Bipartite Unicyclic Graphs with Large Energy

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Abstract

Let $G$ be a graph with $n$ vertices and $\lambda_1, \lambda_2, \cdots, \lambda_n$ be $n$ eigenvalues of its adjacency matrix $A(G)$. The energy of $G$, denoted by $E(G)$, is defined to be the summation $\sum_{i=1}^{n} |\lambda_i|$. Denote by $BU_n$ the set of connected bipartite unicyclic graphs on $n$ vertices. For $n \geq l+1$, let $P_{n}^{l}$ be graph obtained by identifying one pendent vertex of the path $P_{n-l+1}$ with any vertex of the cycle $C_l$. Recently, I. Gutman[7] and Y. Hou[10] determined that $P_{6n}$ is the unique graph with the greatest energy among all graphs in $BU_n \{C_n\}$. Let $BU^{*}_n = BU_n \{C_n, P_{n}^{l}, l = 4, 5, \cdots, n-1\}$. It is proved in this paper that for $n \geq 13$, $M_{n}^{6,3}$ is the graph with maximal energy among all graphs in $BU^{*}_n$, where $M_{n}^{6,3}$ is the graph obtained by joining (by a new edge) any vertex of the hexagon with the vertex 3 of the path $P_{n-6}$.

1 Introduction

Let $G$ be a connected graph with $n$ vertices and $A(G)$ be its adjacency matrix. The characteristic polynomial of $A(G)$ is defined to be

$$\phi(G; x) = |xI - A(G)| = \sum_{i=0}^{n} a_i x^{n-i},$$

which is also said to be the characteristic polynomial of $G$. All $n$ roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ of $\phi(G; x)$ are called to be eigenvalues of $G$. It’s not difficult to see that each $\lambda_i$ ($i = 1, 2, \cdots n$) is real since $A(G)$ is symmetric.

The energy of $G$, denoted by $E(G)$, is defined to be $\sum_{i=1}^{n} |\lambda_i|$. It’s well known that $E(G)$ can


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be expressed as the Coulson integral formula

\[
E(G) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2} \ln \left[ \left( \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^i a_{2i} x^{2i} \right)^2 + \left( \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^i a_{2i+1} x^{2i+1} \right)^2 \right] dx, \tag{1}
\]

where \(a_0, a_1, \ldots, a_n\) are coefficients of the characteristic polynomial of \(G\).

Since the energy of a graph can be used to estimate approximately the total \(\pi\)-electron energy of the molecule, it has been intensively studied by many scholars. For more details see [3-10]; for some recent research along these lines see [11-22]. The interested reader may also refer to [23,24] for the mathematical properties of \(E(G)\).

As usual, we begin with some notations and terminologies. For a graph \(G\), we use \(V(G)\) and \(E(G)\) to denote its set of vertices and edges, respectively. Let \(d_G(v)\) denote the degree of vertex \(v\), namely the number of edges incident with \(v\) in \(G\). By \(d_G(x, y)\) we mean the length of the shortest path connecting vertices \(x\) and \(y\), i.e., the distance between \(x\) and \(y\) in \(G\). Let \(V_p(G)\) denote the set of pendant vertices in \(G\). By \(S_n, C_n\) and \(P_n\) we denote respectively the star graph, the cycle graph and the path graph with \(n\) vertices. Let \(P_n^l(n \geq l + 1)\) be a graph obtained by identifying one pendant vertex of the path \(P_{n-1}^l\) with any vertex of the cycle \(C_l\). Denote by \(K_n^l(n \geq l + 2)\) the graph obtained from \(P_{n-1}^l\) by attaching one pendant edge to one neighbor (lying on \(C_l\)) of the unique 3-degree vertex of \(P_{n-1}^l\). By \(R_n^l(n \geq l + 4)\) we denote the graph obtained by attaching a path of length 2 to one neighbor (lying on \(C_l\)) of the unique 3-degree vertex of \(P_{n-2}^l\). Let \(Q_n^l(n \geq l + 5)\) be a graph obtained by identifying the middle-point of the path \(P_5\) with the unique pendant vertex of \(P_{n-4}^l\). Fig.1. illustrate \(P_n^l, K_n^l, R_n^l\) and \(Q_n^l\), respectively.

Fig.1.

Denote by \(\mathcal{U}_n\) and \(\mathcal{B}\mathcal{U}_n\) the set of connected unicyclic graphs and bipartite unicyclic graphs on \(n\) vertices, respectively. Let \(G\) be any graph in \(\mathcal{U}_n\) and \(v\) the vertex lying on its unique
cycle. If \( d_G(v) \geq 3 \), then \( v \) is said to be a branched vertex. For a given vertex \( x \notin V(C) \) in \( G \), let \( d_G(x, C) = \min \{d_G(x, y) \mid y \in V(C)\} \), where \( C \) is the cycle in \( G \).

Let \( BU^*_n = BU_n \setminus \{C_n, P^*_n, l = 4, 5, \ldots, n-1\} \). For any graph \( G \in BU^*_n \), let \( C \) be the cycle of length \( l \) in \( G \). Then \( n \geq l + 2 \), i.e., \( V_p(G) \neq \emptyset \). Let \( BU^*_n, l = \{G \in BU^*_n \mid \text{there exists } x \in V_p(G) \text{ such that } d_G(x, C) = 1\} \). Set \( BU^*_{n, 2} = BU^*_n \setminus BU^*_{n, 1} \). Let \( BU^*_{n, 2} \) denote the subset of \( BU^*_{n, 2} \) such that for any \( G \in BU^*_{n, 2} \), there’s exactly one branched vertex in the unique cycle of \( G \). Denote by \( BU^*_{n, 2} \) the set \( BU^*_{n, 2} \setminus BU^*_{n, 2} \). By \( BU^*_{n, 2}(l) \) we mean the subset of \( BU^*_{n, 2} \) such that for each graph \( G \) in \( BU^*_{n, 2}(l) \), \( G \) has a unique cycle of length \( l \). Similarly, we can define respectively the sets \( BU^*_{n, 1}(l) \), \( BU^*_{n, 2}(l) \), \( BU^*_{n, 2}(l) \), \( BU^*_{n, 2}(l) \), \( BU^*_{n, 2}(l) \) in this way.

In this paper, we determined the graph with maximal energy among all graphs in \( BU^*_n \).

## 2 Lemmas and Results

Sachs theorem [25] states that

\[
a_i(G) = \sum_{S \in L_i} (-1)^{k(S)} 2^{c(S)},
\]

where \( L_i \) denote the set of Sachs graphs \( G \) with \( i \) vertices, \( k(S) \) is number of components of \( S \) and \( c(S) \) is the number of cycles contained in \( S \).

