (d), 4, 5, 6 and 7,

$$\begin{aligned}
b_{2i}(G) &\geq b_{2i}(G - u_0) + b_{2i-2}(G - u_0 - u_{b-1} - u_t) \\
&\quad + b_{2i-2}(G - u_0 - u_1 - u_t) + b_{2i-2}(G - u_0 - v_{a-1} - u_t) \\
&\geq b_{2i}(T_{n-1,d+2}) + b_{2i-2}(P_{a-t-1} \cup P_{b-t-2} \cup P_{t-1}) \\
&\quad + b_{2i-2}(P_{a-t-1} \cup P_{b-t-1} \cup P_{t-2}) + b_{2i-2}(P_{a-t-2} \cup P_{b-t-1} \cup P_{t-1}) \\
&\geq b_{2i}(T_{n-1,d+1}) + 3b_{2i-2}(P_{a+b-t-6}) \\
&\geq b_{2i}(T_{n-1,d+1}) + b_{2i-2}(P_{n-5}) + 2b_{2i-2}(T_{n-5,d-1}),
\end{aligned}$$

and by Lemma 1 (a), (b) and (c),

$$b_{2i}(B_{n,d+1}) = b_{2i}(T_{n-1,d+1}) + b_{2i-2}(P_d) + 2b_{2i-2}(P_{d-2} \cup S_{n-d-3}). \tag{9}$$

By Lemma 2, $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$.

Combining Cases 1 and 2, we have $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$. Note that $b_4(G) > b_4(B_{n,d+1})$. Thus we have $G \succ B_{n,d+1}$. □

Lemma 15. *Let $G \in \mathcal{B}(n,d)$ with $3 \leq d \leq n-4$. If G contains exactly one pendent vertex u on all diametrical paths of G such that $G - u$ contains no pendent vertices, then $G \succ B_{n,d+1}$.*

Proof. We choose C_a, C_b in G and if there exists the third cycle, then we choose C_c and t as above. Let $b \geq a$. Since $d \leq n - 4$, we have $b \geq 5$. Let v be the neighbor of u.

Case 1. C_a and C_b have no common edges. Then $d = \lfloor a/2 \rfloor + \lfloor b/2 \rfloor + l(G) + 1$.

Subcase 1.1. $b \not\equiv 0 \pmod 4$. If $b \geq 7$, then $d(G - u_1u_2) \geq \lfloor a/2 \rfloor + b + l(G) - 2 \geq d+1$. If $b = 5, 6$, and v lies on C_a, then $d(G - u_1u_2) = \lfloor a/2 \rfloor + b + l(G) - 1 = d+1$. If $a = 5$, $b = 6$, and v lies on C_b, then $d(G - v_1v_2) = \lfloor b/2 \rfloor + a + l(G) - 1 = d + 1$. In these cases, by similar arguments as those in Subcase 1.1 of Lemma 14, $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$. Otherwise, $a = 3, 4$, $b = 5, 6$, and v lies on C_b. If $l(G) = 0$, then it can be checked by

Sachs theorem that $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$. Suppose that $l(G) \geq 1$. If $a = 3$, then by Lemmas 1 (a) and (b), 6 and 9,

$$\begin{aligned} b_{2i}(G) &\geq b_{2i}(G - v_0v_1) + b_{2i-2}(G - v_0 - v_1 - u_0u_1) \\ &\geq b_{2i}(U_{n,d+1}) + b_{2i-2}(T_{n-3,d-1}) \\ &= b_{2i}(U_{n-1,d+1}) + b_{2i-2}(T_{d+1,d-1}) + b_{2i-2}(T_{n-3,d-1}), \end{aligned}$$

which, together with (8) and Lemma 2, implies $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$. If $a = 4$, then $n - 4 = d$, and by Lemmas 1 (a) and (b), 6 and 8,

$$\begin{aligned} b_{2i}(G) &\geq b_{2i}(G - v_0v_1 - v_1v_2 - u_0u_1) + b_{2i-2}(G - v_0v_1 - v_1v_2 - u_0 - u_1) \\ &\quad + b_{2i-2}(G - v_0 - v_1 - v_2v_3 - u_0u_1) + b_{2i-2}(G - v_1 - v_2 - v_3v_0 - u_0u_1) \\ &\geq b_{2i}(T_{n-1,d+1}) + b_{2i-2}(P_{n-b-2} \cup T_{b-1,b-3}) \\ &\quad + b_{2i-2}(T_{n-4,d-2}) + b_{2i-2}(T_{n-3,d-1}) \\ &\geq b_{2i}(T_{n-1,d+1}) + b_{2i-2}(T_{n-4,n-6}) + b_{2i-2}(T_{n-4,d-2}) + b_{2i-2}(T_{n-3,d-1}), \end{aligned}$$

which, together with (9) and Lemma 2, implies $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$.

Subcase 1.2. $b \equiv 0 \pmod 4$. If $b \geq 12$, then $d(G - u_1u_2 - u_2u_3) \geq \lfloor a/2 \rfloor + b + l(G) - 3 \geq d + 2$. If $b = 8$ and v lies on C_a, then $d(G - u_1u_2 - u_2u_3) = \lfloor a/2 \rfloor + b + l(G) - 2 = d + 1$. In these cases, by similar arguments as those in Subcase 1.2 of Lemma 14, $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$. If $a = 5, 6, 7$, $b = 8$, and v lies on C_b, by similar arguments as those in Subcase 1.1 of Lemma 14, $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$. Otherwise, $a = 3, 4$, $b = 8$, and v lies on C_b. If $l(G) \leq 2$, then it can be checked by Sachs theorem that $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$. Suppose that $l(G) \geq 3$. By Lemmas 1 (a) and (b), 8, 9 and 10,

