

# A Proof for the Existence of Five Stereogenicity Types on the Basis of the Existence of Five Types of Subgroups of *RS*-Stereoisomeric Groups. Hierarchy of Groups for Restructuring Stereochemistry (Part 3)

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## Abstract

A proof for the existence of five stereogenicity types, which has once been demonstrated intuitively by means of stereoisograms, is mathematically given on the basis of the five factor groups derived from *RS*-stereoisomeric groups. These factor groups are proved to correspond to five types of subgroups appearing in *RS*-stereoisomeric groups. Thereby, the relationship between the concept of chirality and the concept of stereogenicity is clarified after the latter concept is replaced by a more specified concept, i.e., *RS*-stereogenicity. Moreover, the concepts of holantimers and *RS*-diastereomers gain a sound mathematical basis.

## 1 Introduction

The concept of stereogenicity has been frequently confused with the concept of chirality. For example, the CIP (Cahn-Ingold-Prelog) system of stereochemical nomenclature has

originally proposed to specify the chirality of molecules [1], but later revised to specify the stereogenicity [2]. Although there appeared a convincing discussion so as to differentiate the stereogenicity from the chirality [3], the difference has not been fully demonstrated even in standard textbooks on stereochemistry [4, 5].

We discussed chirality and prochirality both theoretically [6] and intuitively [7] so that various concepts obtained by these discussions were applied to isomer enumerations and classification of symmetries [8]. Recently, we discussed chirality and stereogenicity in terms of observance/violation of chirality fittingness [9] and in terms of stereoisomeric groups [10]. Although these results revealed several points concerned with the difference between stereogenicity and chirality, a more detailed investigation on a common standpoint was desirable so that it enabled us to discuss both stereogenicity and chirality.

For this purpose, the symmetries of molecules of ligancy 4 have been studied theoretically or intuitively in previous articles for tetrahedral molecules [11, 12], in Part 1 of this series for allene molecules [13], and in Part 2 for square-planar complexes [14], where the modes of inclusion for relevant groups (i.e., point groups, *RS*-stereoisomeric groups, stereoisomeric groups, and isoskeletal groups) have been clarified to be altered variously according to the molecular symmetries. Through these studies, the importance of *RS*-stereoisomeric groups has been emphasized as a key for discussing both stereogenicity and chirality. In particular, the *RS*-stereoisomeric groups have been used to specify five stereogenicity types through the concepts of holantimers and stereoisograms [12]. However, the usage has limited within intuitive discussion on molecules of ligancy 4.

As clarified in the previous paragraphs, the purpose of the present article to investigate *RS*-stereoisomeric groups more generally in terms of abstract group theory, where the existence of only five types of subgroups in each *RS*-stereoisomeric group will be proved in general so as to correspond to the existence of five stereogenicity types. Thereby, the concepts of holantimers and stereoisograms [12] will gain a sound theoretical basis to be applied to molecules other than those of ligancy 4.

## 2 *RS*-Stereoisomeric Groups

For the sake of convenience, the present section is devoted to a brief restatement on the definition of *RS*-Stereoisomeric groups. In this paper, the concepts of proligands and promolecules are used as defined in a previous article [15].

### 2.1 Definitions

Consider  $n$  positions of a given skeleton, which are governed by the coset representation  $\mathbf{G}/(\mathbf{G}_i)$  of a point group  $\mathbf{G}$ , where the group  $\mathbf{G}_i$  is a subgroup of  $\mathbf{G}$  and the degree of the coset representation is calculated to be  $n = |\mathbf{G}|/|\mathbf{G}_i|$  [6, 8]. The  $n$  positions accommodate  $n$  (pro)ligands selected from a set of chiral and achiral (pro)ligands so as to generate a (pro)molecule of the point group  $\mathbf{H} (\subset \mathbf{G})$ . When  $\mathbf{G}_i$  runs over all of the subgroups (up to conjugate subgroups) through the subscript  $i$ , all the coset representations of the point group  $\mathbf{G}$  can be examined exhaustively. In general cases in which  $n$  positions are governed by a sum of coset representations  $\sum_i \alpha_i \mathbf{G}/(\mathbf{G}_i)$  ( $\alpha_i$ : the multiplicity of  $\mathbf{G}/(\mathbf{G}_i)$ ), each  $\mathbf{G}/(\mathbf{G}_i)$  can be considered separately without losing generality.