Set \( b_i(G) = |a_i(G)| \ (i = 0, 1, \ldots, n) \). From Eq.(2), we find that \( b_2(G) \) is equal to the number of edges of \( G \). Let \( m(G, k) \) denote the number of \( k \)-matchings of a graph \( G \). If \( G \) contains no cycle, then \( b_{2k}(G) = m(G, k) \) and \( b_{2k+1}(G) = 0 \) for each \( k \geq 0 \). It’s both consistent and convenient to define \( b_k(G) = 0 \) and \( m(G; k) = 0 \) for the case when \( k < 0 \).

In [8], Y. Hou obtained the following result.

**Lemma 1.** Let \( G \in \mathcal{U}_n(l) \). Then \((-1)^k a_{2k} \geq 0 \) for all \( k \geq 0 \); and \((-1)^k a_{2k+1} \geq 0 \) (resp. \(-1)^k a_{2k+1} \leq 0 \) for all \( k \geq 0 \) if \( l = 2r + 1 \) and \( r \) is odd (resp. even ).

From Eq.(1) and lemma 1, we obtain

\[
E(G) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{x^2} ln \left[ \left( \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} b_{2i} x^{2i} \right)^2 + \left( \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} b_{2i+1} x^{2i+1} \right)^2 \right] dx.
\]

It follows from (3) that \( E(G) \) is a strictly increasing function of \( b_i(G) \) for \( i = 0, 1, \ldots, n \). That is to say, for any two unicyclic graphs \( G_1 \) and \( G_2 \), there exists

\[
b_i(G_1) \geq b_i(G_2) \text{ for all } i \geq 0 \Rightarrow E(G_1) \geq E(G_2).
\]
If \( b_i(G_1) \geq b_i(G_2) \) holds for all \( i \geq 0 \), then we write \( G_1 \succeq G_2 \) or \( G_2 \preceq G_1 \).

According to the above relations, the following lemma follows readily.

**Lemma 2.** Let \( G_1 \) and \( G_2 \) be two graphs. Then \( G_1 \succeq G_2 \) implies that \( E(G_1) \geq E(G_2) \) and \( G_1 \succ G_2 \) implies that \( E(G_1) > E(G_2) \).

The following lemma is crucial to the proof of our main result.

**Lemma 3.** Let \( G \) be a unicyclic graph on \( n \) vertices with its cycle being \( C_1 \). Let \( uv \) be an edge in \( E(G) \), we have

(a) If \( uv \in C_1 \), then \( b_i(G) = b_i(G - uv) + b_{i-2}(G - u - v) + 2b_{i-1}(G - C_1) \) if \( l \equiv 0(\text{mod} \ 4) \)
and \( b_i(G) = b_i(G - uv) + b_{i-2}(G - u - v) + 2b_{i-1}(G - C_1) \) if \( l \not\equiv 0(\text{mod} \ 4) \);

(b) If \( uv \notin C_1 \), then \( b_i(G) = b_i(G - uv) + b_{i-2}(G - u - v) \). In particular, if \( uv \) is a pendant edge with pendant vertex \( v \), then \( b_i(G) = b_i(G - v) + b_{i-2}(G - u - v) \).

**Proof.** Recall that

\[
\phi(G; x) = \phi(G - uv; x) - \phi(G - u - v; x) - 2 \sum_{C \in \mathcal{C}_{uv}} \phi(G - C; x),
\]  

where \( \mathcal{C}_{uv} \) denotes the set of cycles containing \( uv \).

One can easily obtain the desired result by equating the coefficients of \( x^{n-i} \) on both sides of Eq.(5). \( \square \)

F. Li and B. Zhou obtained the following result in [21].

**Lemma 4.** Let \( G \) be a unicyclic graph in \( \mathcal{U}_n \) and \( G' \) the graph obtained from \( G \) by deleting at least one edge outside its unique cycle. Then \( G' \prec G \).

I. Gutman [3] show that \( n \)-vertex path \( P_n \) is the unique graph with the maximal energy among all all acyclic graphs on \( n \) vertices. The following lemma could be found in [1] as proposition 9.

**Lemma 5.** Let \( T \) be a tree of order \( n \geq 6 \) not isomorphic to \( P_n \). Then \( E(T) \leq E(T^2_n) \) with equality if and only if \( T \cong T^2_n \), where \( T^2_n \) is the tree obtained by pasting one endpoint of \( P_{n-4} \) to the middle vertex of \( P_5 \). (See Fig.2. for \( T^2_n \).)
In addition to the trees with maximal and second-maximal energy, also the trees with third-
maximal, fourth-maximal, ... energy are determined by F. Zhang and H. Li [6].

\[ T_n^2 \text{ with } n \geq 8 \text{ vertices.} \]

Fig. 2.

Lemmas 6—8 given below are due to Y. Hou in [10].

**Lemma 6.** Let \( G \in \mathcal{U}_n(l) \) with \( l \not\equiv 0(\mod 4) \). If \( G \not\equiv P_n^l \), then \( G \prec P_n^l \).

Let \( \mathcal{C}(n, l) \) be the set of unicyclic graphs obtained from \( C_l \) by attaching to it \( n - l \) pendent vertices.

**Lemma 7.** Let \( G \in \mathcal{U}_n(l) \) with \( l \equiv 0(\mod 4) \). If \( G \not\equiv \mathcal{C}(n, l), P_n^l \), then \( G \prec P_n^l \).

**Lemma 8.** Let \( G \) be any connected graph in \( \mathcal{U}_n \) and \( G \not\equiv C_n \). Then \( E(G) \leq E(P_n^6) \) with equality only if \( l = 6 \).

**Lemma 9.** Suppose \( 4 \leq l \leq n - 6 \). If \( l \not\equiv 4, 6, \) then \( P_{l-2} \cup T_{n-l}^2 \not\succeq P_4 \cup T_{n-6}^2 \not\succeq P_2 \cup T_{n-4}^2 \).

