MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **56** (2006) 593-604

ISSN 0340 - 6253

Trees with m-Matchings and the Third Minimal Hosoya Index *

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(Received March 19, 2006)

Abstract: For a graph G, its *Hosoya* index is defined as the total number of independent edge sets of G. Hou in Discrete Appl. Math. 119(2002)251–257 characterized the trees with a given size of matching and having minimal and second minimal *Hosoya* index. In this paper, we determine the trees with m-matchings and the third minimal *Hosoya* index.

1 Introduction

The *Hosoya* index of a graph, abbreviated by H-index, was first defined by Hosoya [4] in 1971, which is a topological parameter to study the relation between molecular structure and physical and chemical properties of certain hydrocarbon compound. Much related progress can be found in $[2-7]$. All graphs considered here are simple, finite and undirect. Undefined notation and terminology conform to those in [1].

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. Two edges of G are said to be *independent* if they possess no vertex in common. Any subset of $E(G)$ containing no two mutually incident edges is called *an independent edge*

*This work is supported by National Science Foundation of China(10461009) and the Science Foundation of the State Education Ministry of China(205170).

set of G. An independent edge set of k edges in G is said to be *an k-matching* of G. *H-index* of a graph G is defined as follows:

$$Z(G) = \sum_{k=0} m(G,k),$$

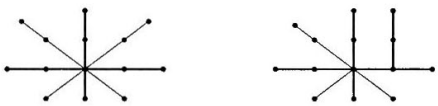
where $m(G,k)$ denotes the number of k-matchings of G. Recall the $m(G,0) = 1$ and $m(G,1) = |E(G)|$. Note that if $m(G,k) = 0$, then $m(G,k+1) = 0$. Furthermore, $m(G,k) = 0$ for $k > \frac{n}{2}$, where $n = |V(G)|$.

For $n \geq 2$, by P_n and S_n we denote the path and the star of n vertices, respectively. For all trees with n vertices, it was proved that the path P_n and star S_n have maximal and minimal H-index in [4, 6], respectively, that is, for any tree T with order n, we have

$$n = Z(S_n) \leq Z(T) \leq Z(P_n) = F_{n+1}.$$

where F_{n+1} is the $(n+1)th$ Fibonacci number with $F_{n+1} = F_n + F_{n-1}$ and $F_1 = F_2 = 1$. The author in [3] characterized the trees with m-matchings and with the minimal and second minimal H-index. In this paper, we shall characterize the trees with m-matchings and having the third minimal H-index.

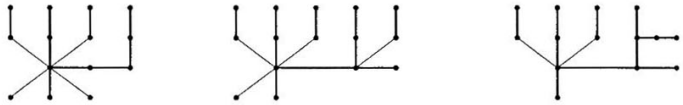
Let k and r be two non-negative integers and let $n = 2k + r + 1$. The tree $S(n,k,r)$ is defined as follows [3]: $S(n,k,r)$ is the graph obtained from star S_{k+r+1} by attaching a pendent edge to k non-central vertices. Note that $S(n,k,r)$ has a matching of $m = k + r'$ edges, where $r' = 0$ if $r = 0$ and $r' = 1$ if $r > 0$, and the center of $S(n,k,r)$ is the center of the star S_{k+r+1}. For $n \geq 3$, let $R(n,k,r)$ denote the graph obtained from $S(n-2,k-1,r)$ by attaching a path of length 2 to one vertex of degree 2. For example, Graphs $S(14,5,3)$ and $R(14,5,3)$ are the graphs shown in Figure 1:



$S(14,5,3)$ $R(14,5,3)$

Figure 1. $S(14,5,3)$ **and** $R(14,5,3)$.

It is obvious that $R(n,k,r)$ also has an m-matching, where $m = k + r'$, $r' = 0$ if $r = 0$ and $r' = 1$ if $r > 0$. The center of $R(n,k,r)$ is the center of $S(n-2,k-1,r)$. Now we define three new families of graphs: $T(n,k,r)$ is the graph obtained by attaching one vertex of degree 1 of P_3 to the pendant vertex of $S(n-2,k,r)$ that is not adjacent to the center, $U(n,k,r)$ is the graph obtained from $R(n-2,k-1,r)$ by attaching one vertex of degree 1 of P_3 to one vertex of degree 3 which is adjacent to the center, and $V(n,k,r)$ is the graph obtained from $R(n-2,k-1,r)$ by attaching one vertex of degree 1 of P_3 to one vertex of degree 2 which is not adjacent to the center. As some examples, $T(14,4,3)$, $U(15,6,2)$ and $V(14,6.1)$ are the graphs shown in Figure 2.