$$\begin{aligned} b_{2i}(G) &\geq b_{2i}(G - u_0u_1 - u_1u_2) + b_{2i-2}(G - u_1 - u_2 - u_3u_4) \\ &\quad + b_{2i-2}(G - u_0 - u_1 - u_2u_3) \\ &\geq b_{2i}(U_{n-1,d+1}) + b_{2i-2}(U_{n-3,d}) + b_{2i-2}(U_{n-9,d-6} \cup T_{6,4}) \\ &\geq b_{2i}(U_{n-1,d+1}) + b_{2i-2}(T_{n-3,d}) + b_{2i-2}(T_{n-9,d-6} \cup T_{6,4}) \\ &\geq b_{2i}(U_{n-1,d+1}) + b_{2i-2}(T_{n-3,d}) + b_{2i-2}(T_{n-4,d-2}), \end{aligned}$$

which, together with (8) and Lemma 2, implies $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$.

Case 2. C_a and C_b have at least one common edge. Then $d = \lfloor c/2 \rfloor + 1 = \lfloor (a + b)/2 \rfloor - t + 1$. Since $b \geq 5$, assume that $w_0, w_1 \neq v$.

If $b, c \not\equiv 0 \pmod 4$ and $b \geq 6$, then $d(G - w_0w_1) \geq \lfloor a/2 \rfloor + b - t - 1 \geq d + 1$ and $d(G - w_0 - w_1) \geq c - 3 \geq d - 1$. If $b \not\equiv 0 \pmod 4$, $c \equiv 0 \pmod 4$ and $b \geq 9$, then $d(G - w_0w_1 - w_1w_2) \geq \lfloor a/2 \rfloor + b - t - 2 \geq d + 1$, $d(G - w_1 - w_2 - w_3w_4) \geq d - 1$

and $d(G - w_0 - w_1 - w_2w_3) \geq c - 4 \geq d + 1$. If $b \equiv 0 \pmod 4$ and $c \not\equiv 0 \pmod 4$, then $d(G - w_0w_1 - w_1w_2) \geq d + 1$, $d(G - w_1 - w_2 - w_3w_4) \geq d - 1$ and $d(G - w_0 - w_1 - w_2w_3) \geq d$. If $a, b, c \equiv 0 \pmod 4$ and $a \neq 4$ or $b \neq 8$, then $n - d \geq 7$ and $d(G - w_0) \geq c - 2 \geq d + 1$. If $b, c \equiv 0 \pmod 4$, $a \not\equiv 0 \pmod 4$ and $a \geq 10$, then $d(G - w_0w_{c-1} - w_{c-1}w_{c-2}) \geq d + 2$, $d(G - w_{c-1} - w_{c-2} - w_{c-3}w_{c-4}) \geq d$ and $d(G - w_0 - w_{c-1} - w_{c-2}w_{c-3}) \geq d + 1$. In these cases, by similar arguments as those in Subcases 2.1–2.4 of Lemma 14, $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$.

If $b, c \equiv 0 \pmod 4$ and $a = 6$, then $d(G - w_{c-1}w_0 - w_0w_1) \geq c - 2 \geq d + 1$. By Lemmas 1 (a), (b) and (c), 6 and 7,

$$\begin{aligned} b_{2i}(G) &\geq b_{2i}(G - w_{c-1}w_0 - w_0w_1) + b_{2i-2}(G - w_0 - w_1 - w_2w_3) \\ &\quad + b_{2i-2}(G - w_{c-1} - w_0 - w_1w_2) \\ &\geq b_{2i}(T_{n,d+1}) + 2b_{2i-2}(T_{n-4,d-1}). \end{aligned}$$

By Lemma 1 (b) and (c),

$$b_{2i}(B_{n,d+1}) = b_{2i}(T_{n,d+1}) + 2b_{2i-2}(P_{d-2} \cup S_{n-d-3}). \tag{10}$$

Hence, by Lemma 2, we have $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$.

Now we are left with the cases: (i) $b, c \not\equiv 0 \pmod 4$ and $b = 5$, (ii) $b \not\equiv 0 \pmod 4$, $c \equiv 0 \pmod 4$ and $b = 5, 6, 7$, or (iii) $c \equiv 0 \pmod 4$, $a = 4$ and $b = 8$. It can be checked directly by Sachs theorem that $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$.

Combining Cases 1 and 2, we have $b_{2i}(G) \geq b_{2i}(B_{n,d+1})$. Note that $b_4(G) > b_4(B_{n,d+1})$. Thus we have $G \succ B_{n,d+1}$. $\square$

Theorem 1. *Let $G \in \mathcal{B}(n, d)$ with $3 \leq d \leq n - 3$. If there are two vertex-disjoint cycles in G, then $G \succ B_{n,d}$.*

Proof. We prove this theorem by induction on $n - d$.

By Lemma 13, the result holds for $n - d = 3$. Let $h \geq 4$ and suppose that the result holds for $n - d < h$. Now suppose that $n - d = h$ and $G \in \mathcal{B}(n, d)$.

Case 1. There is no pendent vertex in G . By Lemmas 12 and 14, $G \succ B_{n,d}$.

Case 2. There is a pendent vertex outside some diametrical path $P(G) = x_0x_1 \ldots x_d$. Let u, adjacent to v, be a pendent vertex outside $P(G)$ in G. Then $G - u \in \mathcal{B}(n-1, d)$. By the induction hypothesis, $G - u \succ B_{n-1,d}$. By Lemma 1 (a),

$$b_{2i}(B_{n,d}) = b_{2i}(B_{n-1,d}) + b_{2i-2}(T_{d+1,d-2}). \tag{11}$$

Hence it suffices to show that $b_{2i}(H) \geq b_{2i}(T_{d+1,d-2})$, where $H = G - u - v$. We choose C_a and C_b in G as above.

