MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **58** (2007) 511-521

ISSN 0340 - 6253

A NOTE ON GF(5)-REPRESENTABLE MATROIDS

A. BETTEN, R. J. KINGAN, AND S. R. KINGAN

(Received July 16, 2006)

Abstract. Rota's conjecture states that the number of minimal excluded minors for the class of $GF(q)$-representable matroids is finite. The conjecture holds for $q = 2$, 3, and 4, but remains unresolved for fields of order 5 and higher. At present only six 7-element minimal excluded minors for $GF(5)$-representation are known. We found two 8-element, nine 9-element, one 10-element, and one 11-element rank-3 minimal excluded minors. There are no 12-element rank-3 minimal excluded minors for $GF(5)$-representable matroids. Our list is exhaustive up to 12 elements.

1. Introduction

The matroid terminology follows Oxley [6]. A *matroid* M is defined as an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E, called *independent sets*, that satisfy the following axioms: the empty set is independent; any subset of an independent set is independent; and if I_1 and I_2 are independent sets such that $|I_1| < |I_2|$, then there exists $e \in I_2 - I_1$ such that $I_1 \cup \{e\}$ is independent. A matroid in which all the one and two element sets are independent is called a *simple matroid* or *combinatorial geometry*.

A maximal independent set is called a *basis*. For a subset X of E, the *rank* of X, denoted by $r(X)$, is the size of a maximal independent subset in X. The *rank* of M is the size of a basis set. A subset X of E is *closed* if $r(X \cup \{e\}) = r(X) + 1$ for all elements e not in X. A *flat* is a closed set. The flats of a matroid satisfy the following axioms: the intersection of any two flats is a flat; and if F is a flat and $F_1, \ldots, F_k$ are flats that cover F, then $F_1 - F, \ldots, F_k - F$ partition $E - F$.

Flats of rank 1, 2, 3, and so on can be represented geometrically as points, lines, planes, etc. A rank 1 simple matroid is a point. A rank 2 n-element simple matroid is a line with n points. We denote it as $U_{2,n}$. A rank 3 n-element simple matroid is a 2-dimensional figure consisting of points and lines that satisfy the following axioms: any two distinct points belong to exactly one line; any line contains at least two distinct points; and there are at least three non-colinear points [6, pg. 42]. The first two axioms are the flat axioms expressed for points and lines. The third axiom is the trivial condition required to ensure the matroid has rank 3. Simple matroids of rank at most 3 are also called *linear spaces*.

A rank r simple matroid M with ground set E is *representable* over a field F if there is a rank preserving map $\phi : E \longrightarrow V(r, F)$. If M is representable, then we can find a matrix A with entries from F that represents M. We can write the matrix in standard form $A = [I_r|D]$, where the columns of A correspond to a non-zero representative vector from one-dimensional subspaces of $V(r, F)$. So the columns of A can be viewed as a subset of $PG(r-1, F)$. In this paper we are interested in matroids representable over finite fields, $GF(q)$, where q is a prime power. For a fixed q, our goal is to find the rank 3 obstructions for representability that are in a sense minimal. We need to make precise what we mean by minimal.

The matroid obtained by *deleting* an element x from M, denoted by $M\backslash x$, is defined as the matroid on $E - \{x\}$, in which $I \subseteq E - \{x\}$ is an independent set if I is independent in M. The matroid obtained by *contracting* x from M, denoted by M/x, is defined as the matroid on $E - \{x\}$, in which $I \subseteq E - \{x\}$ is an independent set if $I \cup \{x\}$ is independent in M. A matroid N is a *minor* of a matroid M if N can be obtained from M by deleting and/or contracting elements. We say a class of matroids is *closed under minors* if every minor of a matroid in the class is also in the class. For example, the classes of $GF(q)$-representable matroids for a specific prime power (matrices with entries from $GF(q)$), graphic matroids (graphs), and regular matroids are all closed under minors. Regular matroids are matroids represented by matrices over the reals with the property that every square submatrix has determinant in $\{0, 1, -1\}$.

The dual of a matroid M on set E with basis set $\mathcal{B}$ is defined as the matroid on E with basis set $\{E - B : B \in \mathcal{B}\}$. The dual matroid is denoted by M^*. We say a class of matroids is *closed under duality* if the dual of every matroid in the class is also in the class. For example, the classes of $GF(q)$-representable matroids and regular matroids are closed under duality, but not the class of graphic matroids since the duals of non-planar graphs are not graphs.