$T(14,4,3)$ $U(15,6,2)$ $V(14,6,1)$

Figure 2. $T(14,4,3)$, $U(15,6,2)$ **and** $V(14,6,1)$.

2 Preliminaries

Let T be a tree with n vertices and A its adjacent matrix. Let $B(T) = A + I$, where I is unit matrix of order n. Recall the definition of permanent [8] of a matrix $B = (b_{ij})$:

$$per(B) = \sum_{\sigma} \prod_{i=1}^{n} b_{i\sigma(i)},$$

where the summation is taken over the symmetric group of order n.

Lemma 1([3]). Let T be a tree. Then

$$Z(T) = per(A + I).$$

Let M denote a matching of graph, $v \in M$ means the vertex v is incident to an edge of M and $v \notin M$ means the vertex v is not incident to any edges of M.

Lemma 2([3]). Let T be a tree with a perfect matching. Then T has a pendant edge which is incident to a vertex of degree 2.

Lemma 3([3]). Let T be a tree of n vertices with an m-matching, where $n > 2m$. Then there exists an m-matching M and a pendant vertex v such that $v \notin M$.

Lemma 4([3]). Let T be a tree of n vertices with an m-matching, where $m \geq 1$. Then

$$Z(T) \geq 2^{m-2}(2n - 3m + 3),$$

where the equality holds if and only if $T = S(n, m-1, n-2m+1)$.

Lemma 5([3]). Let T be a tree of n vertices with an m-matching, where $m \geq 1$. If $T \neq S(n, m-1, n-2m+1)$, then

$$Z(T) \geq 2^{m-4}(10n - 15m + 9),$$

where the equality holds if and only if $T = R(n, m-1, n-2m+1)$.

Remark: There is an error that the original version of Lemma 5 in [3] is $Z(T) \geq 5 \cdot 2^{m-4}(2n - 3m)$.

Lemma 6. $Z(T(n,k,r)) = 2^{k-2}(5k + 10r + 11)$.

Proof: We choose an appropriate ordering of the vertices for $T(n,k,r)$, such that

$$B(T) = A + I = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & & & & & & & \\ 0 & 1 & 1 & 1 & 0 & 0 & & & & & & & \\ 0 & 0 & 1 & 1 & 1 & 0 & & & & & & & \\ 0 & 0 & 0 & 1 & 1 & 0 & & & & & & & \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & & & & & & \\ \vdots & & & & & & & \ddots & & & & & \\ 1 & & & & & & & & 1 & 1 & & & \\ 0 & & & & & & & & 1 & 1 & & & \\ & & & & & & & & & & 1 & & \\ \vdots & & & & & & & & & & & \ddots & \\ 1 & & & & & & & & & & & & 1 \end{pmatrix},$$

where the unwritten entries are all zeroes. Calculating the permanent by an expansion along the first row, we then obtain

$$Z(T(n,k,r)) = 5{\cdot}2^{k-1} + 3{\cdot}2^{k-1} + 5(k-1)2^{k-2} + 5r2^{k-1} = 2^{k-2}(5k + 10r + 11).$$

As an analogue to the proof of Lemma 6, we have the following lemma.

Lemma 7. (i) $Z(U(n,k,r)) = 2^{k-3}(6k + 12r - 2)$.
(ii) $Z(V(n,k,r)) = 2^{k-3}(6k + 12r - 1)$.

Lemma 8. Let T be the graph obtained from G by attaching one vertex of degree 1 of P_3 to a vertex of G, where $G \in \{S(n-2,m-2,1), R(n-2,m-2,1), U(n-2,m-2,1), V(n-2,m-2,1)\}$ and $T \notin \{S(n,m-1,1), R(n,m-1,1), U(n,m-1,1), V(n,m-1,1)\}$. Then

$$Z(T) \geq Z(T(n,m-2,1)).$$

Proof: We only prove the lemma for $G = U(n-2,m-2,1)$. It is easy to see that $n = 2m$ and

$$Z(U(n,m-1,1)) = 2^{m-4}(12n - 18m + 4),\ Z(T(n,m-2,1)) = 2^{m-4}(5m + 11).$$

Note that $m \geq 5$ for $U(n-2,m-2,1)$ and $U(8,3,1) \cong S(8,3,1)$ if $m = 5$. By the definition of T, we have that T is one of the following graphs:

$S(10,4,1)$, or $R(10,4,1) \cong U(10,4,1)$, or $T(10,3,1)$, or H_1,
where H_1 is the graph shown in Figure 3.