The coset representation  $\mathbf{G}/(\mathbf{G}_i)$  is regarded as a permutation group  $\mathbf{G}_{C\sigma}$ , where a

permutation corresponding to an improper rotation (rotoreflexion) is designated by a symbol with an overbar in order to represent the inversion of ligand chirality.

**Def. 1** *Three types of groups, i.e., achiral/chiral groups (point groups); ascleral/scleral groups (inversion groups); and RS-astereogenic/RS-stereogenic groups (RS-permutation groups), are defined on the basis of a point group  $\mathbf{G}$ . Then, RS-stereoisomeric groups are defined by combining these three types of groups.*

**Achiral/chiral groups (point groups)** When the point  $\mathbf{G}$  is an achiral point group, the group  $\mathbf{G}_{C\sigma}$  is obtained in terms of coset representations of  $\mathbf{G}$  as follows:

$$\mathbf{G}_{C\sigma} = \mathbf{G}_C + \sigma\mathbf{G}_C, \quad (1)$$

where the permutation group  $\mathbf{G}_C$  corresponds to the maximum chiral subgroup of  $\mathbf{G}$  and  $\sigma$  represents a permutation corresponding to a rotoreflexion. Since the group  $\mathbf{G}_{C\sigma}$  is isomorphic to the point group  $\mathbf{G}$ , it is also called *point group* for the sake of simplicity. As a subgroup of the point group  $\mathbf{G}$  is classified into a chiral group or an achiral group, a subgroup of the group  $\mathbf{G}_{C\sigma}$  is classified into a chiral group or an achiral group. Such a chiral subgroup is a subgroup of the maximum chiral subgroup  $\mathbf{G}_C$ . Note that each operation of the coset  $\sigma\mathbf{G}_C$  represented by a permutation with an overbar provides a reflection of the skeleton as well as the reflections of ligands. The coset  $\sigma\mathbf{G}_C$  corresponds to a set of enantiomers. The representative  $\sigma$  of the coset  $\sigma\mathbf{G}_C$  can be selected according to the purpose of discussions.

**Ascleral/scleral groups (inversion groups)** Let the symbol  $\tilde{I}$  represent an operation which provides the reflection of ligands, but does not provide the reflection of the skeleton. Thereby, another group  $\mathbf{G}_{C\tilde{I}}$  called *inversion group* is obtained as follows:

$$\mathbf{G}_{C\tilde{I}} = \mathbf{G}_C + \tilde{I}\mathbf{G}_C. \quad (2)$$

A subgroup of the inversion group is classified into a scleral group or an ascleral group. A scleral subgroup is a subgroup of the maximum scleral group, which is identical with the chiral group  $\mathbf{G}_C$ . An ascleral subgroup contains at least one element of  $\tilde{I}\mathbf{G}_C$ .

The coset  $\tilde{I}\mathbf{G}_C$  corresponds to a set of holantimers [12]. It should be noted that the element  $\tilde{I}$  is recognized as a definite element. If the identity element  $I$  is represented by the permutation  $(1)(2)\cdots(n)$ , the element  $\tilde{I}$  is represented by the permutation  $(\bar{1})(\bar{2})\cdots(\bar{n})$ , where each number with an overbar represents a 1-cycle with the reflection of a ligand.

**RS-Astereogenic/RS-stereogenic groups (RS-permutation groups)** Consider a permutation  $\bar{\sigma}$ , which satisfies  $\bar{\sigma} = \tilde{I}\sigma = \sigma\tilde{I}$ . The operation  $\bar{\sigma}$  is represented by the same permutation as  $\sigma$ , which in turn has no overbar because it does not provide the reflection of ligands. Thereby, another group called an *RS-permutation group* is obtained as follows:

$$\mathbf{G}_{C\bar{\sigma}} = \mathbf{G}_C + \bar{\sigma}\mathbf{G}_C \quad (3)$$