Subcase 2.1. v lies on some cycle, say C_a. Then $H \supseteq C_b$.

First, suppose that $P(G)$ and C_b have no common vertices. Then $H \supseteq P_k \cup P_{d-k} \cup C_b$. By Lemmas 2, 3, 4 and 5,

$$\begin{aligned} b_{2i}(H) &\geq b_{2i}(P_k \cup P_{d-k} \cup C_b) \geq b_{2i}(P_{d-1} \cup C_b) \geq b_{2i}(P_{d-1} \cup S_b) \\ &\geq b_{2i}(P_{d-1} \cup P_3) \geq b_{2i}(P_{d+1}) \geq b_{2i}(T_{d+1,d-2}). \end{aligned}$$

Next, suppose that $P(G)$ and C_b have common vertices $x_l, \ldots, x_{l+q}$, where $q \geq 0$. If v lies outside $P(G)$, then $H \supseteq G_1$, where $G_1 \in \mathcal{U}(s_1, d)$, $s_1 \geq d+2$. By Lemmas 2, 5, 9 and 10,

$$b_{2i}(H) \geq b_{2i}(G_1) \geq b_{2i}(U_{s_1,d}) \geq b_{2i}(T_{s_1,d}) \geq b_{2i}(P_{d+1}) \geq b_{2i}(T_{d+1,d-2}).$$

Suppose that v lies on $P(G)$. Then $P(G)$ and C_a have common vertices, say $x_k, \ldots, x_{k+p}$, where $p \geq 0$, $k+p < l$.

If $p = 0$, then $k \geq 1$, $H \supseteq P_2 \cup P_k \cup G_2$, where $G_2 \in \mathcal{U}(s_2, d_2)$, $s_2 \geq d_2 + 2$, $d_2 \geq d - k - 1 \geq 1$. If $d_2 = 1$, then $k = d-2$ and $G_2 = C_3$, and by Lemmas 2, 3, 4 and 5,

$$b_{2i}(H) \geq b_{2i}(P_2 \cup P_{d-2} \cup C_3) \geq b_{2i}(P_{d-1} \cup P_3) \geq b_{2i}(P_{d+1}) \geq b_{2i}(T_{d+1,d-2}).$$

If $d_2 = 2$, then $k \geq d-3$, $s_2 \geq 4$, and by Lemmas 2, 3, 4, 7 and 8,

$$\begin{aligned} b_{2i}(H) &\geq b_{2i}(P_2 \cup P_k \cup G_2) \geq b_{2i}(P_2 \cup P_k \cup S_{s_2}) \geq b_{2i}(P_2 \cup P_k \cup T_{4,2}) \\ &\geq b_{2i}(P_{k+1} \cup T_{4,2}) \geq b_{2i}(T_{k+4,k+2}) \geq b_{2i}(T_{d+1,d-1}) \geq b_{2i}(T_{d+1,d-2}). \end{aligned}$$

If $d_2 \geq 3$, then by Lemmas 2, 4, 7, 8, 9 and 10,

$$\begin{aligned} b_{2i}(H) &\geq b_{2i}(P_2 \cup P_k \cup U_{s_2,d_2}) \geq b_{2i}(P_2 \cup P_k \cup T_{s_2,d_2}) \geq b_{2i}(T_{s_2+k,d_2+k}) \\ &\geq b_{2i}(T_{d_2+k+2,d_2+k}) \geq b_{2i}(T_{d+1,d-1}) \geq b_{2i}(T_{d+1,d-2}). \end{aligned}$$

Suppose that $p = 1$. If $v = x_k$, then $k \geq 1$, $H \supseteq P_k \cup G_3$, where $G_3 \in \mathcal{U}(s_3, d_3)$, $s_3 \geq d_3 + 2$, $d_3 \geq d - k \geq 3$, and by Lemmas 2, 7, 8, 9 and 10,

$$b_{2i}(H) \geq b_{2i}(P_k \cup U_{s_3,d_3}) \geq b_{2i}(P_k \cup T_{s_3,d_3}) \geq b_{2i}(T_{s_3+k-1,d_3+k-1}) \geq b_{2i}(T_{d+1,d-2}).$$

If $v = x_{k+1}$, then $H \supseteq P_{k+2} \cup G_4$, where $G_4 \in \mathcal{U}(s_4, d_4)$, $s_4 \geq d_4 + 2$, $d_4 \geq d-k-2 \geq 1$. If $d_4 = 1$, then $k = d-3$, $G_4 = C_3$, and by Lemmas 2, 3, 4 and 5,

$$b_{2i}(H) \geq b_{2i}(P_{d-1} \cup C_3) \geq b_{2i}(P_{d-1} \cup P_3) \geq b_{2i}(P_{d+1}) \geq b_{2i}(T_{d+1,d-2}).$$

If $d_4 = 2$, then $k \geq d-4$, $s_4 \geq 4$, and by Lemmas 2, 3, 7 and 8,

$$b_{2i}(H) \geq b_{2i}(P_{k+2} \cup S_{s_4}) \geq b_{2i}(P_{k+2} \cup T_{4,2}) \geq b_{2i}(T_{d+1,d-2}).$$

If $d_4 \geq 3$, then by Lemmas 2, 7, 8, 9 and 10,

$$b_{2i}(H) \geq b_{2i}(P_{k+2} \cup U_{s_4,d_4}) \geq b_{2i}(P_{k+2} \cup T_{s_4,d_4}) \geq b_{2i}(T_{d+1,d-2}).$$