Figure 3. Graph H_1.

By calculation, we have $Z(H_1) = 76 > Z(T(10,3,1)) = 72$. From the condition of the lemma, the result holds.

Now we need only consider the case of $m \geq 6$. For T, v_1, v_2 and v_3 are the vertices of T shown in Figure 4.

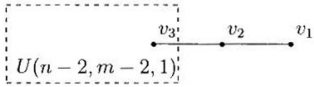


Figure 4. Graph T with the vertices v_1, v_2 and v_3.

Then

$$B(T) = A + I = \begin{pmatrix} 1 & 1 & 0 & O_{n-3} \\ 1 & 1 & 1 & O_{n-3} \\ 0 & 1 & & \\ O_{n-3}^T & O_{n-3}^T & B(U(n-2, m-2, 1)) & \end{pmatrix},$$

where O_{n-3} is a zero vector of length $n-3$ and O_{n-3}^T is the transpose of O_{n-3}.

$$perB(T) = 2perB(U(n-2, m-2, 1)) + perB(T'),$$

where T' is the graph obtained from T by deleting v_1, v_2 and v_3. From the structure of $U(n-2, m-2, 1)$ and the condition of the lemma, we have that $d_T(v_3) = 2$, or 3, or 5. We distinguish the following cases:

Case 1. $d_T(v_3) = 2$. Then T' is a tree of $n-3$ vertices with $(m-2)$–matchings. It follows from Lemma 4 that

$$perB(T') \geq 2^{m-4}(2n - 3m + 3).$$

Together with $perB(U(n-2, m-2, 1)) = 2^{m-5}(12n - 18m - 2)$, we have that $perB(T) \geq$
$2^{m-4}(14n - 21m + 1)$ and $perB(T) - perT(n, m-2, 1) \geq 2^{m-4}(2m - 10)$. So

$$perB(T) \geq perB(T(n, m-2, 1)).$$

Case 2. $d_T(v_3) = 3$. Then T' is the union of K_1 and a tree of $n-4$ vertices with $(m-2)$–matchings. From Lemma 4 we have

$$perB(T') \geq 2^{m-4}(2n - 3m + 1).$$

So

$$perB(T) \geq 2^{m-4}(14n - 21m - 1).$$

If $m = 5$, then $T = R(10, 4, 1)$. If $m \geq 6$, then

$$perB(T) - perB(T(n, m-2, 1)) \geq 2^{m-4}(2m - 12),$$

and so

$$perB(T) \geq perB(T(n, m-2, 1)).$$

Case 3. $d_T(v_3) = 5$. Then T is the graph shown in Figure 5.

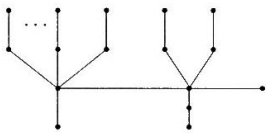


Figure 5. Graph T with $d_T(v_3) = 5$.

If $m = 6$ or 7, then T is $R(12, 5, 1)$ or $U(14, 6, 1)$, which contradicts to the condition of the theorem. If $m \geq 8$, by calculation we have $Z(T) = 2^{m-6}(28m - 12)$, and so

$$Z(T) - Z(T(n, m-2, 1)) = 2^{m-6}(28m - 12) - 2^{m-4}(5m + 11) > 0.$$

This completes the proof.

Lemma 9. For $n \geq 2m+1$, let T be the graph obtained from G by attaching one vertex of P_2 to any vertex of G, where $G \in \{S(n-1, m-1, n-2m), R(n-1, m-1, n-2m), U(n-1, m-1, n-2m), V(n-1, m-1, n-2m)\}$ and $T \notin \{S(n, m-1, n-2m+1), R(n, m-1, n-2m+1), U(n, m-1, n-2m+1), V(n, m-1, n-2m+1)\}$. Then

$$Z(T) \geq Z(T(n, m-2, n-2m+1)).$$

Proof: We only prove the lemma for $G = U(n-1, m-1, n-2m)$. For $U(n-1, m-1, n-2m)$, we have $m \geq 4$. By calculation we have