The group  $\mathbf{G}_{C\bar{\sigma}}$  is isomorphic to the point group  $\mathbf{G}$ , so long as the reflection of ligand chirality is not taken into consideration. A subgroup of the *RS-permutation*

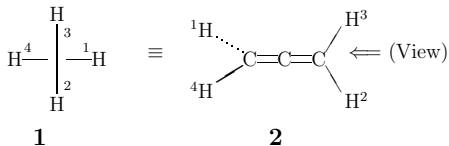


Figure 1: Convention for drawing allene derivatives

group is classified into an *RS*-stereogenic group or an *RS*-astereogenic group. An *RS*-stereogenic subgroup is a subgroup of the maximum *RS*-stereogenic group that is identical with a chiral group  $\mathbf{G}_C$ . An *RS*-astereogenic subgroup contains at least one element of  $\bar{\sigma}\mathbf{G}_C$ . The coset  $\bar{\sigma}\mathbf{G}_C$  corresponds to a set of *RS*-diastereomers.

***RS*-Stereoisomeric groups** By combining eqs. 1, 2, and 3, a supergroup  $\mathbf{G}_{C\sigma\bar{\sigma}\bar{I}}$  is obtained:

$$\mathbf{G}_{C\sigma\bar{\sigma}\bar{I}} = \mathbf{G}_C + \sigma\mathbf{G}_C + \bar{\sigma}\mathbf{G}_C + \bar{I}\mathbf{G}_C, \quad (4)$$

which represents a coset decomposition. This group is called an *RS-stereoisomeric group*.

It should be noted that the normal subgroup  $\mathbf{G}_C$  is a chiral group, a scleral group, and an *RS*-stereogenic group at the same time, as found in eqs. 1, 2, and 3. Because  $\mathbf{G}_{C\sigma}$  represented by eq. 1 is one of the largest subgroups of  $\mathbf{G}_{C\sigma\bar{\sigma}\bar{I}}$  (eq. 4), the former is called *the maximum point group* of the latter. Because  $\mathbf{G}_{C\bar{I}}$  represented by eq. 2 is another one of the largest subgroups of  $\mathbf{G}_{C\sigma\bar{\sigma}\bar{I}}$  (eq. 4), the former is called *the maximum inversion group* of the latter. Because  $\mathbf{G}_{C\bar{\sigma}}$  represented by eq. 3 is a further one of the largest subgroups of  $\mathbf{G}_{C\sigma\bar{\sigma}\bar{I}}$  (eq. 4), the former is called *the maximum RS-permutation group* of the latter.

As for the elements  $\sigma$  (eq. 1) and  $\bar{I}$  (eq. 2), there are two cases, i.e.,  $\sigma \neq \bar{I}$  and  $\sigma = \bar{I}$ . If we place  $\sigma = \bar{I}$  ( $\exists \sigma \in \sigma\mathbf{G}_C$ ), the group  $\mathbf{G}_{C\sigma}$  coincides with the group  $\mathbf{G}_{C\bar{I}}$ . In this paper, we focus our attention on the case of  $\sigma \neq \bar{I}$  ( $\forall \sigma \in \sigma\mathbf{G}_C$ ) as a more general case than the case of  $\sigma = \bar{I}$  ( $\exists \sigma \in \sigma\mathbf{G}_C$ ).

*Example 1:* Throughout the present paper, an allene skeleton of  $\mathbf{D}_{2d}$  is taken as an example (Fig. 1). The four positions of the allene skeleton are governed by the coset representation  $\mathbf{D}_{2d}/(\mathbf{C}_s)$ , the degree of which is calculated to be  $|\mathbf{D}_{2d}|/|\mathbf{C}_s| = 8/2 = 4$ .

