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On a Pair of Equienergetic Graphs *

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Abstract

In this paper we correct an error in[3] and completely solve the problem of constructing a pair of equienergetic graphs on p vertices for all $p \geq 10$.

1 Introduction

Let G be a simple graph with p vertices. The eigenvalues of the adjacency matrix of G, denoted by λ_i, $i = 1, 2, \ldots, p$, are the eigenvalues of G, and form the spectrum of G [2]. The energy $E(G)$ of a graph G with p vertices is defined as $E(G) = \sum_{i=1}^{p} |\lambda_i|$.

Two non-isomorphic graphs of the same order are cospectral if they have the same spectrum. Two non-isomorphic graphs G_1 and G_2 of the same order are said to be equienergetic if $E(G_1) = E(G_2)$. Certainly, two cospectral graphs are obviously equienergetic.

For connected graphs it can be seen that there are no equienergetic graphs of order $p \leq 5$. In[3], the authors point out: for $p = 7, 9$, there are such graphs but they are cospectral. In [1] a class of equienergetic graphs for $p \equiv 0(mod4)$ is constructed using the tensor product of graphs. Stevanović [4] recently constructed such equienergetic graphs for $p \equiv 0(mod5)$. In[3] equienergetic graphs for $p = 6, 14, 18$ and $p \geq 20$ are constructed. Thus the existence of pairs of equienergetic graphs for $p = 11, 13, 17, 19$ remains open. Equienergetic graphs were constructed also in [5-11].

In this paper first we show a pair of equienergetic non-cospectral trees of order 9. Thus the conclusion in[3], that there is no equienergetic graphs of order 9, is not true. Then we construct pairs of connected, non-cospectral equienergetic graphs for every $p \geq 10$. In [10], the authors also gave non-regular equienergetic graphs for $p \geq 10$, but we and they complete the results independently. Our method is different from them, we use matrix. And our equienergetic graphs are also different from their graphs. Furthermore we give connected, non-cospectral, non-regular equienergetic graphs for $p \geq 9$. But the pair of equienergetic graphs with 9 vertices in [10] are connected regular graphs of degree four.

2 A pair of equienergetic graphs of order 9

Theorem 2.1

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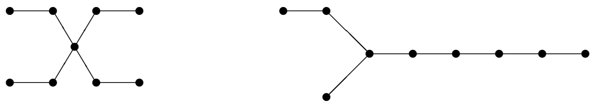
There exists a pair of equienergetic non-cospectral graphs of order 9.

Proof. We consider the following two connected graphs of order 9 (T_1, T_2 are graphs 2.79, 2.91 in appendix of [2] respectively).



T_1 T_2

Let $P_1(\lambda)$, $P_2(\lambda)$ be characteristic polynomials of T_1, T_2 respectively.
Then

$$P_1(\lambda) = \lambda^9 - 8\lambda^7 + 18\lambda^5 - 16\lambda^3 + 5\lambda$$
$$= \lambda(\lambda^2 - 1)^3(\lambda^2 - 5).$$

The spectra of T_1 is $(0, \pm 1, \pm 1, \pm 1, \pm\sqrt{5})$.
Thus, $E(T_1) = 3 \times 2 + 2\sqrt{5} = 6 + 2\sqrt{5}$.

$$P_2(\lambda) = \lambda^9 - 8\lambda^7 + 20\lambda^5 - 17\lambda^3 + 4\lambda$$
$$= \lambda(\lambda^2 - 4)(\lambda^2 - 1)(\lambda^2 - \frac{3+\sqrt{5}}{2})(\lambda^2 - \frac{3-\sqrt{5}}{2}).$$

The spectra of T_2 is $(0, \pm 2, \pm 1, \pm\sqrt{\frac{3+\sqrt{5}}{2}}, \pm\sqrt{\frac{3-\sqrt{5}}{2}})$.
Thus, $E(T_2) = 6 + 2\sqrt{\frac{3+\sqrt{5}}{2}} + 2\sqrt{\frac{3-\sqrt{5}}{2}}$
$= 6 + \sqrt{6 + 2\sqrt{5}} + \sqrt{6 - 2\sqrt{5}}$
$= 6 + \sqrt{5} + 1 + \sqrt{5} - 1$
$= 6 + 2\sqrt{5}$.
Hence, $E(T_1) = E(T_2)$.
Clearly, T_1 and T_2 are equienergetic non-cospectral graphs. □

3 Equienergetic graphs for every $p \geq 10$

Definition 3.1

Let G be an $r-$regular graph on p vertices labeled as $\{v_1, v_2, \ldots, v_p\}$. Then introduce a set of k ($k \geq 1$) isolated vertices and make all of them adjacent to all vertices of G. The resultant graph is denoted by H.