A matroid M is a *minimal excluded minor* for a minor-closed class of matroids, if M is not in the class, but every minor of M is in the class. Excluded minor results are quite popular in graphs and matroids. The Kuratowski-Wagner characterization of planar graphs is generally considered to be the first excluded minor result. It states that a graph is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$, where K_5 is the complete graph on five vertices and $K_{3,3}$ is the complete bipartite graph with three vertices in each of the two vertex classes. So K_5 and $K_{3,3}$ are the minimal obstructions to planarity in graphs.

The study of excluded minor results in matroids was initiated by Tutte when he characterized the class of binary matroids and regular matroids in terms of excluded minors. Specifically, he proved that a matroid is binary if and only if it had no minor isomorphic to $U_{2,4}$, the four-point line [6, 9.1.5]. He also proved that a matroid is regular if and only if it has no minor isomorphic to $U_{2,4}$, the Fano matroid and its dual [6, 13.1.1]. Observe that the Fano matroid is $PG(2,2)$, the well-known design consisting of 7 points and 7 lines (see Figure 1) usually denoted by F_7. It is representable only over fields of characteristic two . Note that for a class of matroids closed under duality, if M is a minimal excluded minor for the class, then so is M^*.

In 1971 Rota conjectured that the class of $GF(q)$-representable matroids has a finite list of minimal excluded minors [6, 14.1.1]. This conjecture remains unsolved. Several years later Bixby and Seymour independently showed that a matroid is ternary if and only if it has no minor isomorphic to $U_{2,5}$, $U_{3,5}$, F_7, or F_7^* [6, 10.3.1]. The matroid $U_{2,5}$ is the five-point line and its dual $U_{3,5}$ can be represented by five freely placed points in the plane. In 2000 Geelen, Gerards, and Kapoor proved that a matroid is quaternary if and only if it has no minor isomorphic to $U_{2,6}$, $U_{4,6}$, P_6, $\bar{F}_7$, $\bar{F}_7^*$, P_8, and P_8'' [2]. The matroid $U_{2,6}$ is the six-point line and its dual is $U_{4,6}$. The self-dual 6-element, rank-3 matroid P_6 has a single non-trivial line passing through 3 points. The relaxed Fano matroid, $\bar{F}_7$, is almost like the Fano matroid except the curved line is missing. It is representable only over fields of characteristic other than two. The matroids P_8 and P_8'' have rank 4. The upper bound on the number of elements in a minimal excluded minor for a field of order $q \leq 4$ is small, so once the upper bound is determined, it is feasible to work out the minimal excluded minors by hand. For $q \geq 5$ the upper bound is expected to be large, so using a computer may be the only way to get the minimal excluded minors. Moreover, at present the bound for $q \geq 5$ is unknown. The results of this paper suggest that there will be a finite upper bound and that for rank 3 matroids the upper bound will be $2q + 1$.

Using computers to search for combinatorial objects is an established area of research in other areas of combinatorics. For example, design theorists look for projective planes and other designs with certain properties. We are applying similar computing ideas to matroid theory with a view to getting certain types of matroids that may be of interest to matroid theorists.

We found several rank 3 minimal excluded minors for $GF(5)$-representation. Figure 1 shows the known 7-element minimal excluded minors. Figures 2, 3, and 4 show previously unknown minimal excluded minors. We found no 12-element minimal excluded minor. Our list is exhaustive up to 12 elements. In Figure 2 observe that the second 8-element matroid is the well-known 8_3-configuration. In Figure 3, M_8 is the well-known Pappus matroid and M_6 and M_7 are not representable over any field.

2. The method

For $n \geq 4$, let $\mathcal{M}_n$ denote the set of all non-isomorphic rank 3, n-element simple matroids and let $\mathcal{M}_n^5$ denote the set of all non-isomorphic rank 3, n-element, $GF(5)$-representable simple matroids. The number of non-isomorphic rank 3 simple matroids of sizes 3 to 12 are, respectively, 1, 2, 4, 9, 23, 68, 383, 5249, 232,928, and 28,872,972 [1]. Using the generation technique in [3] and adjusting for inequivalent representations, we found that the number of non-isomorphic rank 3, $GF(5)$-representable simple matroids of sizes 3 to 12 are, respectively, 1, 2, 4, 9, 18, 34, 82, 168, 296, and 476.