Suppose that $p \geq 2$. If $v \neq x_k, x_{k+p}$, then $H \supseteq G_5$, where $G_5 \in \mathcal{U}(s_5, d)$, $s_5 \geq d+2$. It is easy to show as above that $b_{2i}(H) \geq b_{2i}(T_{d+1,d-2})$. If $v = x_k$, then $k \geq 1$, $H \supseteq P_k \cup G_6$, where $G_6 \in \mathcal{U}(s_6, d_6)$, $s_6 \geq d_6 + 3$, $d_6 \geq d - k - 1 \geq 3$, and by Lemmas 2, 8, 9 and 10,

$$b_{2i}(H) \geq b_{2i}(P_k \cup U_{s_6,d_6}) \geq b_{2i}(P_k \cup T_{s_6,d_6}) \geq b_{2i}(T_{d+1,d-2}).$$

If $v = x_{k+p}$, then $H \supseteq T_1 \cup G_7$ or $P_{k+p+1} \cup G_7$, where $G_7 \in \mathcal{U}(s_7, d_7)$, $s_7 \geq d_7 + 2$, $d_7 \geq d - k - p - 1 \geq 1$, $T_1 \in \mathcal{T}(k + p + 1, k + p - 1)$. If $d_7 = 1$, then $k + p = d - 2$, $G_7 = C_3$, and by Lemmas 2, 3, 4, 5, 6, 7 and 8,

$$b_{2i}(H) \geq b_{2i}(T_{d-1,d-3} \cup C_3) \geq b_{2i}(T_{d-1,d-3} \cup P_3) \geq b_{2i}(T_{d+1,d-2}),$$

or

$$b_{2i}(H) \geq b_{2i}(P_{d-1} \cup C_3) \geq b_{2i}(T_{d+1,d-2}).$$

If $d_7 = 2$, then $k + p \geq d - 3$, $s_7 \geq 4$, and by Lemmas 2, 3, 6, 7 and 8,

$$b_{2i}(H) \geq b_{2i}(T_{k+p+1,k+p-1} \cup S_{s_7}) \geq b_{2i}(T_{k+p+1,k+p-1} \cup T_{4,2}) \geq b_{2i}(T_{d+1,d-2}),$$

or

$$b_{2i}(H) \geq b_{2i}(P_{k+p+1} \cup S_{s_7}) \geq b_{2i}(P_{k+p+1} \cup T_{4,2}) \geq b_{2i}(T_{d+1,d-2}).$$

If $d_7 \geq 3$, then by Lemmas 2, 6, 7, 8, 9 and 10,

$$b_{2i}(H) \geq b_{2i}(T_{k+p+1,k+p-1} \cup U_{s_7,d_7}) \geq b_{2i}(T_{k+p+1,k+p-1} \cup T_{s_7,d_7}) \geq b_{2i}(T_{d+1,d-2}),$$

or

$$b_{2i}(H) \geq b_{2i}(P_{k+p+1} \cup U_{s_7,d_7}) \geq b_{2i}(P_{k+p+1} \cup T_{s_7,d_7}) \geq b_{2i}(T_{d+1,d-2}).$$

Subcase 2.2. v lies outside any cycle. Then $H \supseteq C_a \cup C_b$.

First, suppose that v lies on $P(G)$ and take $v = x_k$. If $P(G)$ and any cycle have no common vertices, then $H \supseteq C_a \cup C_b \cup P_k \cup P_{d-k}$. By Lemmas 2, 3, 4 and 5,

$$b_{2i}(H) \geq b_{2i}(C_a \cup C_b \cup P_k \cup P_{d-k}) \geq b_{2i}(P_3 \cup P_{d-1}) \geq b_{2i}(P_{d+1}) \geq b_{2i}(T_{d+1,d-2}).$$

If $P(G)$ and exactly one cycle, say C_a, have no common vertices, then $H \supseteq C_a \cup P_k \cup G_1$, where $G_1 \in \mathcal{U}(s_1, d_1)$, $s_1 \geq d_1 + 2$, $d_1 \geq d - k - 1 \geq 1$. By Lemmas 2 and 3,

$$b_{2i}(H) \geq b_{2i}(C_a \cup P_k \cup G_1) \geq b_{2i}(P_2 \cup P_k \cup G_1) \geq b_{2i}(T_{d+1,d-2}).$$

If $P(G)$ and both cycles have common vertices, then $H \supseteq P_k \cup G_2$ or $G_3 \cup G_4$, where $G_2 \in \mathcal{B}(s_2, d_2)$, $G_3 \in \mathcal{U}(s_3, d_3)$, $G_4 \in \mathcal{U}(s_4, d_4)$, $d_2 + 3 \leq s_2 \leq n - 2 - k$, $d_2 \geq d - k - 1 \geq 2$, $s_3 \geq d_3 + 2$, $d_3 \geq k - 1 \geq 2$, $s_4 \geq d_4 + 2$, $d_4 \geq d - k - 1 \geq 1$. Suppose that $H \supseteq P_k \cup G_2$. Since $s_2 - d_2 < h$ and $d_2 \geq 4$, by the induction hypothesis, $G_2 \succ B_{s_2,d_2}$. From (10), $b_{2j}(B_{s_2,d_2}) \geq b_{2j}(T_{s_2,d_2})$ for all j. By Lemmas 2 and 8,

$$b_{2i}(H) \geq b_{2i}(P_k \cup G_2) \geq b_{2i}(P_k \cup B_{s_2,d_2}) \geq b_{2i}(P_k \cup T_{s_2,d_2}) \geq b_{2i}(T_{d+1,d-2}).$$