**Proof.** From [3], we know that \( P_2 \cup P_{n-2} \succeq P_4 \cup P_{n-4} \succeq P_i \cup P_{n-i} \) for any integer \( 1 \leq i \leq n - 1 \) and \( i \neq 2, 4 \). Note that

\[
m(P_{l-2} \cup T_{n-l}^2; k) = m(P_{l-2} \cup P_2 \cup P_{n-l-2}; k) + m(P_{l-2} \cup P_2 \cup P_{n-l-5}; k - 1),
\]

\[
m(P_4 \cup T_{n-6}^2; k) = m(P_4 \cup P_2 \cup P_{n-8}; k) + m(P_4 \cup P_2 \cup P_{n-11}; k - 1),
\]

\[
m(P_2 \cup T_{n-4}^2; k) = m(P_2 \cup P_2 \cup P_{n-6}; k) + m(P_2 \cup P_2 \cup P_{n-9}; k - 1).
\]

Hence the result follows. \( \square \)

**Lemma 10.** Suppose \( (i, j, k) \) is a 3-element ordered pair with \( 1 \leq i \leq j \leq k \) and \( i+j+k = n \). If \( (i, j, k) \neq (2, 2, n-4), (2, 4, n-6) \), then \( P_i \cup P_j \cup P_k \not\succeq P_2 \cup P_4 \cup P_{n-6} \not\succeq P_2 \cup P_2 \cup P_{n-4} \).
Proof. If \( j \neq 2 \), then
\[
P_1 \cup (P_j \cup P_k) \leq P_1 \cup (P_4 \cup P_{j+k-4}) = P_4 \cup (P_i \cup P_{j+k-4}) \leq P_4 \cup (P_2 \cup P_{i+j+k-6}) = P_2 \cup (P_4 \cup P_{n-6}).
\]

Similarly, if \( i \neq 2 \), we can show that \( P_i \cup P_j \cup P_k \leq P_2 \cup P_4 \cup P_{n-6}. \) Since \( P_2 \cup P_4 \cup P_{n-6} \leq P_2 \cup P_2 \cup P_{n-4} \), then the result follows. \( \square \)

**Theorem 11.** Let \( G \in \mathcal{BU}^*_{n,1} \) with \( n \geq 8 \) vertices. If \( G \ncong K^6_n \), then \( G < K^6_n \).

**Proof.** Let \( G \) be any graph in \( \mathcal{BU}^*_{n,1} \) and \( C_l \) be the unique cycle in \( G \). Since \( G \ncong P^l_{n^*} \), \( G \) has at least two pendent vertices. Let \( v \) be the pendent vertex in \( G \) such that \( d_G(v, C_l) = 1 \) and \( v \) its unique neighbor. Note that \( G - v - u \) is a acyclic graph on \( n - 2 \) vertices. So \( G - v - u \leq P_{n-2} \). Since \( G - v \ncong C_{n-1} \), then \( G - v \leq P^6_{n-1} \) by lemma 8. According to lemma 3(b), we get
\[
b_{2k}(G) = b_{2k}(G - v) + b_{2k-2}(G - v - u) \leq b_{2k}(P^6_{n-1}) + b_{2k-2}(P_{n-2}) = b_{2k}(K^6_n).
\]

If \( G \ncong K^6_n \), we can always find a positive integer \( k_0 \) such that \( b_{2k_0}(G) < b_{2k_0}(K^6_n) \). This completes the proof. \( \square \)

**Lemma 12.** Let \( G \in \mathcal{BU}^b_{n,2}(l) \) with \( n = l + 3 \), then \( G \leq K^l_n \).

**Proof.** Obviously \( G_1 \) is the single element in \( \mathcal{BU}^b_{n,2}(l) \)(see Fig.3, for \( G_1 \)). In view of lemma 3(b), we obtain
\[
b_{2k}(K^l_{l+3}) - b_{2k}(G_1) = b_{2k-2}(P_{l+1}) - b_{2k-2}(C_l) = m(P_{l+1}; k - 1) - m(P_l; k - 1) - m(P_{l-2}; k - 2) \pm 2,
\]
where the last term ”\( \pm 2 \)” should be erased if \( 2k - 2 \neq l \).

When \( 2k - 2 \neq l \), \( b_{2k}(K^l_{l+3}) - b_{2k}(G_1) = m(P_{l-3}; k - 3) \geq 0 \). When \( 2k - 2 = l \) and \( l \equiv 0(\text{mod} \ 4) \), we have \( b_{2k}(K^l_{l+3}) - b_{2k}(G_1) = m(P_{l-3}; k - 3) + 2 > 0 \). When \( 2k - 2 = l \) and \( l \equiv 0(\text{mod} \ 4) \), we have \( b_{2k}(K^l_{l+3}) - b_{2k}(G_1) = m(P_{l-3}; k - 3) - 2 = m(P_{l-3}; \frac{l}{2} - 2) - 2 \geq 0 \).

Consequently, the result follows. \( \square \)
**Lemma 13.** Let $G \in \mathcal{B}^{*b}_{n,l}(l)$ with $n = l + 4$, then $G \preceq R^l_{l_n}$.

**Proof.** It’s evident that $G$ must be one of graphs $G_2 - G_5$ as shown in Fig. 3.

According to lemmas 3(b) and 4, one can easily obtain that $G_2 \succ G_4$. In the following, we will show that $R^l_{l_n} \succ G_2, G_3, G_5$. Apply lemma 3(b) once again, we obtain

$$b_{2k}(R^l_{l+4}) - b_{2k}(G_2) = b_{2k}(P^l_{l+2}) + b_{2k-2}(P^l_{l+2}) + b_{2k-2}(P_{l+1}) - b_{2k}(P^l_{l+3})$$
$$-b_{2k-2}(P_2 \cup C_l)$$
$$= b_{2k-2}(P^l_{l+2}) + b_{2k-2}(P_{l+1}) - b_{2k-2}(P^l_{l+1}) - b_{2k-2}(C_l)$$
$$-b_{2k-4}(C_l)$$
$$= \ldots$$
$$= b_{2k-2}(P_{l+1}) - b_{2k-2}(C_l).$$

Similar to the proof of lemma 12, we can show that $G_2 \preceq R^l_{l+4}$.

Similarly,

$$b_{2k}(R^l_{l+4}) - b_{2k}(G_3) = b_{2k}(P^l_{l+2}) + b_{2k-2}(P^l_{l+2}) + b_{2k-2}(P_{l+1}) - b_{2k}(P^l_{l+3})$$
$$-b_{2k-2}(P^l_{l+1})$$
$$= \ldots$$
$$= b_{2k-4}(C_l) + b_{2k-2}(P_{l+1}) - b_{2k-4}(P_{l-1}) - b_{2k-2}(C_l).$$

If $2k - 4 \neq l$ and $2k - 2 \neq l$, then $b_{2k}(R^l_{l+4}) - b_{2k}(G_3) = m(P_{l-3}; k - 3) + 2m(P_{l-2}; k - 3) \geq 0$.