Lemma 3.1 *[2]*

Let G be a connected $r-$regular graph on p vertices with its adjacency matrix A having m distinct eigenvalues $\lambda_1 = r$, $\lambda_2, \ldots \lambda_m$. Then there exists a polynomial $P(x) = p \times \frac{(x-\lambda_2)(x-\lambda_3)\ldots(x-\lambda_m)}{(r-\lambda_2)(r-\lambda_3)\ldots(r-\lambda_m)}$, such that $P(A) = J$ where J is the square matrix of order p whose all entries are one, so that $P(r) = p$ and $P(\lambda_i) = 0$ $\forall \lambda_i \neq r$.

Theorem 3.1

Let G be a connected $r-$regular graph on p vertices and H be the graph obtained by the above operation. Then $E(H) = E(G) + \sqrt{r^2 + 4kp} - r$

Proof. Let A be the adjacency matrix of G and J the all one matrix. Then the adjacency matrix of H is given by

$$\begin{pmatrix} A_{p\times p} & J_{p\times k} \\ J_{k\times p} & 0 \end{pmatrix}.$$

Therefore the characteristic polynomial of H is

$$P_H(\lambda) = \begin{vmatrix} \lambda I_p - A & -J_{p\times k} \\ -J_{k\times p} & \lambda I_k \end{vmatrix}.$$

Then we have,

$$P_H(\lambda) = \begin{vmatrix} \lambda I_k & -J_{k\times p} \\ -J_{p\times k} & \lambda I_p - A \end{vmatrix}$$

$$= \lambda^k \left|\lambda I_p - A - \tfrac{k}{\lambda} J_p\right|$$

$$= \lambda^{k-p} \left|\lambda^2 I_p - A\lambda - kJ_p\right|$$

$$= \lambda^{k-p} \textstyle\prod_{i=1}^{p} [x^2 - \lambda_i x - kP(\lambda_i)]$$

where $P(x)$ is the polynomial in Lemma 3.1.
Thus the eigenvalues of H are
$x = 0; \quad k-1 \quad \textit{times}$
$x = \frac{r\pm\sqrt{r^2+4kp}}{2}; \quad \textit{corresponing} \quad \textit{to} \quad \lambda_1 = r$
$x = \lambda_i; \quad \textit{corresponing} \quad \textit{to} \quad \lambda_i \neq r.$

Hence, $E(H) = E(G) + \sqrt{r^2 + 4kp} - r.$ □

Theorem 3.2

There exists a pair of equienergetic graphs for $p \geq 10$.

Proof.

The cubic graphs 3.16 and 3.26 ([2]) on 10 vertices are equienergetic and non-cospectral. Let H be the graphs obtained from the above cubic graphs as in Definition 3.1 for $k \geq 1$. Using Theorem 3.1, we obtain the equienergetic graphs for $p \geq 10$. □

Corollary 3.1

There exists a pair of equienergetic graphs for $p = 11, 13, 17, 19$.

Thus we can conclude as follows.

There exists a pair of equienergetic non-cospectral graphs of order p if and only if $p = 6$ and $p \geq 8$.

Remark. If unconnected graphs are allowed as equienergetic (non-cospectral) graphs, then we can construct pairs of equienergetic (non-cospectral) graphs of order p for $p \geq 4$. Because $C_4 \bigcup rK_1$ and $C_3 \bigcup (r+1)K_1$ and $P_2 \bigcup P_2 \bigcup rK_1$, $r = 0, 1, 2, \ldots$, are equienergetic and non-cospectral.

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