$$Z(U(n, m-1, n-2m+1)) = 2^{m-4}(12n - 18m + 4)$$

and

$$Z(T(n, m-2, n-2m+1)) = 2^{m-4}(10n - 15m + 11).$$

For T, v_1 is the vertex of T with $v_1 \in V(P_2)$ and $d_T(v_1) = 1$, and v_2 is the vertex of T with $v_2 \in V(P_2)$ and $d_T(v_2) \geq 2$. Then

$$B(T) = A + I = \begin{pmatrix} 1 & 1 & O_{n-2} \\ 1 & & \\ O_{n-2}^T & & B(U(n-1, m-1, n-2m)) \end{pmatrix},$$

where O_{n-2} denotes the zero vector of length of $n-2$ and O_{n-2}^T is the transpose of O_{n-2}. From the theory of matrix, it follows that

$$perB(T) = perB(U(n-1, m-1, n-2m)) + perB(T'),$$

where T' is the graph obtained from T by deleting v_1 and v_2. From the structure of $U(n-1, m-1, n-2m)$ and the condition of the lemma, it is easy to see that $d_T(v_2) = 2$, or 3, or 5. We distinguish the following cases:

Case 1. $d_T(v_2) = 2$. Then T' is a tree of $n-2$ vertices with $(m-1)$–matchings. From Lemma 4 we have

$$perB(T') \geq 2^{m-3}(2n - 3m + 2).$$

Since

$$perB(U(n-1,m-1,n-2m)) = 2^{m-3}(6n-9m-4),$$

we have $perB(T) \geq 2^{m-3}(8n-12m-2)$ and $perB(T) - perT(n,m-2,n-2m+1) \geq 2^{m-4}(6n-9m-15)$. By $n \geq 2m+1$, we have

$$perB(T) \geq perB(T(n,m-2,n-2m+1)).$$

Case 2. $d_T(v_2) = 3$. Then $T'$is the union of K_1 and a tree of $n-3$ vertices with $(m-1)$−matching. By Lemma 4, it is not hard get that

$$perB(T') \geq 2^{m-3}(2n-3m).$$

So

$$perB(T) \geq 2^{m-3}(8n-12m-4),$$

$$perB(T) - perB(T(n,m-2,n-2m+1)) \geq 2^{m-4}(6n-9m-19).$$

We arrive that

$$perB(T) \geq perB(T(n,m-2,n-2m+1))$$

for $m \geq 5$. For $m = 4$, we obtain the result by calculation.

Case 3. $d_T(v_2) = 5$. Then T is the graph shown in Figure 6.

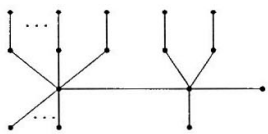


Figure 6. Graph T with $d_T(v_2) = 5$.

If $m = 5$ and $n = 11$, then $T = R(11,4,2)$, contradicting with the condition of the theorem. If $m = 5$ and $n > 11$ or $m \geq 6$, with a similar proof of Lemma 6 we have $Z(T) = 2^{m-4}(16n-24m-12)$. It is easy to get that

$$Z(T) - Z(T(n,m-2,n-2m+1)) = 6n-9m-23.$$

Note that $n \geq 2m+1$. Thus we have that $Z(T) - Z(T(n,m-2,n-2m+1)) = 6n-9m-23 > 0$ for $m = 5$ and $n > 11$, and $Z(T) - Z(T(n,m-2,n-2m+1)) = 6n-9m-23 > 3m-17 > 0$ for $m \geq 6$. This completes the proof.

3 Main Results and Proof

Theorem 1. Let T be a tree of n vertices with m-matchings, where $m \geq 1$. If $T \notin \{S(n,m-1,n-2m+1), R(n,m-1,n-2m+1), U(n,m-1,n-2m+1), V(n,m-1,n-2m+1)\}$, then

$$Z(T) \geq 2^{m-4}(10n-15m+11)$$

with equality if and only if $T = T(n, m-2, n-2m+1)$, where $n \geq 2m$.

Proof: From the condition of the theorem, we have $n \geq 2m$. Suppose that $n = 2m$, that is, T has a perfect matching. We prove the theorem by induction on m. When $m = 1$, $T = P_2 = S(2, 0, 1)$; when $m = 2$, the tree with perfect 2−matchings is $P_4 = S(4, 1, 1)$; If $m = 3$, the trees with perfect 3−matchings are $S(6, 2, 1)$, or P_6. If $m = 4$, the trees with perfect 4−matchings are P_8, $S(8, 3, 1)$, $R(8, 3, 1)$, $T(8, 2, 1)$ or the graph H_2 shown in Figure 7:

Figure 7: Graph H_2.