As found in Fig. 2, we place  $\mathbf{G}_{C\sigma} = \mathbf{D}_{2d}/(\mathbf{C}_s)$  and  $\sigma = \sigma_{d(1)} = (1\ 2)(3\ 4)$ . Note that we take account of the coset representation  $\mathbf{D}_{2d}/(\mathbf{C}_s)$  in place of the  $\mathbf{D}_{2d}$ -point group itself. The *RS*-permutation group  $\mathbf{G}_{C\bar{\sigma}}$  for this case is represented by  $\mathbf{D}_{2\bar{\sigma}}$ , which is isomorphic with  $\mathbf{S}_7^{[4]}$ , i.e., a subgroup of the symmetric group of degree 4 ( $\mathbf{S}^{[4]}$ ). The inversion group  $\mathbf{G}_{C\bar{I}}$  for this case is the combination of the diagonal parts appearing in Fig. 2. This group is designated by the symbol  $\mathbf{D}_{2\bar{I}}$  as follows:

$$\mathbf{D}_{2\bar{I}} = \{ (1)(2)(3)(4), (1\ 4)(2\ 3), (1\ 2)(3\ 4), (1\ 3)(2\ 4); \\ (1)(2)(3)(4), (\bar{1}\ 4)(2\ 3), (\bar{1}\ 2)(3\ 4), (\bar{1}\ 3)(2\ 4) \} \quad (5)$$

		$\mathbf{D}_{2\tilde{\sigma}} (= \mathbf{S}_9^{[4]})$	
		$\mathbf{S}_7^{[4]}$	$(1)(2\ 3)(4)\mathbf{S}_7^{[4]}$
}	$\mathbf{D}_2$	$I \sim$	$(1)(2)(3)(4) \quad (1)(2\ 3)(4)$
		$C_{2(3)} \sim$	$(1\ 4)(2\ 3) \quad (1\ 4)(2)(3)$
		$C_{2(1)} \sim$	$(1\ 2)(3\ 4) \quad (1\ 2\ 4\ 3)$
		$C_{2(2)} \sim$	$(1\ 3)(2\ 4) \quad (1\ 3\ 4\ 2)$
}	$\mathbf{D}_2\sigma_{d(1)}$	$\sigma_{d(1)} \sim$	$\overline{(1)(2\ 3)(4)} \quad \overline{(1)(2)(3)(4)}$
		$\sigma_{d(2)} \sim$	$\overline{(1\ 4)(2)(3)} \quad \overline{(1\ 4)(2\ 3)}$
		$S_4 \sim$	$\overline{(1\ 2\ 4\ 3)} \quad \overline{(1\ 2)(3\ 4)}$
		$S_4^3 \sim$	$\overline{(1\ 3\ 4\ 2)} \quad \overline{(1\ 3)(2\ 4)}$
		point group: $\mathbf{D}_{2d}/\langle C_s \rangle$	
$RS$ -stereoisomeric group: $\mathbf{D}_{2d\tilde{\sigma}1} (= \mathbf{S}_9^{[4]} \times \{I, \sigma\})$			

Figure 2:  $RS$ -Stereoisomeric Group and Its Subgroups for characterizing an allene skeleton.

Finally, the  $RS$ -stereoisomeric group  $\mathbf{G}_{C\sigma\sigma\tilde{1}}$  for this case can be constructed so as to be designated by the symbol  $\mathbf{D}_{2d\tilde{\sigma}1}$ , as shown in Fig. 2.  $\square$

## 2.2 Homomer Sets

The  $n$  positions of a  $\mathbf{G}_{C\sigma\sigma\tilde{1}}$ -skeleton generate a set of skeletons on the action of permutations of the  $\mathbf{G}_{C\sigma\sigma\tilde{1}}$  group. Then, the  $n$  positions of each permuted skeleton accommodate  $n$  ligands or proligands according to a function  $f: f(1), f(2), \dots, f(n)$  so as to a set of  $RS$ -stereoisomers.