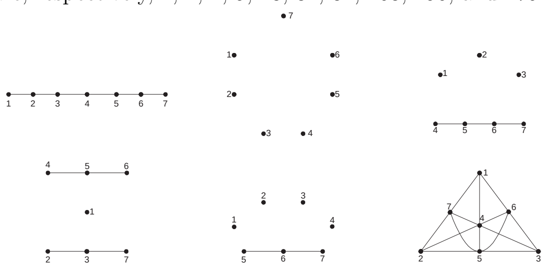

Figure 1: The 7-element minimal excluded minors

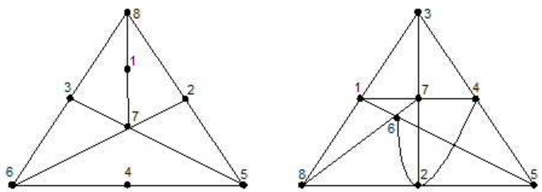

Figure 2: The 8-element rank 3 minimal excluded minors

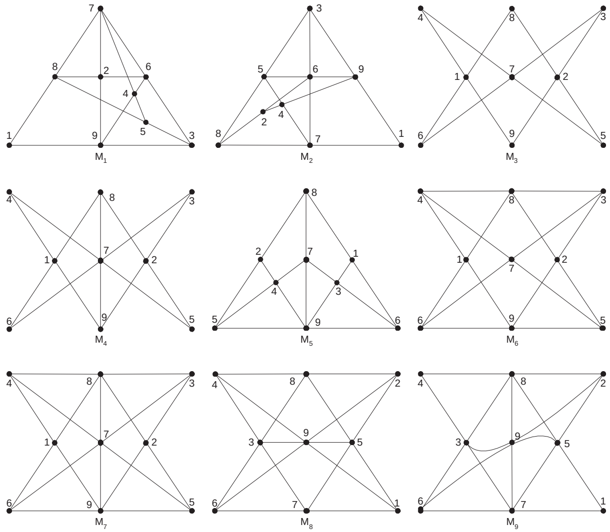

Figure 3: The 9-element rank 3 minimal excluded minors

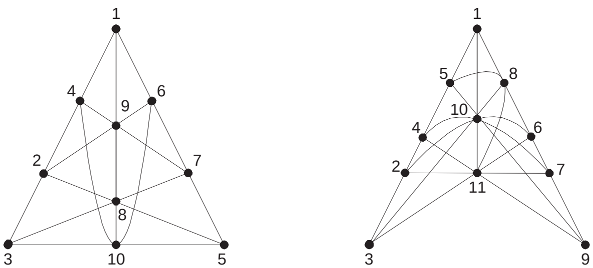

Figure 4: The 10-element and 11-element rank 3 minimal excluded minors

Our minimal excluded minor algorithm proceeds through a series of filtering steps. First, observe that $U_{2,q+2}$ is a minimal excluded minor for $GF(q)$-representability since the lines in $PG(r-1,q)$ have $q+1$ points [6, 6.5.3]. Note that $U_{2,q+2}$ is the line with $q+2$ points. So we can eliminate from consideration the matroids $M \in \mathcal{M}_n$ that have a $U_{2,q+2}$ minor. Since M is a rank 3 simple matroid, a single-element contraction of M will have rank 2 and size $n-1$. Further, observe that if M has a $U_{2,q+2}$ minor, then it must have at least $q+3$ elements.

The contraction algorithm follows the general rules for contraction listed in [6, p. 120]. Given a rank 3 simple matroid M, and a point e in M, let L be a line (possibly trivial) not containing e. Observe that there will be at least one such line because M has rank 3. The elements of M/e are the original elements of L, together with images of the elements of $E-(L\cup e)$ under projection from e. To project a point $f\in E-(L\cup e)$ onto L, we must find the line L' (also possibly trivial) containing f and e. If $L'\cap L=g$, then f and all other points on $L'-e$ project onto g. If $L'\cap L=\phi$, then f and all other points on $L'-e$ project onto a new point which is added to L.