If $2k - 4 = l$ or $2k - 2 = l$, then

$$b_{2k}(R^l_{l+4}) - b_{2k}(G_3) \geq m(P_{l-3}; k - 3) + 2m(P_{l-2}; k - 3) - 2$$
$$\geq \begin{cases} 
2m(P_{l-2}; \frac{l}{2} - 1) - 2 = 0, & 2k - 4 = l \\
2m(P_{l-2}; \frac{l}{2} - 2) - 2 \geq 0, & 2k - 2 = l 
\end{cases}$$

Thus $G_3 \preceq R^l_{l+4}$. 

It is easy to obtain that $G_5 \prec R_{l+4}^l$ by means of lemma 3. This completes the proof. □

**Lemma 14.** Let $n \geq 10$ and $4 \leq l \leq n - 4$. If $l \neq 6$, then $R_n^l \prec R_n^6$.

**Proof.** By lemma 3(b), we have

\[
\begin{align*}
b_{2k}(R_n^l) &= b_{2k}(K_{n-1}^l) + b_{2k-2}(P_{n-2}^l) \\
&= b_{2k}(P_{n-2}^l) + b_{2k-2}(P_{n-2}^l) + b_{2k-2}(P_{n-3}),
\end{align*}
\]

\[
\begin{align*}
b_{2k}(R_n^6) &= b_{2k}(P_{n-2}^6) + b_{2k-2}(P_{n-2}^6) + b_{2k-2}(P_{n-3}).
\end{align*}
\]

Since $n - 2 \geq l + 2$, the lemma follows as expected by lemma 8. □

By the same reasoning as employed in lemma 14, we can prove:

**Lemma 15.** Suppose $n \geq 8$ and $4 \leq l \leq n - 2$. If $l \neq 6$, then $K_n^l \prec K_n^6$.

**Lemma 16.** For $n \geq 10$, we have $K_n^6 \prec R_n^6$.

\[\begin{array}{ccc}
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&
\begin{array}{c}
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4 \\
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5
\end{array}
\end{array}\]

\[M_n^{l,i} \quad M_n^{n-5,3} \quad M_n^{n-5,2}\]

\[2 \leq i \leq n - l - 1 \quad \text{and} \quad n - l \geq 6 \quad M_n^{n-5,3} (= Q_n^{n-5}) \quad M_n^{n-5,2}
\]

Fig.4.

For $2 \leq i \leq n - l - 1$ and $n - l \geq 5$, we use $M_n^{l,i}$ to denote the graph obtained by joining a vertex of $C_l$ by a new edge with the $i^{th}$ vertex of path $P_{n-l}$, where the vertices of $P_{n-l}$ are labelled according to their natural orderings.

**Theorem 17.** Let $G \in \mathcal{B} \cup_{n,2}^{*b}$ with $n \geq 13$. If $G \not\cong M_n^{n-5,2}$, $M_n^{n-5,3}$, $M_n^{6,3}$ and $Q_n^6$, then $G \prec M_n^{6,3}$ or $Q_n^6$.

**Proof.** Let $G$ be any graph in $\mathcal{B} \cup_{n,2}^{*b}$ and $C_l$ be the unique cycle in it. Since $G \in \mathcal{B} \cup_{n,2}^{*b}$, then $n \geq l + 3$. 

If \( n = l + 3 \) or \( l + 4 \), the result is evidently true from the combination of lemmas 12–16. So we may suppose that \( n \geq l + 5 \) herein. We shall prove the theorem by distinguishing between two cases.

**Case 1.** \( l = 4 \).

By means of lemmas 3(a) and 5, we have

\[
 b_{2k}(G) = m(G; k) - 2b_{2k-4}(G - C_4) \\
\leq m(G; k) \\
\leq m(T^2_n; k) + m(P_2 \cup T^2_{n-4}; k - 1) \\
= m(Q^4_n; k).
\]

In the following, we shall prove that \( b_{2k}(Q^6_n) \geq m(Q^4_n; k) \) for all \( k \geq 0 \).

In view of lemma 3(a),

\[
 b_{2k}(Q^6_n) = m(T^2_n; k) + m(P_4 \cup T^2_{n-6}; k - 1) + 2m(T^2_{n-6}; k - 3).
\]

Thus

\[
b_{2k}(Q^6_n) - m(Q^4_n; k) = m(P_4 \cup T^2_{n-6}; k - 1) + 2m(T^2_{n-6}; k - 3) - m(P_2 \cup T^2_{n-4}; k - 1) \\
= m(P_2 \cup P_2 \cup T^2_{n-6}; k - 1) + m(T^2_{n-6}; k - 2) + 2m(T^2_{n-6}; k - 3) \\
- m(P_2 \cup P_2 \cup T^2_{n-7}; k - 2) \\
= m(T^2_{n-6}; k - 2) - m(T^2_{n-7}; k - 2) + 2m(T^2_{n-6}; k - 3) - m(T^2_{n-7}; k - 3) \\
\geq 0.
\]

So \( b_{2k}(Q^6_n) \geq b_{2k}(G) \) and \( b_{2k}(Q^6_n) \geq b_{2k}(Q^4_n) \) for all \( k \geq 0 \) in this case. In particular, \( b_0(Q^6_n) > b_0(G) \) and \( b_0(Q^6_n) > b_0(Q^4_n) \). Hence \( G < Q^6_n \) and \( Q^4_n < Q^6_n \).

**Case 2.** \( l \geq 6 \).

**Case 2.1.** \( G \cong M^{l, i}_n \) for some \( 2 \leq i \leq n - l - 1 \). (See Fig.4. for \( M^{l, i}_n \))

In this case, we claim that \( G < M^{6, 3}_n \). Since \( G \not\cong M^{n-5, 2}_n \cong M^{n-5, 4}_n, M^{n-5, 3}_n \), then \( n - l \geq 6 \).

Firstly, we prove that if \( i \neq 3, n - l - 2 \), then \( M^{l, i}_n < M^{l, 3}_n \).

Note that

\[
 b_{2k}(M^{l, i}_n) = b_{2k}(C_l \cup P_{n-l}) + b_{2k-2}(P_{l-1} \cup P_{l-1} \cup P_{n-l-i}), \\
 b_{2k}(M^{l, 3}_n) = b_{2k}(C_l \cup P_{n-l}) + b_{2k-2}(P_{l-1} \cup P_{2} \cup P_{n-l-3}).
\]

By means of lemma 10, it’s not difficult to show that \( P_{l-1} \cup P_{l-1} \cup P_{n-l-i} \prec P_{l-1} \cup P_{2} \cup P_{n-l-3} \). So there exists some \( k_0 \) such that \( b_{2k_0}(M^{l, 3}_n) > b_{2k_0}(M^{l, i}_n) \) and then \( M^{l, i}_n < M^{l, 3}_n \).