By calculation, we have that $Z(H_2) = 32 > Z(T(8, 2, 1)) = 31$.

If $m = 5$, From the reference [9], we know that the trees with perfect 5−matchings are

$$P_{10}, S(10, 4, 1), R(10, 4, 1), T(10, 3, 1), V(10, 4, 1), G_i (1 \leq i \leq 10),$$

where G_i are the graphs in Figure 8.

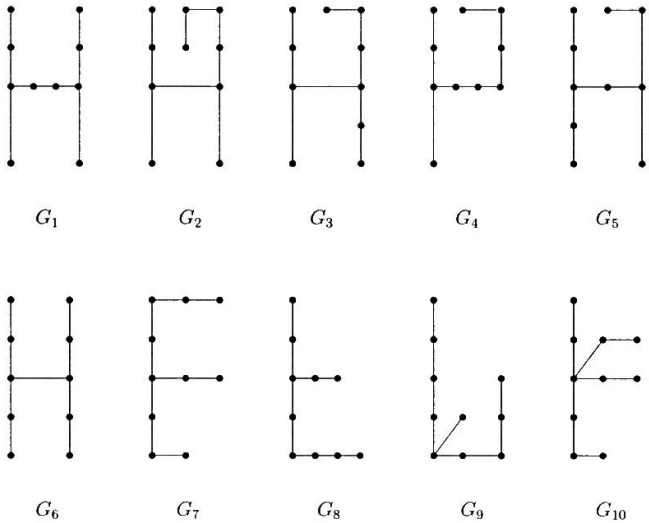


Figure 8. Some graphs G_i with perfect 5−matchings.

By calculation it is not hard to see that the results holds.

We now suppose that $m > 5$ and proceed by induction. From Lemma 2, it follows that T has a pendant edge uw such that $d(u) = 1$ and $d(w) = 2$.

Then there exists only one another edge wv such that we obtain the graph T' with $2(m-1)$ vertices and $(m-1)-$matching, by deleting u and w. Note that $T' \in \{S(n-2,m-2,1), R(n-2,m-2,1), U(n-2,m-2,1), V(n-2,m-2,1)\}$. It follows, from Lemma 8 and the condition of the theorem, that

$$Z(T) \geq 2^{m-4}(10n-15m+11).$$

If $T' \notin \{S(n-2,m-2,1), R(n-2,m-2,1), U(n-2,m-2,1), V(n-2,m-2,1)\}$, by the induction hypothesis we have

$$Z(T') \geq 2^{m-5}(10(n-2)-15(m-1)+11) = 2^{m-5}(5(m-1)+11), \tag{1}$$

with equality if and only if $T' = T(2m-2, m-3, 1)$.

Labelling the vertices by the order of u, w and v, we have

$$B(T) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & X \\ 0 & 0 & X^T & C \end{pmatrix},$$

$$B(T') = \begin{pmatrix} 1 & X \\ X^T & C \end{pmatrix}$$

and

$$Z(T) = perB(T) = perC + 2perB(T') = perC + 2Z(T'). \tag{2}$$

Since T' has at least one perfect $(m-1)-$matching, we label the vertices except for u, w, v such that

$$C = \begin{pmatrix} 1 & 1 & & & & \\ 1 & 1 & & * & & \\ & & \ddots & & & \\ & & & 1 & 1 & \\ & * & & 1 & 1 & \\ & & & & & 1 \end{pmatrix},$$

where there are $(m-2)$ blocks $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ in C. If the entries in $*$ of C are all 0, then the two entries in X^T of $B(T')$ which have the same rows as some block $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ in C are not all 1 or 0. If the two entries are all 1, then T contains a cycle; If they are all 0, then T is disconnected. Hence, one of them is 1 and the

other is 0. Noting that the label of T', we have

$$B(T') = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & & & & & & & & \\ 0 & 1 & 1 & & & & & & & & \\ 1 & 0 & 0 & 1 & 1 & & & * & & & \\ 0 & 0 & 0 & 1 & 1 & & & & & & \\ 1 & & & & & & \ddots & & & & \\ 0 & & & & & & & \ddots & & & \\ \vdots & & & & & & & & \ddots & & \\ 0 & & & * & & & & & & \ddots & \\ 1 & & & & & & & & & & 1 \end{pmatrix}$$