If $n\geq q+3$, then we can first check the matroid for an element contained in no non-trivial line (*i.e.* a line with at least three points). If it has such a point, then it will have a $U_{2,q+2}$-minor since when such a point is contracted all the other points fall on a line without giving rise to parallel points. If it doesn't have such a point, then we must check each single-element contraction of M for a $U_{2,q+2}$-minor. The contraction will consist of one line with $n-1$ points, but some of the points may be parallel to other points. We need to know which points from M will contribute to the length of the line, and which will become parallel points to other points. The matroid M/e has a $U_{2,q+2}$-minor if and only if the simplified line has length at least $q+2$. The complexity of this algorithm is $\mathcal{O}(n^3)$.

In Figure 5 we see that contracting the point 1 onto line $\{3,4,8,9\}$ gives a line with only 6 distinct classes of points because point 9 is parallel to point 6 and point 8 is parallel to point 7. So $M/1$ does not have a $U_{2,7}$-minor. However, contracting point 5 onto line $\{3,4,8,9\}$ gives a $U_{2,7}$-minor.

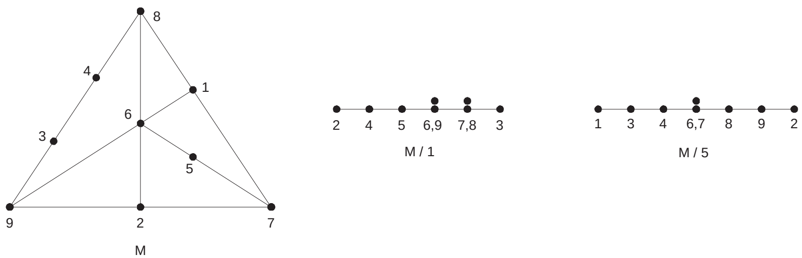

Figure 5: Examples of single-element contractions

Running the contraction algorithm with $q = 5$ and the exhaustive, isomorph-free sets of rank 3, simple matroids of sizes 8, 9, 10, 11, and 12 gives 44, 149, 492, 1302, and 2279 matroids, respectively, with no $U_{2,7}$-minor.

In the second step we eliminate from consideration the rank 3, simple matroids left which are representable over $GF(q)$. This gives us the excluded minors for $GF(q)$-representation with no $U_{2,q+2}$-minors. However, they may not be minimal. Running this step with $q = 5$ and the output from Step 1 gives 10, 67, 224, 1006, and 1803 matroids of size 8, 9, 10, 11, and 12 as the excluded minors for $GF(5)$-representation with no $U_{2,7}$-minor.

In the third step we eliminate the matroids which have a single-element deletion that is not $GF(q)$-representable. To check single-element deletions we remove one element at a time from M and compare the resulting matroid with the matroids in $\mathcal{M}^q_{n-1}$. Only those matroids, all of whose single-element deletions occur in $\mathcal{M}^q_{n-1}$ are retained. These will be the minimal excluded minors for $GF(q)$-representation.

In the second and third step an implementation of the partition backtrack method [4] by the first author was used to put the point-line incidence matrix of a matroid into canonical form. Using this algorithm dramatically decreased the time required to identify $GF(q)$-representable matroids from an older algorithm that required pairwise isomorphism checking. For a discussion on canonical form see [5].

We illustrate this with an example. The matroids M and N in Figure 6 are not identical, but isomorphic under the map $(7, 2, 1, 4, 5, 3, 8, 6, 10, 11, 9)$, where $1 \longrightarrow 7$, $2 \longrightarrow 2$, $3 \longrightarrow 1$, and so on. We will put them both in canonical form.

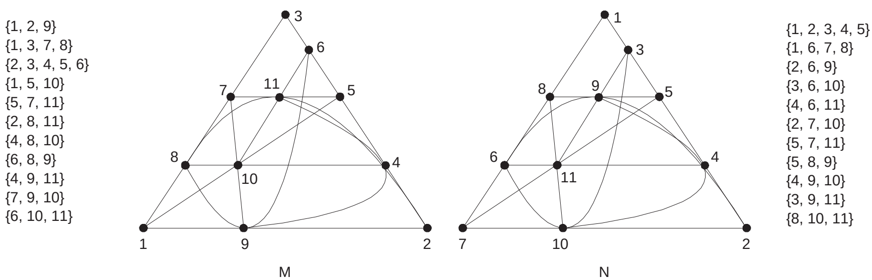

Figure 6: Two non-identical, but isomorphic matroids with 11 elements

The *point-line incidence matrix* of a matroid is the matrix with points along the rows and non-trivial lines along the columns. Entry $a_{ij} = 1$ if point i is on line j and 0 otherwise. It uniquely defines the matroid. The point-line incidence matrices for M and N are shown below. Each has 11 rows, 11 columns and 36 nonzero entries. Rows and columns are labeled $0, 1, \ldots, 10$.