Suppose that $H \supseteq G_3 \cup G_4$. If $d_3 = 2$ and $d_4 = 1$, then $d = 5$, $s_3 \geq 4$, and by Lemmas 2, 3, 7 and 8,

$$b_{2i}(H) \geq b_{2i}(S_{s_3} \cup C_3) \geq b_{2i}(T_{4,2} \cup P_3) \geq b_{2i}(T_{6,4}) \geq b_{2i}(T_{d+1,d-2}).$$

If $d_3 = 2$, $d_4 = 2$, then $d \leq 6$, $s_3 \geq 4$, $s_4 \geq 4$, and by Lemmas 2, 3 and 8,

$$b_{2i}(H) \geq b_{2i}(S_{s_3} \cup S_{s_4}) \geq b_{2i}(T_{4,2} \cup T_{4,2}) \geq b_{2i}(T_{7,4}) \geq b_{2i}(T_{d+1,d-2}).$$

If $d_3 \geq 3$, $d_4 = 1$, then $d_3 \geq d - 3$, and by Lemmas 2, 3, 7, 8, 9 and 10,

$$b_{2i}(H) \geq b_{2i}(U_{s_3,d_3} \cup C_3) \geq b_{2i}(T_{s_3,d_3} \cup P_3) \geq b_{2i}(T_{d+1,d-1}) \geq b_{2i}(T_{d+1,d-2}).$$

If $d_3 \geq 3$, $d_4 = 2$, then $d_3 \geq d - 4$, $s_4 \geq 4$, and by Lemmas 2, 3, 8, 9 and 10,

$$b_{2i}(H) \geq b_{2i}(U_{s_3,d_3} \cup S_{s_4}) \geq b_{2i}(T_{s_3,d_3} \cup T_{4,2}) \geq b_{2i}(T_{d+1,d-2}).$$

If $d_3 \geq 3$, $d_4 \geq 3$, then $d_3 + d_4 \geq d - 2$, and by Lemmas 2, 8, 9 and 10,

$$b_{2i}(H) \geq b_{2i}(U_{s_3,d_3} \cup U_{s_4,d_4}) \geq b_{2i}(T_{s_3,d_3} \cup T_{s_4,d_4}) \geq b_{2i}(T_{d+1,d-2}).$$

Next, suppose that v lies outside $P(G)$. Then $G \supseteq C_a \cup C_b \cup P(G)$, $C_a \cup G_1$ or G_2, where $G_1 \in \mathcal{U}(s_1, d)$ with $s_1 \geq d + 2$ and $G_2 \in \mathcal{B}(s_2, d)$ with $d + 3 \leq s_2 \leq n - 2$. It is easy to show as above that $b_{2i}(H) \geq b_{2i}(T_{d+1,d-2})$.

Case 3. Any diametrical path of G contains all pendent vertices. Let $P(G) = x_0 x_1 \dots x_d$ be a diametrical path of G. Suppose that $y_0 y_1 \dots, y_p$ is a path whose internal vertices $y_1, y_2, \dots, y_{p-1}$ all have degree two and y_p is a pendent vertex. Then we say that it is a pendent path, denoted by (y_0, y_p).

Subcase 3.1. There are exactly two pendent vertices, x_0 and x_d. Suppose that $\deg_G(x_k)$, $\deg_G(x_l) \geq 3$ and that (x_k, x_0) and (x_l, x_d) are distinct pendent paths. Let $s = l - k$.

Suppose that $s = 0$, i.e. $x_k = x_l$. Then $k \geq 3$ and $l \leq d - 3$. Hence it suffices to prove that $G_1, G_2 \succ B_{n-d+3,3}$, $G_3, G_4 \succ S_{n-d+2}$, where $G_1 = G - (x_{k-3}, x_0) -$

(x_{l+2}, x_d), $G_3 = G - (x_{k-3}, x_0) - (x_{l+1}, x_d)$, $G_2 = G - (x_{k-2}, x_0) - (x_{l+3}, x_d)$ and $G_4 = G - (x_{k-2}, x_0) - (x_{l+2}, x_d)$. By Lemma 3, $G_3, G_4 \succ S_{n-d+2}$. Let $d_1 = d(G_1)$. Since $d_1 \geq 4$, $n - d + 3 - d_1 < h$. By the induction hypothesis and Lemma 12, $G_1 \succ B_{n-d+3,d_1} \succeq B_{n-d+3,3}$. Similarly, $G_2 \succ B_{n-d+3,3}$.

If $s = 1$ or 2, then by similar arguments as above, we have the desired result.

Suppose that $s \geq 3$. We need only consider the case $k \geq 2$ and $l \leq d - 2$. It suffices to prove that $G_5, G_6 \succ B_{n-d+s+1,s+1}$, $G_7 \succ B_{n-d+s+2,s+2}$ and $G_8 \succ B_{n-d+s,s}$, where $G_5 = G - (x_{k-2}, x_0) - (x_{l+1}, x_d)$, $G_7 = G - (x_{k-2}, x_0) - (x_{l+2}, x_d)$, $G_8 = G - (x_{k-1}, x_0) - (x_{l+1}, x_d)$ and $G_6 = G - (x_{k-1}, x_0) - (x_{l+2}, x_d)$. Let $d_j = d(G_j)$ and $n_j = |V(G_j)|$, where $j = 5, 6, 7, 8$. Then $d_j \geq 4$. If $n_j - d_j < h$, then by the induction hypothesis and Lemma 12, we have the desired result.