Secondly, we will demonstrate that if \( l \neq 6 \), i.e., \( l \geq 8 \), then \( M^{l, 3}_n < M^{6, 3}_n \).
By lemma 3(a), we deduce that

\[ b_{2k}(M_n^{l,3}) = b_{2k}(T_1) + b_{2k} - 2(P_{l - 2} \cup P_{n - l}) + 2b_{2k-l}(P_{n-l}), \]

\[ b_{2k}(M_n^{6,3}) = b_{2k}(T_2) + b_{2k} - 2(P_4 \cup P_{n-6}) + 2b_{2k-6}(P_{n-6}). \]

where \( T_1 \) (resp. \( T_2 \)) is the acyclic graph of order \( n \) obtained from \( M_n^{l,3} \) (resp. \( M_n^{6,3} \)) by deleting one edge on \( C_l \) (resp. \( C_6 \)) incident with the unique 3-degree vertex of \( C_l \) (resp. \( C_6 \)).

Moreover,

\[ b_{2k}(T_1) = b_{2k}(P_2 \cup P_{n-2}) + b_{2k} - 2(P_1 \cup P_{l - 3}), \]

\[ b_{2k}(T_2) = b_{2k}(P_2 \cup P_{n-2}) + b_{2k} - 2(P_1 \cup P_6 \cup P_{n-9}). \]

Furthermore,

\[ b_{2k-6}(P_{n-6}) = m(P_{n-6}; k - 3) = m(P_{n-7}; k - 3) + m(P_{n-8}; k - 4) \]
\[ \geq m(P_{n-8}; k - 4) \]
\[ \geq \cdots \]
\[ \geq m(P_{n-6-l}; k - 3 - \frac{l - 6}{2}) \]
\[ = m(P_{n-l}; k - \frac{l}{2}) = b_{2k-l}(P_{n-l}). \]

When \( n - l \neq 7 \), we clearly have \( P_1 \cup P_1 \cup P_{n-l-3} \preceq P_1 \cup P_6 \cup P_{n-9} \) since \( l \geq 8 \). Thus \( T_1 \preceq T_2 \) and then \( M_n^{l,3} \preceq M_n^{6,3} \).

When \( n - l = 7 \),

\[ b_{2k}(M_n^{6,3}) - b_{2k}(M_n^{l,3}) \geq b_{2k} - 2(P_4 \cup P_{n-6}) - b_{2k} - 2(P_7 \cup P_{n-9}) + b_{2k} - 2(P_6 \cup P_{n-9}) \]
\[ - b_{2k} - 2(P_1 \cup P_{n-7}) \]
\[ = m(P_4 \cup P_{n-8}; k - 2) - m(P_5 \cup P_{n-9}; k - 2) \geq 0. \]

So \( M_n^{l,3} \preceq M_n^{6,3} \).

Since \( b_0(P_{n-6}) = 1 > 0 = b_{l+1}(P_{n-l}) \), then \( b_0(M_n^{6,3}) > b_0(M_n^{l,3}) \). This gives \( M_n^{l,3} \preceq M_n^{6,3} \).

**Case 2.2.** \( G \nless M_n^{l,i} \) for any \( 2 \leq i \leq n-l-1 \).

Since \( G \in \mathcal{BC}_n^{n_2} \), \( C_l \) has exactly one branched vertex. Let \( u \) be such a branched vertex and \( w \) be one of its neighbors lying on \( C_l \). By lemma 3(a),

\[ b_{2k}(G) = b_{2k}(G - uw) + b_{2k} - 2(G - u - w) + 2b_{2k-l}(G - C_l) \]
\[ \leq b_{2k}(T_n^2) + b_{2k} - 2(P_{l - 2} \cup T_{n-l}) + 2b_{2k-l}(T_{n-l}), \]

where \( T_{n-l} \) denotes the forest obtained by deleting the cycle \( C_l \) from \( G \). As \( T_{n-l} \nless P_{n-l} \) (otherwise \( G \cong P_n^l \) or \( M_n^{l,i} \), a contradiction), we have \( T_{n-l} \preceq T_{n-l}^2 \) by lemma 5. Because \( l \geq 6 \), we have \( P_{l-2} \cup T_{n-l}^2 \preceq P_4 \cup T_{n-6}^2 \) by lemma 9.
When \( l \equiv 0 (\text{mod} \, 4) \), we have

\[
b_{2k}(G) \leq b_{2k}(T_n^2) + b_{2k-2}(P_4 \cup T_{n-6}^2) + 2b_{2k-6}(T_{n-6}^2) = b_{2k}(Q_n^6).
\]

Moreover, there exists some \( k_0 \) such that \( b_{2k_0}(Q_n^6) > b_{2k_0}(G) \) since \( G \not\subset Q_n^6 \).

When \( l \not\equiv 0 (\text{mod} \, 4) \), we have

\[
m(T_{n-6}^2; k - 3) = m(T_{n-7}^2; k - 3) + m(T_{n-8}^2; k - 4) \\
\geq m(T_{n-8}^2; k - 4) \\
\geq \cdots \\
\geq m(T_{n-6-(l-6)}^2; k - 3 - \frac{l - 6}{2}) \\
= m(T_{n-l}^2; k - \frac{l}{2}).
\]

Hence \( b_{2k}(G) \leq b_{2k}(T_n^2) + b_{2k-2}(P_{l-2} \cup T_{n-l}^2) + 2b_{2k-l}(T_{n-l}^2) \leq b_{2k}(Q_n^6) + b_{2k-2}(P_4 \cup T_{n-6}^2) + 2b_{2k-6}(T_{n-6}^2) = b_{2k}(Q_n^6) \). If \( l \neq 6 \), there must exist some \( k'_0 \) such that \( b_{2k'_0}(Q_n^6) > b_{2k'_0}(G) \).

From the combination of cases 1 and 2 it follows the present theorem as expected. \( \square \)

**Lemma 18.** Let \( G \in \mathcal{B} \mathcal{U}^{a}_{n,2}(l) \) with \( n = l + 4 \) or \( l + 5 \). If \( G \not\subset R_n^l \), then \( G < R_n^l \).