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} N = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Applying the canonical form generator gives us the following permutations of the rows and columns of M: $[0, 3, 10, 4, 1, 5, 2, 8, 6, 7, 9]$ and $[10, 8, 6, 7, 5, 9, 3, 0, 1, 4, 2]$. Similarly, we get the following permutations of the rows and columns of N: $[10, 3, 5, 4, 1, 8, 0, 2, 9, 6, 7]$ and $[10, 8, 4, 3, 2, 7, 6, 9, 1, 0, 5]$. Observe that in canonical form M_c and N_c are identical. So M and N are isomorphic. By replacing pairwise isomorphism checks of matroids with pairwise comparisons of point-line incidence matrices in canonical form we were able to identify isomorphic matroids much more quickly.

$$M_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} N_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The matroids are read and output in sparse matrix format. For the matrix M above, the sparse matrix format is obtained by recording the positions of the nonzero entries, in row-wise fashion, counting from zero. The first row has nonzero entries in positions 0, 1 and 3, so the first three entries in the list are 0, 1 and 3. The second row, starting with position 11, has nonzero entries in positions 0, 2 and 5, so the next three entries are $11 = 0 + 11$, $13 = 2 + 11$ and $16 = 5 + 11$. When this process is completed we get:

`0 1 3 11 13 16 23 24 35 39 41 46 47 48 57 62 65 67 70 75 78 82 83 84 88 95 96 97 102 105 108 109 114 115 118 120`

The output for this matroid in canonical form is:

`6 7 8 17 20 21 27 30 31 37 40 43 45 46 54 55 58 65 67 69 71 73 77 79 82 83 90 91 92 96 99 100 103 108 118 120`

Similarly matrix N in sparse matrix format is:

`0 1 11 13 16 22 25 31 33 37 41 44 50 51 56 57 58 59 67 71 72 78 84 87 90 95 96 97 102 104 107 109 114 116 119 120`

and in canonical form is:

`6 7 8 17 20 21 27 30 31 37 40 43 45 46 54 55 58 65 67 69 71 73 77 79 82 83 90 91 92 96 99 100 103 108 118 120`

Observe that comparison is now just a straightforward comparison of two strings of numbers.

We end with a brief summary of the minimal excluded minor algorithm and its output.

Algorithm: For $n \geq 5$, to determine a set of rank 3, n-element minimal excluded minors for $GF(q)$-representable matroids, where q is a prime power.
Input: $\mathcal{M}_n$, $\mathcal{M}_n^q$ and $\mathcal{M}_{n-1}^q$
Output: The list of size n minimal excluded minors for $GF(q)$-representable matroids.

(1) For each matroid M in $\mathcal{M}_n$, retain M if M has no minor isomorphic to $U_{2,q+2}$. Let $\mathcal{X}$ be the set of matroids retained.
(2) For each matroid M in $\mathcal{X}$, retain M if M is not isomorphic to a matroid in $\mathcal{M}_n^q$. Let $\mathcal{Y}$ be the set of matroids retained. Observe that matroids in $\mathcal{Y}$ are excluded minors for $GF(q)$. But they are not necessarily minimal excluded minors.
(3) For each matroid M in $\mathcal{Y}$, retain M if, for every element e, $M\backslash e \in \mathcal{M}_{n-1}^q$. Let $\mathcal{Z}$ be the set of matroids retained. The matroids in $\mathcal{Z}$ are the minimal excluded minors for $GF(q)$-representation.

We tested the code for $GF(2)$, $GF(3)$, and $GF(4)$, since the minimal excluded minors for these fields are known, and found that it gave correct answers in all instances. It also correctly determined the known five rank 3, 7-element minimal excluded minors for $GF(5)$-representation. Below is a list of the non-trivial lines (lines with at least three points) in the minimal excluded minors found by the program.