Suppose that $n_j - d_j = h$. If x_{k-1} lies on all diametrical paths of G_5, then by Lemmas 12 and 15, $G_5 \succ B_{n-d+s+1,s+1}$. Otherwise, by similar arguments as those in Case 2, we also have $G_5 \succ B_{n-d+s+1,s+1}$. Similarly, $G_6 \succ B_{n-d+s+1,s+1}$. By Lemmas 12 and 14, $G_8 \succ B_{n-d+s,s}$. If there exists some diametrical path $P(G_7)$ such that x_{k-1} or x_{l+1} lies outside $P(G_7)$, then by similar arguments as those in Case 2, $G_7 \succ B_{n-d+s+2,s+2}$. Otherwise, by Lemmas 1 (a), 2, 9, 10, 11 and 15, $G_7 - x_{k-1} \succ B_{n-d+s+1,s+2}$, $G_7 - x_{k-1} - x_k \succeq U_{n-d+s,s} \succ T_{n-d+s,s} \succ T_{s+3,s}$, and then $G_7 \succ B_{n-d+s+2,s+2}$.

Subcase 3.2. There is only one pendent vertex. By similar arguments as those in Subcase 3.1, we have the desired result. □

Theorem 2. *Let $G \in \mathcal{B}(n, d)$ with $3 \leq d \leq n - 3$ and $G \neq B_{n,d}$. If there is no vertex-disjoint cycles, then $G \succ B_{n,d}$.*

Proof. We prove this theorem by induction on $n - d$.

By Lemma 13, the result holds for $n - d = 3$. Let $h \geq 4$ and suppose that the result holds for $n - d < h$. Now suppose that $n - d = h$ and $G \in \mathcal{B}(n, d)$.

Case 1. There is no pendent vertex in G. By Lemmas 12 and 14, $G \succ B_{n,d}$.

Case 2. There is a pendent vertex outside some diametrical path $P(G) = x_0x_1 \ldots x_d$. Let u, adjacent to v, be a pendent vertex outside $P(G)$. Then $G - u \in \mathcal{B}(n - 1, d)$. If $G - u = B_{n-1,d}$, then it can be checked that $G - u - v \succ T_{d+1,d-2}$ (or $T_{4,2}$ if $d = 3$) and thus from (11), we have $G \succ B_{n,d}$. Otherwise, by the induction hypothesis, $G - u \succ B_{n-1,d}$. So it suffices to show $b_{2i}(H) \geq b_{2i}(T_{d+1,d-2})$ (or $T_{4,2}$ if $d = 3$), where $H = G - u - v$. We choose C_a, C_b as above in G and if there exists the third cycle, we denote it by C_c.

Subcase 2.1. v lies on some cycle, say C_a. Obviously, if $v = u_0$ or u_t, then H contains no cycles. Otherwise, $H \supseteq C_b$ or C_c.

First, suppose that $v = u_0$ or u_t. If v lies outside $P(G)$, then $H \supseteq P(G)$. By Lemmas 2 and 5, $b_{2i}(H) \geq b_{2i}(P_{d+1}) \geq b_{2i}(T_{d+1,d-2})$.

Suppose that v lies on $P(G)$, take $v = x_k$. If C_a and C_b have exactly one common vertex, then $H \supseteq P_2 \cup P_2 \cup P_k \cup P_{d-k}$, $P_2 \cup P_k \cup P_{d-k+1}$, $P_2 \cup P_k \cup T_1$, $P_{k+1} \cup P_{d-k+1}$, $P_{k+1} \cup T_1$ or $T_1 \cup T_2$, where $T_1 \in \mathcal{T}(d-k+1, d-k-1)$, $T_2 \in \mathcal{T}(k+1, k-1)$. If C_a and C_b have at least two common vertices, then $H \supseteq P_3 \cup P_k \cup P_{d-k}$, $P_k \cup P_{d-k+2}$, $P_k \cup T_3$, $P_k \cup T_4$ or $P(G)$, where $T_3 \in \mathcal{T}(d-k+2, d-k-1)$, $T_4 \in \mathcal{T}(d-k+2, d-k)$. If $H \supseteq P_2 \cup P_2 \cup P_k \cup P_{d-k}$, $P_2 \cup P_k \cup P_{d-k+1}$, $P_{k+1} \cup P_{d-k+1}$, $P_3 \cup P_k \cup P_{d-k}$, $P_k \cup P_{d-k+2}$ or $P(G)$, then by Lemmas 2, 4 and 5,

$$b_{2i}(H) \geq b_{2i}(P_{d+1}) \geq b_{2i}(T_{d+1,d-2}).$$

If $H \supseteq P_2 \cup P_k \cup T_1$ or $P_{k+1} \cup T_1$, then by Lemmas 2, 4, 6, 7 and 8,

$$b_{2i}(H) \geq b_{2i}(P_{k+1} \cup T_{d-k+1,d-k-1}) \geq b_{2i}(T_{d+1,d-1}) \geq b_{2i}(T_{d+1,d-2}).$$

If $H \supseteq T_1 \cup T_2$, then by Lemmas 2, 6 and 8,

$$b_{2i}(H) \geq b_{2i}(T_{k+1,k-1} \cup T_{d-k+1,d-k-1}) \geq b_{2i}(T_{d+1,d-2}).$$

If $H \supseteq P_k \cup T_3$, then by Lemmas 2, 6 and 8,

$$b_{2i}(H) \geq b_{2i}(P_k \cup T_{d-k+2,d-k-1}) \geq b_{2i}(T_{d+1,d-2}).$$

If $H \supseteq P_k \cup T_4$, the by Lemmas 2, 6, 7 and 8,

$$b_{2i}(H) \geq b_{2i}(P_k \cup T_{d-k+2,d-k}) \geq b_{2i}(T_{d+1,d-1}) \geq b_{2i}(T_{d+1,d-2}).$$