**Theorem 1** *The group  $\mathbf{G}_C$  is a normal subgroup of  $\mathbf{G}_{C\sigma\sigma\tilde{1}}$ .*

*Proof:* Because the group  $\mathbf{G}_C$  is the maximum chiral group of the achiral point group  $\mathbf{G}_{C\sigma} (= \mathbf{G}_C + \sigma\mathbf{G}_C)$ , the group  $\mathbf{G}_C$  is a normal subgroup of the latter. Hence, we obtain  $g\mathbf{G}_Cg^{-1} = \mathbf{G}_C$  for  $g (\in \mathbf{G}_C)$  and for  $g (\in \sigma\mathbf{G}_C)$ . Because the group  $\mathbf{G}_{C\tilde{\sigma}} (= \mathbf{G}_C + \tilde{\sigma}\mathbf{G}_C)$  is isomorphic to  $\mathbf{G}_{C\sigma}$ , the group  $\mathbf{G}_C$  is a normal subgroup of  $\mathbf{G}_{C\tilde{\sigma}}$ . Hence, we obtain  $g\mathbf{G}_Cg^{-1} = \mathbf{G}_C$  for  $g (\in \tilde{\sigma}\mathbf{G}_C)$ . Let be  $\tilde{g} = \tilde{I}g (\in \tilde{I}\mathbf{G})$  for  $g (\in \mathbf{G}_C)$ . Then, we have  $\tilde{g}\mathbf{G}_C\tilde{g}^{-1} = \tilde{I}g\mathbf{G}_Cg^{-1}\tilde{I}^{-1} = \tilde{I}\mathbf{G}_C\tilde{I}^{-1} = \mathbf{G}_C\tilde{I}\tilde{I}^{-1} = \mathbf{G}_C$ , where we use  $\tilde{I}\mathbf{G}_C = \mathbf{G}_C\tilde{I}$ . According to eq. 4, we can summarize the results as follows:  $g\mathbf{G}_Cg^{-1} = \mathbf{G}_C$  for  $g (\in \mathbf{G}_{C\sigma\sigma\tilde{1}})$ .  $\square$

**Theorem 2** *Each of the cosets appearing in the right-hand side of eq. 4 represents a set of homomers.*

*Proof:* Since the group  $\mathbf{G}_C$  is a normal subgroup of  $\mathbf{G}_{C\sigma\bar{\sigma}I}$ , the  $\mathbf{G}_C$  is a stabilizer of each coset appearing in eq. 4, i.e.,  $\mathbf{G}_C (= \mathbf{G}_C I)$ ,  $\sigma\mathbf{G}_C$ ,  $\bar{\sigma}\mathbf{G}_C$ , or  $\bar{I}\mathbf{G}_C$ . This means that each of the four cosets appearing in eq. 4 corresponds to a homomer set which contains homomers generated by the action of the elements of the coset.  $\square$

**Def. 2** A set of homomers that corresponds to the coset  $\mathbf{G}_C (= \mathbf{G}_C I)$  is called the original (reference) molecule. A set of homomers that corresponds to the coset  $\sigma\mathbf{G}_C$  is defined as its enantiomer; a set of homomers that corresponds to the coset  $\bar{\sigma}\mathbf{G}_C$  is defined as its *RS*-diastereomer; and a set of homomers that corresponds to the coset  $\bar{I}\mathbf{G}_C$  is defined as its holantimer.

Practically speaking, anyone molecule can be selected as a representative from each set of homomers. Then, it represents the original molecule, the enantiomer, the *RS*-diastereomer, or the holantimer according to Def. 2.

In terms of Defs. 1 and 2, an achiral molecule is regarded as being self-enantiomeric; an *RS*-astereogenic molecule is regarded as being self-*RS*-diastereomeric; and an ascleral molecule is regarded as being self-holantimeric.

*Example 2:* Suppose that the four positions of an allene skeleton (Fig. 1) are permuted on the action of the *RS*-stereoisomeric group  $D_{2d\sigma\bar{\sigma}I}$  (Fig. 2). Then, the function,  $f(1) = A$ ,  $f(2) = B$ ,  $f(3) = C$ , and  $f(4) = p$ , is applied to the resulting numbered skeletons so that sixteen molecules are generated, as shown in Fig. 3.