**Proof.** We consider only the case that \( n = l + 4 \). Since \( G \in \mathcal{B} \mathcal{U}^{a}_{n,2} \), \( G \) must have a pendant vertex \( v \) such that \( d_G(v, C_l) = 2 \) and \( d_G(u) = 2 \), where \( u \) is the unique neighbor of \( v \). Note that \( G - v \in \mathcal{B} \mathcal{U}^{a}_{n-1,1} \), one can easily verify that \( G - v \leq K_{n-1}^l \) by lemma 3(b). Similarly, we can demonstrate that \( G - u - v \leq P_n^{l-2} \) since \( G - u - v \not\subset C(n-2, l) \) and \( G - u - v \not\subset C_{n-2} \).

By lemma 3(b),

\[
b_{2k}(G) = b_{2k}(G - v) + b_{2k-2}(G - u - v) \leq b_{2k}(K_{n-1}^l) + b_{2k-2}(P_n^{l-2}) = b_{2k}(R_n^l).
\]

If \( G \not\subset R_n^l \), we can always find a positive integer \( k_0 \) such that \( b_{2k_0}(R_n^l) > b_{2k_0}(G) \).

When \( n = l + 5 \), the lemma can be proved by the same reasoning as used above. So the result follows. \( \square \)

**Lemma 19.** Let \( G \in \mathcal{B} \mathcal{U}^{a}_{n,2}(l) \) with \( l \not\equiv 0 (\text{mod} \, 4) \). If \( G \not\subset R_n^l \), then \( G < R_n^l \).

**Proof.** Let \( G \) be any graph in \( \mathcal{B} \mathcal{U}^{a}_{n,2} \) and \( C_l \) be the unique cycle in \( G \). Since \( G \in \mathcal{B} \mathcal{U}^{a}_{n,2} \), then \( n \geq l + 4 \). We shall prove this lemma by induction on \( n - l \). When \( n - l = 4 \) or 5, the lemma is immediate from Lemma 18. Suppose that \( n - l \geq 6 \) and the lemma is true for graphs in \( \mathcal{B} \mathcal{U}^{a}_{n-1,2} \) or \( \mathcal{B} \mathcal{U}^{a}_{n-2,2} \). Now, let \( G \) be graph in \( \mathcal{B} \mathcal{U}^{a}_{n,2} \) with \( n - l \geq 6 \). There’re two cases we should distinguish between.

**Case 1.** \( d_G(v, C_l) = 2 \) for any \( v \in V_p(G) \).
Let $S$ be the set of vertices adjacent to pendent vertices in $G$. If $d_G(u) = 2$ for some vertex $u \in S$, then by the same method as used in proving lemma 18, we can show that $G \sim R^l_n$ (Here $G \not\sim R^l_n$). Suppose that $d_G(u) \geq 3$ for all vertices $u \in S$. Let $u$ be any vertex in $S$ and $v$ be one pendent vertex adjacent to it. Then $G - v \in \mathcal{BU}^a_{n-1,2}$ and thus $G - v \sim R^l_{n-1}$ by induction assumption. Since $d_G(u) \geq 3$, all connected components not containing $C_l$ of $G - v - u$ must be isolated vertices. So by lemma 4, $G - v - u \sim G'$, where $G'$ is the graph by attaching all isolated vertices of $G - v - u$ to any vertex of $C_l$. Evidently, $G' \in \mathcal{BU}^*_{n-2,1}$ and it’s not difficult to obtain that $G' \sim K^l_{n-2}$. By lemmas 3(b) and (6), $K^l_{n-2} \sim R^l_{n-2}$ since $n - 2 \geq l + 4$ and $l \not\equiv 0$(mod 4).

Therefore $G \sim R^l_n$.

**Case 2** There exists some pendent vertex $v$ in $V_p(G)$ such that $d_G(v, C_l) \geq 3$.

Let $w \in V_p(G)$ be the pendent vertex in $G$ such that $d_G(w, C_l) = \max \{d_G(x, C_l) | x \in V_p(G)\}$. Obviously $G - w \in \mathcal{BU}^a_{n-1,2}$ and thus $G - w \sim R^l_{n-1}$ by induction assumption.

Let $u$ be the unique neighbor of $w$. If $G - w - u$ is connected, then $G - w - u \in \mathcal{BU}^a_{n-2,2}(d_G(w, C_l) \geq 4)$ or $\mathcal{BU}^*_{n-2,1}(d_G(w, C_l) = 3)$.

If $G - w - u \in \mathcal{BU}^*_{n-2,1}$, then $G - w - u \sim K^l_{n-2} \sim R^l_{n-2}$ (as $n - 2 \geq l + 4$ and $l \not\equiv 0$(mod 4)).

If $G - w - u$ is disconnected, then $G - w - u \sim G'' \sim K^l_{n-2} \sim R^l_{n-2}$, where $G''$ is the graph by attaching all isolated vertices of $G - w - u$ to any vertex of $C_l$.

Combining cases 1 and 2, the proof is completed. □

Let $G$ be any graph in $\mathcal{U}_n$ and $C_l$ the unique cycle in $G$. Given that all vertices of the cycle $C_l$ are ordered successively as $v_1, v_2, \ldots, v_l$. For any $v_i \in V(C_l)$, let $T_{[v_i]}$ denote the connected component containing $v_i$ of $G - v_{i-1}v_i - v_iv_{i+1}$.

**Lemma 20.** Let $G \in \mathcal{BU}^a_{n,2}(l)$ with $l \equiv 0$(mod 4), $4 \leq l \leq n - 4$ and $n \geq 12$. Then $G \sim Q^6_n$ or $R^6_n$.

**Proof.** Since $G \in \mathcal{BU}^a_{n,2}$, then $n \geq l + 4$. We consider the following two cases.

**Case 1.** For some branched vertex $v_i \in V(C_l)$, $n(T_{[v_i]}) = 3$, where $n(T_{[v_i]})$ is the order of $T_{[v_i]}$.

Since $G \in \mathcal{BU}^a_{n,2}(l)$, then $T_{[v_i]} \cong P_3$ and $v_i$ is one end-point of $P_3$. Let the vertices of $T_{[v_i]}(or P_3)$ be ordered successively as $v_i, v_i', v_i''$ such that $d(v_i') = 2$ and $d(v_i'') = 1$. Then $G - v_i'' \in \mathcal{BU}^*_{n-1,1}$ and thus $G - v_i'' \sim K^l_{n-1} \sim K^6_{n-1}$ by theorem 11. Moreover, $G - v_i'' - v_i' \sim P^6_{n-2}$ by lemma 8 since $G - v_i'' - v_i' \not\cong C_{n-2}$. So $G \sim R^6_n$ in this case.