Next, suppose that $v \neq u_0$ and u_t. If v lies outside $P(G)$, then $H \supseteq G_1$ or $P(G) \cup C_s$, where $G_1 \in \mathcal{U}(s_1, d)$, $s_1 \geq d+2$, $s = b$ or c. If $H \supseteq G_1$, then by Lemmas 2, 5, 9 and 10,

$$b_{2i}(H) \geq b_{2i}(G_1) \geq b_{2i}(U_{s_1,d}) \geq b_{2i}(T_{s_1,d}) \geq b_{2i}(P_{d+1}) \geq b_{2i}(T_{d+1,d-2}).$$

If $H \supseteq P(G) \cup C_s$, then by Lemmas 2, 3 and 5,

$$b_{2i}(H) \geq b_{2i}(P_{d+1} \cup S_s) \geq b_{2i}(P_{d+1}) \geq b_{2i}(T_{d+1,d-2}).$$

Suppose that v lies on $P(G)$. Then $P(G)$ and C_a have vertices, say $x_k, \ldots, x_{k+p}$, in common, where $p \geq 0$.

If $p = 0$, then $k \geq 1$, $H \supseteq P_k \cup P_{d-k} \cup C_s$, where $s = b$ or c, and by Lemmas 2, 3, 4 and 5,

$$b_{2i}(H) \geq b_{2i}(P_k \cup P_{d-k} \cup S_s) \geq b_{2i}(P_{d-1} \cup P_3) \geq b_{2i}(P_{d+1}) \geq b_{2i}(T_{d+1,d-2}).$$

Suppose that $p \geq 1$. If $v \neq x_k, x_{k+p}$, then $H \supseteq G_2$, where $G_2 \in \mathcal{U}(s_2, d)$, $s_2 \geq d+2$. It is easy to show as above that $b_{2i}(H) \geq b_{2i}(T_{d+1,d-2})$. Otherwise, for $v = x_k$ or $v = x_{k+p}$, say $v = x_k$, and then $k \geq 1$, $H \supseteq P_k \cup G_3$, $P_k \cup G_4$ or $P_k \cup G'$, where $G_3 \in \mathcal{U}(s_3, d_3)$, $G_4 \in \mathcal{U}(s_4, d_4)$, $s_3 \geq d_3 + 3$, $d_3 \geq d - k - 1 \geq 2$, $s_4 \geq d_4 + 2$, $d_4 \geq d - k \geq 2$ and G' is the graph obtained by attaching a path P_{d-k-2} to a vertex of $C_b = C_3$. Suppose that $H \supseteq P_k \cup G_3$. If $d_3 = 2$, then $k = d - 3$, $s_3 \geq 5$, and by Lemmas 2, 3 and 8,

$$b_{2i}(H) \geq b_{2i}(P_k \cup S_{s_3}) \geq b_{2i}(P_k \cup T_{5,2}) \geq b_{2i}(T_{d+1,d-2}).$$

If $d_3 \geq 3$, then by Lemmas 2, 8, 9 and 10,

$$b_{2i}(H) \geq b_{2i}(P_k \cup U_{s_3,d_3}) \geq b_{2i}(P_k \cup T_{s_3,d_3}) \geq b_{2i}(T_{d+1,d-2}).$$

Suppose that $H \supseteq P_k \cup G_4$. If $d_4 = 2$, then $k = d - 2$, $s_4 \geq 4$, and by Lemmas 2, 3, 7 and 8,

$$b_{2i}(H) \geq b_{2i}(P_k \cup S_{s_4}) \geq b_{2i}(P_k \cup T_{4,2}) \geq b_{2i}(T_{d+1,d-1}) \geq b_{2i}(T_{d+1,d-2}).$$

If $d_4 \geq 3$, then by Lemmas 2, 7, 8, 9 and 10,

$$b_{2i}(H) \geq b_{2i}(P_k \cup U_{s_4,d_4}) \geq b_{2i}(P_k \cup T_{s_4,d_4}) \geq b_{2i}(T_{d+1,d-1}) \geq b_{2i}(T_{d+1,d-2}).$$

Suppose that $H \supseteq P_k \cup G'$. If $d - k - 2 = 0$, then $k = d - 2$, and by Lemmas 2, 7 and 8,

$$b_{2i}(H) \geq b_{2i}(P_k \cup C_3) \geq b_{2i}(P_k \cup T_{4,2}) \geq b_{2i}(T_{d+1,d-1}) \geq b_{2i}(T_{d+1,d-2}),$$

since it can be checked that $b_{2j}(C_3) \geq b_{2j}(T_{4,2})$. If $d - k - 2 \geq 1$, then by Lemmas 1 (a) and (b), 2 and 4,

$$\begin{aligned}
b_{2i}(H) &\geq b_{2i}(P_k \cup G' - u_0u_1) + b_{2i-2}(P_k \cup G' - u_0 - u_1) \\
&= b_{2i}(P_k \cup T_{d-k+1,d-k-1}) + b_{2i-2}(P_k \cup P_{d-k-1}) \\
&= b_{2i}(P_k \cup P_{d-k-1}) + 2b_{2i-2}(P_k \cup P_{d-k-2}) + b_{2i-2}(P_k \cup P_{d-k-1}) \\
&\geq b_{2i}(P_{d-2}) + 3b_{2i-2}(P_{d-3}) = b_{2i}(T_{d+1,d-2}).
\end{aligned}$$

Subcase 2.2. v lies outside any cycle. Then H contains two cycles C_a, C_b with at least one common vertex. Let $C_a \cdot C_b$ denote the subgraph of G induced by $V(C_a) \cup V(C_b)$.