The 8-element minimal excluded minors:

i) $\{1,7,8\},\{2,5,8\},\{2,6,7\},\{3,5,7\},\{3,6,8\},\{4,5,6\}$

ii) $\{1,3,8\},\{1,4,7\},\{1,5,6\},\{2,3,7\},\{2,4,6\},\{2,5,8\},\{3,4,5\},\{6,7,8\}$

The 9-element minimal excluded minors:

i) $\{1,3,9\}$, $\{1,7,8\}$, $\{2,6,8\}$, $\{2,7,9\}$, $\{3,5,8\}$, $\{3,6,7\}$, $\{4,5,7\}$, $\{4,6,9\}$

ii) $\{1,3,9\}$, $\{1,7,8\}$, $\{2,4,9\}$, $\{2,6,8\}$, $\{3,5,8\}$, $\{3,6,7\}$, $\{4,5,7\}$, $\{5,6,9\}$

iii) $\{1,4,9\}$, $\{1,6,8\}$, $\{2,3,9\}$, $\{2,5,8\}$, $\{3,6,7\}$, $\{4,5,7\}$

iv) $\{1,4,9\}$, $\{1,6,8\}$, $\{2,3,9\}$, $\{2,5,8\}$, $\{3,6,7\}$, $\{4,5,7\}$, $\{7,8,9\}$

v) $\{1,3,9\}$, $\{1,6,8\}$, $\{2,4,9\}$, $\{2,5,8\}$, $\{3,6,7\}$, $\{4,5,7\}$, $\{5,6,9\}$, $\{7,8,9\}$

vi) $\{1,4,9\}$, $\{1,6,8\}$, $\{2,3,9\}$, $\{2,5,8\}$, $\{3,4,8\}$, $\{3,6,7\}$, $\{4,5,7\}$, $\{5,6,9\}$

vii) $\{1,4,9\}$, $\{1,6,8\}$, $\{2,3,9\}$, $\{2,5,8\}$, $\{3,4,8\}$, $\{3,6,7\}$, $\{4,5,7\}$, $\{5,6,9\}$, $\{7,8,9\}$

viii) $\{1,4,9\}$, $\{1,5,8\}$, $\{1,6,7\}$, $\{2,4,8\}$, $\{2,5,7\}$, $\{2,6,9\}$, $\{3,4,7\}$, $\{3,5,9\}$, $\{3,6,8\}$

ix) $\{1,4,9\}$, $\{1,5,8\}$, $\{1,6,7\}$, $\{2,3,9\}$, $\{2,4,8\}$, $\{2,5,7\}$, $\{3,4,7\}$, $\{3,6,8\}$, $\{5,6,9\}$, $\{7,8,9\}$

The 10-element minimal excluded minor is:

i) $\{1,2,3,4\}$, $\{1,5,6,7\}$, $\{1,8,9,10\}$, $\{2,5,8\}$, $\{2,6,9\}$, $\{3,5,10\}$, $\{3,7,8\}$, $\{4,6,10\}$, $\{4,7,9\}\}$.

The 11-element minimal excluded minor is:

i) $\{1,2,3,4,5\},\{1,6,7,8,9\},\{1,10,11\},\{2,6,10\},\{2,7,11\},\{3,6,11\},\{3,8,10\},\{4,7,10\}$, $\{4,9,11\}$, $\{5,8,11\}$, $\{5,9,10\}$

No 12 element minimal excluded minor for $GF(5)$-representablity was found. We conjecture that no more rank 3 minimal excluded minors will be found.

References

[1] A. Betten and D. Betten (1999), Linear spaces with at most 12 points, *J. Combin. Des.* *7* **no. 2**, 119–145.

[2] J. Geelen, B. Gerards, and A. Kapoor (2000). The excluded minors for GF(4)-representable matroids, Journal of Combinatorial Theory, Series B. **79**, 247-299.

[3] R. J. Kingan, S. R. Kingan, and Wendy Myrvold (2003), On matroid generation, *Congressus Numerantium* **164**, 95-109.

[4] J. S. Leon (1997), Partitions, refinements, and permutation group computation, *Groups and computation, II*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **28**, 123-158, Amer. Math. Soc., Providence, RI.

[5] McKay, B. D. (1998), Isomorph-free exhaustive generation *J. Algorithms*, **26**, 306–324.
[6] J. G. Oxley (1992), *Matroid Theory.* Oxford University Press, NY.

Department of Mathematics Colorado State University Fort Collins, CO 80523
E-mail address: betten@math.colostate.edu

Department of Mathematics Clayton State University Morrow, GA 30260
E-mail address: rkingan@clayton.edu

Department of Mathematics Clayton State University Morrow, GA 30260
E-mail address: skingan@clayton.edu