**Case 2.** For each branched vertex $v_i \in V(C_l)$, $n(T_{[v_i]}) \geq 4$.

Let $v_i$ be any branched vertex on $C_l$. We can always find one neighbor, say $v_i'$, of $v_i$.
on $C_l$) such that

$$b_{2k}(G - v_lv'_l) + b_{2k-2}(G - v_l - v'_l) \leq b_{2k}(T^2_n) + b_{2k-2}(P_4 \cup T^2_{n-6}) \text{(by lemmas 5 and 9)}$$

or

$$b_{2k}(G - v_lv'_l) + b_{2k-2}(G - v_l - v'_l) \leq b_{2k}(P_n) + b_{2k-2}(P_2 \cup P_4 \cup P_{n-8}) \text{(by lemma 10)}.$$ 

So

$$b_{2k}(G) = b_{2k}(G - v_lv'_l) + b_{2k-2}(G - v_l - v'_l) - 2b_{2k-2}(G - C_l)$$

$$\leq b_{2k}(T^2_n) + b_{2k-2}(P_4 \cup T^2_{n-6}) + 2b_{2k-6}(P_2 \cup P_{n-8}) = b_{2k}(Q^6_n).$$

or

$$b_{2k}(G) = b_{2k}(G - v_lv'_l) + b_{2k-2}(G - v_l - v'_l) - 2b_{2k-2}(G - C_l)$$

$$\leq b_{2k}(P_n) + b_{2k-2}(P_2 \cup P_4 \cup P_{n-8}) + 2b_{2k-6}(P_2 \cup P_{n-8}) = b_{2k}(R^6_n).$$

In either cases, there exists some $k_0$ such that $b_{2k_0}(G) < b_{2k_0}(R^6_n)$ or $b_{2k_0}(G) < b_{2k_0}(Q^6_n)$.

This proves the lemma. □

**Theorem 21.** Let $G \in \mathcal{B}_{n,2}^*$ with $n \geq 12$. Then $G \not \subset Q^6_n$ or $R^6_n$.

**Proof.** Let $G$ be any graph in $\mathcal{B}_{n,2}^*$ and $C_l$ be the unique cycle in $G$. If $l \equiv 0 (\text{mod } 4)$, the theorem is true by lemma 20. If $l \not \equiv 0 (\text{mod } 4)$, then $G \not \subset R^6_n$ by lemma 19. Since $n \geq 12$, we can easily verify that $R^6_n \preceq R^6_n$ and the theorem follows as desired. □

**Lemma 22.** For $n \geq 13$, we have $M^5_{n-5,2} \prec M^5_{n-5,3} \prec R^6_n$.

**Proof.** In full analogy with the proof of subcase 2.1 of theorem 17, we can obtain that $M^5_{n-5,2} \prec M^5_{n-5,3}$. In what follows we shall verify that $M^5_{n-5,3} \prec R^6_n$.

By means of lemma 3(a), we have

$$b_{2k}(R^6_n) = b_{2k}(P_n) + b_{2k-2}(P_2 \cup P_4 \cup P_{n-8}) + 2b_{2k-6}(P_2 \cup P_{n-8})$$

$$= b_{2k}(P_2 \cup P_{n-2}) + b_{2k-2}(P_4 \cup P_{n-3}) + b_{2k-2}(P_2 \cup P_4 \cup P_{n-8})$$

$$+ 2b_{2k-6}(P_{n-8}) + 2b_{2k-8}(P_{n-8}),$$

$$b_{2k}(M^5_{n-5,3}) = b_{2k}(T^2_n) + b_{2k-2}(P_5 \cup P_{n-7}) \pm 2b_{2k-(n-5)}(P_5)$$

$$= b_{2k}(P_2 \cup P_{n-2}) + b_{2k-2}(P_2 \cup P_{n-5}) + b_{2k-2}(P_5 \cup P_{n-7})$$

$$\pm 2b_{2k-(n-5)}(P_5).$$
So

\[ b_{2k}(R_n^6) - b_{2k}(M_n^{n-5,3}) = b_{2k-4}(P_1 \cup P_1 \cup P_{n-6}) + b_{2k-2}(P_2 \cup P_4 \cup P_{n-8}) \]

\[ - b_{2k-2}(P_2 \cup P_3 \cup P_{n-7}) - b_{2k-4}(P_1 \cup P_2 \cup P_{n-7}) + 2b_{2k-6}(P_{n-8}) + 2b_{2k-8}(P_{n-8}) \pm 2b_{2k-(n-5)}(P_5) \]

\[ \geq (\ast) b_{2k-4}(P_{n-6}) - b_{2k-4}(P_{n-7}) - b_{2k-6}(P_{n-7}) + 2b_{2k-6}(P_{n-8}) + 2b_{2k-8}(P_{n-8}) \pm 2b_{2k-(n-5)}(P_5) \]

\[ = - b_{2k-8}(P_{n-9}) + 2b_{2k-6}(P_{n-8}) + 2b_{2k-8}(P_{n-8}) \pm 2b_{2k-(n-5)}(P_5) \]

\[ \geq 2b_{2k-6}(P_{n-8}) + b_{2k-8}(P_{n-8}) \pm 2b_{2k-(n-5)}(P_5) \]

\[ \geq (\ast) 0. \]

where the inequality (\ast) holds due to the fact that \( P_2 \cup P_4 \cup P_{n-8} \geq P_2 \cup P_3 \cup P_{n-7}. \)

If \( n \) is even, \( b_{2k-(n-5)}(P_5) = 0 \) and the inequality (\ast) is evidently true. Suppose that \( n \) is odd. If \( n - 5 \equiv 0 \pmod{4} \), the inequality (\ast) holds clearly. If \( n - 5 \not\equiv 0 \pmod{4} \) and \( 2k - (n - 5) \geq 6 \), the result is obvious. If \( n - 5 \not\equiv 0 \pmod{4} \) and \( 2k - (n - 5) = 4 \), \( b_{2k-6}(P_{n-8}) = 0 \) and \( b_{2k-8}(P_{n-8}) = b_{n-9}(P_{n-8}) = m(P_{n-8}; \frac{n-9}{2}) = m(P_{n-9}; \frac{n-9}{2}) + m(P_{n-10}; \frac{n-9}{2} - 1) = 1 + m(P_{n-10}; \frac{n-11}{2}) \geq 1 + 2 = b_{2k-(n-5)}(P_5). \) If \( n - 5 \not\equiv 0 \pmod{4} \) and \( 2k - (n - 5) = 2 \), then \( b_{2k-6}(P_{n-8}) = b_{n-9}(P_{n-8}) = m(P_{n-8}; \frac{n-9}{2}) = \cdots = 1 + m(P_{n-10}; \frac{n-11}{2}) \geq 1 + 2 = 3 \) and \( b_{2k-8}(P_{n-8}) = b_{n-11}(P_{n-8}) = m(P_{n-8}; \frac{n-11}{2}) = m(P_{n-9}; \frac{n-11}{2}) + m(P_{n-10}; \frac{n-13}{2}) > 2. \) Hence \( 2b_{2k-6}(P_{n-8}) + b_{2k-8}(P_{n-8}) \pm 2b_{2k-(n-5)}(P_5) > 2 \times 3 + 2 \times 4 = 0. \) If \( n - 5 \not\equiv 0 \pmod{4} \) and \( 2k - (n - 5) = 0, \) the inequality (\ast) is immediate by the same method as used above.