Obviously, $T' = S(2m-2, m-2, 1)$ which is a contradiction. So the number of 1's in $*$ of C is at least 2(C is symmetric). We choose two blocks $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, which have the same row or column as that of some 1. By expanding $perC$ along the four rows in which the two blocks lie, we have

$$PerC \geq perD \cdot perE, \tag{3}$$

where D is one of the following four matrices,

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and E is obtained from C by deleting the row and column in which the entries of D lie. It is obvious that $perE \geq 2^{m-4}$ and $perD = 5$. Thus

$$perC \geq 5 \cdot 2^{m-4}, \tag{4}$$

$$Z(T) = perC + 2Z(T') \geq 5 \cdot 2^{m-4} + 2 \cdot 2^{m-5}(5(m-1)+11) = 2^{m-4}(5m+11) \tag{5}$$

with equality in (5) holds if and only if $Z(T') = 2^{m-5}(5(m-1)+11)$, that is, $T' = T(2m-2, m-3, 1)$ and $perC = 5 \cdot 2^{m-4}$. Then there exists only one minor like as D in C, the other entries is 0, the minor like as D implies that there is path P_4 contained in T'. Since T is a tree, there is only one 1 of the entries in X^T which have the same rows as D. The degree of all vertices in $T' = T(2m-2, m-3, 1)$ is less than 3 except the vertex v. Thus, the number of $1'$s in every row of C is not more than 4. So the entries in X^T, which have the same row as one of the rows in which the two $1's$ of D lie, are 1, then one of pendant vertex of P_4 is adjacent to v of T'.

Suppose that $n > 2m$, we will prove the theorem by induction on the order of T. From Lemma 3, it follows that T has an m-matching M and a pendant vertex v such that $v \notin M$. Let w be adjacent to v in T and T' denotes the graph from T by deleting v. Then T' is a tree of $n-1$ vertices with m-matchings. If $T' \in \{S(n-1, m-1, n-2m), R(n-1, m-1, n-2m), U(n-1, m-1, n-$

$2m), V(n-1, m-1, n-2m)\}$, by the condition of the theorem and Lemma 9 we have

$$Z(T) \geq 2^{m-4}(10n - 15m + 11).$$

If $T' \notin \{S(n-1, m-1, n-2m), R(n-1, m-1, n-2m), U(n-1, m-1, n-2m), V(n-1, m-1, n-2m)\}$, by the induction hypothesis we have

$$Z(T') \geq 2^{m-4}(10(n-1) - 15m + 11) \tag{6}$$

with equality if and only if $T' = T(n-1, m-2, n-2m)$.

Ordering the vertices of T as $v, w, \cdots$, we have

$$B(T) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & X \\ 0 & X^T & C \end{pmatrix}, \quad B(T') = \begin{pmatrix} 1 & X \\ X^T & C \end{pmatrix}.$$

By expanding the permanent along the first row, we have

$$Z(T) = perB(T) = perB(T') + perC.$$

As an analogue to the above proof, we can obtain

$$perC \geq 5 \cdot 2^{m-3}, \tag{7}$$

$$Z(T') \geq 2^{m-4}(10(n-1) - 15m + 11). \tag{8}$$

Thus

$$Z(T) \geq 5 \cdot 2^{m-3} + 2^{m-4}(10(n-1) - 15m + 11) = 2^{m-4}(10n - 15m + 11). \tag{9}$$

If the equality holds in (9), so do (6) and (7). By induction hypothesis, we have $T' = T(n-1, m-2, n-2m)$.

Similarly, by induction on m, we can show the case of $n = 2m$, and w is the center of $T' = T(n-1, m-2, n-2m)$ and

$$T = T(n, m-2, n-2m+1).$$

This completes the proof.

Remark: From Lemma 7, if $m \geq 6$, then

$$Z(U(n, m-1, n-2m+1)) \geq Z(T(n, m-2, n-2m+1)).$$

If $m \geq 7$, then

$$Z(V(n, m-1, n-2m+1)) \geq Z(T(n, m-2, n-2m+1)).$$

We only characterize the tree with m-matchings and having the third minimal *Hosoya* index when the number of matchings is more than 6.

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