First, suppose that v lies on $P(G)$, take $v = x_k$. If vertices on $P(G)$ lie outside any cycle, then $H \supseteq C_a \cdot C_b \cup P_k \cup P_{d-k}$. By Lemmas 2, 3, 4 and 5,

$$b_{2i}(H) \geq b_{2i}(C_a \cdot C_b \cup P_k \cup P_{d-k}) \geq b_{2i}(P_3 \cup P_k \cup P_{d-k}) \geq b_{2i}(T_{d+1,d-2}).$$

If some vertex of $P(G)$ lies on one cycle, then $H \supseteq P_k \cup G_1$, where $G_1 \in \mathcal{B}(s_1, d_1)$, $d_1 + 2 \leq s_1 \leq n - 2 - k$, $d_1 \geq \max\{d - k - 1, 2\}$. Suppose that $s_1 \geq d_1 + 3$. If $d_1 = 2$, then $k \geq d - 3$, $s_1 \geq 5$, and by Lemmas 2, 3 and 8,

$$b_{2i}(H) \geq b_{2i}(P_k \cup S_{s_1}) \geq b_{2i}(P_k \cup T_{5,2}) \geq b_{2i}(T_{d+1,d-2}).$$

Otherwise, $d_1 \geq 3$, $s_1 - d_1 < h$, and by the induction hypothesis, $G_1 \succeq B_{s_1,d_1}$. Hence by Lemmas 2 and 8 and bearing in mind (10),

$$b_{2i}(H) \geq b_{2i}(P_k \cup G_1) \geq b_{2i}(P_k \cup B_{s_1,d_1}) \geq b_{2i}(P_k \cup T_{s_1,d_1}) \geq b_{2i}(T_{d+1,d-2}).$$

Now suppose that $s_1 = d_1 + 2$. Then G_1 is obtained by attaching respectively paths P_l and P_{d_1-l-2} to the two non-adjacent vertices in $K_4 - e$. If $d_1 = 2$, then $k \geq d - 3$, and by Lemmas 2 and 8,

$$b_{2i}(H) \geq b_{2i}(P_k \cup K_4 - e) \geq b_{2i}(P_k \cup T_{5,2}) \geq b_{2i}(T_{d+1,d-2}),$$

since it can be checked that $b_{2j}(K_4 - e) \geq b_{2j}(T_{5,2})$. If $d_1 \geq 3$, then by Lemmas 1 (a) and (c), 2, 4, 9 and 10,

$$\begin{aligned} b_{2i}(H) &\geq b_{2i}(P_k \cup G_1 - u_0u_1) + b_{2i-2}(P_k \cup G_1 - u_0 - u_1) \\ &\geq b_{2i}(P_k \cup U_{s_1,d_1}) + b_{2i-2}(P_k \cup P_{l+1} \cup P_{d_1-l-1}) \\ &\geq b_{2i}(P_k \cup T_{d-k+1,d-k-1}) + b_{2i-2}(P_k \cup P_{d-k-2}) \\ &\geq b_{2i}(T_{d+1,d-2}). \end{aligned}$$

Next, suppose that v lies outside $P(G)$. Then $G \supseteq C_a \cdot C_b \cup P(G)$ or G_1, where $G_1 \in \mathcal{B}(s, d)$ with $d + 2 \leq s \leq n - 2$. It is easy to show as above that $b_{2i}(H) \geq b_{2i}(T_{d+1,d-2})$.

Case 3. Any diametrical path of G contains all pendent vertices. By similar arguments as those in Case 3 of Theorem 1, $G \succ B_{n,d}$. $\square$

Combining Theorems 1 and 2, and using the increasing property (2), we obtain the following main result of this paper.

Theorem 3. *Let $G \in \mathcal{B}(n, d)$ with $3 \leq d \leq n - 3$ and $G \neq B_{n,d}$. Then $E(G) > E(B_{n,d})$.*

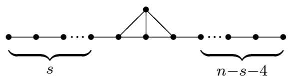

Figure 4: Graph B_n^s with $0 \le s \le \lfloor n/2 \rfloor - 2$.

Finally we discuss the case $d = n-2$. Any graph G in $\mathcal{B}(n, n-2)$ is of the form B_n^s, which is shown in Figure 4, where $0 \le s \le \lfloor n/2 \rfloor - 2$. By direct calculation of the eigenvalues, we find

Conjecture 1. *For $n \ge 6$, let*

$$s(n) = \begin{cases} 1 & \text{if } n \equiv 1, 3 \pmod 4, \\ n/2 - 3 & \text{if } n \equiv 2 \pmod 4, \\ n/2 - 2 & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Then for $0 \le s \le \lfloor n/2 \rfloor - 2$ and $s \ne s(n)$, $E(B_n^s) > E(B_n^{s(n)})$.

However, from the the characteristic polynomials, we cannot use the relation "$\succ$" to deduce $E(B_n^s) > E(B_n^{s(n)})$ for $s \ne s(n)$, as may be seen in Table 1 for $6 \le n \le 9$.

Table 1: Characteristic polynomials and energies of B_n^s with $0 \le s \le \lfloor n/2 \rfloor - 2$ and $6 \le n \le 9$.

n	s	coefficients							$E(B_n^s)$
		a_3	a_4	a_5	a_6	a_7	a_8	a_9	
6	$0 = s(6)$	-4	8	6	0				7.73240
	1	-4	7	4	-1				7.80642
7	0	-4	14	10	-3	-2			9.25165
	$1 = s(7)$	-4	13	8	-4	-2			9.21222
8	0	-4	21	14	-11	-8	0		10.32058
	1	-4	20	12	-11	-6	1		10.37466
	$2 = s(8)$	-4	20	12	-12	-8	0		10.29253
9	0	-4	29	18	-25	-18	3	2	11.78220
	$1 = s(9)$	-4	28	16	-24	-14	5	2	11.73846
	2	-4	28	16	-25	-16	4	2	11.74918

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