From above arguments we conclude that \( b_{2k}(R_n^6) \geq b_{2k}(M_n^{n-5,3}) \) and \( b_6(R_n^6) > b_6(M_n^{n-5,3}) \), which proved the lemma. \( \square \)

**Theorem 23.** Let \( G \in BU_n \) with \( n \geq 13. \) Then \( M_n^{6,3} \) has the maximal energy among all graphs in \( BU_n. \)

**Proof.** According to theorems 11, 17 and 21 and lemmas 16 and 22, we need only to prove that \( M_n^{6,3} \succ R_n^6, Q_n^6. \)

Using lemma 3, we obtain

\[ b_{2k}(M_n^{6,3}) = b_{2k}(P_2 \cup P_{n-2}) + b_{2k-2}(P_1 \cup C_6 \cup P_{n-9}), \] (6)

\[ b_{2k}(P_n^6) = b_{2k}(P_2 \cup P_{n-2}) + b_{2k-2}(P_1 \cup P_{n-3}), \] (7)

\[ b_{2k}(Q_n^6) = b_{2k}(P_2 \cup P_{n-2}) + b_{2k-2}(P_1 \cup P_{2} \cup P_{n-5}). \] (8)

To prove that \( M_n^{6,3} \succ R_n^6, \) it’s sufficient to prove that \( C_6 \cup P_{n-9} \succ P_{n-3} \) by Eqs.(6) and (7). In view of lemma 3, we obtain

\[ b_{2k}(C_6 \cup P_{n-9}) = b_{2k}(P_6 \cup P_{n-9}) + b_{2k-2}(P_4 \cup P_{n-9}) + 2b_{2k-6}(P_{n-9}), \]
\[ b_{2k}(P_{n-3}) = b_{2k}(P_6 \cup P_{n-9}) + b_{2k-2}(P_5 \cup P_{n-10}). \]

It’s easy to see that \( b_6(C_6 \cup P_{n-9}) > b_6(P_{n-3}) \). Therefore \( C_6 \cup P_{n-9} \succ P_{n-3} \) and then \( M_n^{6,3} \succ R_n^6 \).

Next, we shall prove that \( M_n^{6,3} \succ Q_n^6 \). Combining Eqs. (6) and (8), we need only to prove that \( C_6 \cup P_{n-9} \succ P_2 \cup P_{n-5}^6 \). In view of lemma 3(b), we obtain

\[
\begin{align*}
b_{2k}(C_6 \cup P_{n-9}) &= b_{2k}(C_6 \cup P_2 \cup P_{n-11}) + b_{2k-2}(C_6 \cup P_1 \cup P_{n-12}), \\
b_{2k}(P_2 \cup P_{n-5}^6) &= b_{2k}(C_6 \cup P_2 \cup P_{n-11}) + b_{2k-2}(P_2 \cup P_5 \cup P_{n-12}).
\end{align*}
\]

In what follows, we shall prove that \( C_6 \cup P_1 \cup P_{n-12} \succ P_2 \cup P_3 \cup P_{n-12} \). Once again by lemma 3, we have

\[
\begin{align*}
b_{2k}(C_6 \cup P_1 \cup P_{n-12}) &= b_{2k}(P_6 \cup P_1 \cup P_{n-12}) + b_{2k-2}(P_4 \cup P_1 \cup P_{n-12}) + 2b_{2k-6}(P_1 \cup P_{n-12}) \\
&= b_{2k}(P_1 \cup P_2 \cup P_4 \cup P_{n-12}) + b_{2k-2}(P_1 \cup P_1 \cup P_3 \cup P_{n-12}) + \\
&\quad + b_{2k-2}(P_1 \cup P_2 \cup P_2 \cup P_{n-12}) + b_{2k-4}(P_{n-12}) + 2b_{2k-6}(P_{n-12}), \\
b_{2k}(P_2 \cup P_5 \cup P_{n-12}) &= b_{2k}(P_2 \cup P_2 \cup P_3 \cup P_{n-12}) + b_{2k-2}(P_2 \cup P_1 \cup P_2 \cup P_{n-12}) \\
&= b_{2k}(P_1 \cup P_1 \cup P_2 \cup P_3 \cup P_{n-12}) + b_{2k-2}(P_2 \cup P_3 \cup P_{n-12}) + \\
&\quad + b_{2k-2}(P_1 \cup P_2 \cup P_2 \cup P_{n-12}).
\end{align*}
\]

Obviously, \( P_1 \cup P_2 \cup P_4 \cup P_{n-12} \succ P_1 \cup P_1 \cup P_2 \cup P_3 \cup P_{n-12} \). So \( b_{2k}(C_6 \cup P_1 \cup P_{n-12}) - b_{2k}(P_2 \cup P_5 \cup P_{n-12}) \)

\[
\begin{align*}
&\geq b_{2k-2}(P_1 \cup P_1 \cup P_3 \cup P_{n-12}) - b_{2k-2}(P_2 \cup P_3 \cup P_{n-12}) \\
&\quad + b_{2k-4}(P_{n-12}) + 2b_{2k-6}(P_{n-12}) \\
&= b_{2k-2}(P_3 \cup P_{n-12}) - b_{2k-2}(P_3 \cup P_{n-12}) - b_{2k-4}(P_3 \cup P_{n-12}) \\
&\quad + b_{2k-4}(P_{n-12}) + 2b_{2k-6}(P_{n-12}) \\
&= \cdots = 0.
\end{align*}
\]

It’s evident that there exists some \( k_0 \) such that \( b_{2k_0}(C_6 \cup P_1 \cup P_{n-12}) > b_{2k_0}(P_2 \cup P_5 \cup P_{n-12}) \). So \( C_6 \cup P_1 \cup P_{n-12} \succ P_2 \cup P_5 \cup P_{n-12} \) and then \( C_6 \cup P_{n-9} \succ P_2 \cup P_{n-5}^6 \). This completes the proof. \( \square \)

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References