		$D_{2d\sigma\bar{\sigma}} (= S_9^{[4]})$							
$D_{2d}$	$p^4$	$\begin{array}{c} C \\   \\ \text{---}^3_1 A \\   \\ B \\ \text{---}^2_3 \mathbf{3a} \end{array}$	$\begin{array}{c} P \\   \\ \text{---}^4_2 B \\   \\ A \\ \text{---}^1_3 \mathbf{3b} \end{array}$	$\begin{array}{c} A \\   \\ \text{---}^1_3 C \\   \\ P \\ \text{---}^2_4 \mathbf{3c} \end{array}$	$\begin{array}{c} B \\   \\ \text{---}^2_4 p \\   \\ C \\ \text{---}^3_1 \mathbf{3d} \end{array}$	$\begin{array}{c} B \\   \\ \text{---}^2_1 A \\   \\ C \\ \text{---}^3_4 \mathbf{4a} \end{array}$	$\begin{array}{c} A \\   \\ \text{---}^1_2 B \\   \\ P \\ \text{---}^4_3 \mathbf{4b} \end{array}$	$\begin{array}{c} P \\   \\ \text{---}^4_3 C \\   \\ A \\ \text{---}^1_2 \mathbf{4c} \end{array}$	$\begin{array}{c} C \\   \\ \text{---}^3_4 p \\   \\ B \\ \text{---}^2_1 \mathbf{4d} \end{array}$
		$I$	$C_{2(1)} \sim$	$C_{2(2)} \sim$	$C_{2(3)} \sim$	$\tilde{\sigma}_{d(1)} \sim$	$\tilde{S}_4 \sim$	$\tilde{S}_4^3 \sim$	$\tilde{\sigma}_{d(2)} \sim$
		$(1)(2)(3)(4)$	$(12)(34)$	$(13)(24)$	$(14)(23)$	$(1)(2)(3)(4)$	$(1243)$	$(1342)$	$(14)(2)(3)$
		$\bar{p}^4$	$\begin{array}{c} B \\   \\ \text{---}^2_1 A \\   \\ C \\ \text{---}^3_4 \mathbf{3a} \end{array}$	$\begin{array}{c} A \\   \\ \text{---}^1_2 B \\   \\ \bar{P} \\ \text{---}^4_3 \mathbf{3b} \end{array}$	$\begin{array}{c} \bar{P} \\   \\ \text{---}^4_3 C \\   \\ A \\ \text{---}^1_2 \mathbf{3c} \end{array}$	$\begin{array}{c} C \\   \\ \text{---}^3_4 \bar{p} \\   \\ B \\ \text{---}^2_1 \mathbf{3d} \end{array}$	$\begin{array}{c} C \\   \\ \text{---}^3_1 A \\   \\ B \\ \text{---}^2_4 \mathbf{4a} \end{array}$	$\begin{array}{c} \bar{P} \\   \\ \text{---}^4_2 B \\   \\ A \\ \text{---}^1_3 \mathbf{4b} \end{array}$	$\begin{array}{c} A \\   \\ \text{---}^1_3 C \\   \\ \bar{P} \\ \text{---}^4_2 \mathbf{4c} \end{array}$
		$\sigma_{d(1)} \sim$	$S_4 \sim$	$S_4^3 \sim$	$\sigma_{d(2)} \sim$	$\bar{I}$	$\bar{C}_{2(1)} \sim$	$\bar{C}_{2(2)} \sim$	$\bar{C}_{2(3)} \sim$
		$(1)(2)(3)(4)$	$(1243)$	$(1342)$	$(14)(2)(3)$	$(1)(2)(3)(4)$	$(12)(34)$	$(13)(24)$	$(14)(2)(3)$

Figure 3: Homomer Sets for ABCp under the *RS*-stereoisomeric group  $\mathbf{G}_{C\sigma\bar{\sigma}I}$ .

As found easily, (1) molecules  $\mathbf{3a}$  to  $\mathbf{3d}$  appearing the upper-left part of Fig. 3 are homomeric so as to give a homomer set that is assigned to the original molecule. This part corresponds to the coset  $\mathbf{D}_2$ . (2) Molecules  $\mathbf{4a}$  to  $\mathbf{4d}$  appearing the upper-right part of Fig. 3 are homomeric so as to a homomer set that is assigned to the *RS*-diastereomeric molecule. This part corresponds to the coset  $\tilde{\sigma}_{d(1)}\mathbf{D}_2$ . (3) Molecules  $\bar{\mathbf{3a}}$  to  $\bar{\mathbf{3d}}$  appearing the bottom-left part of Fig. 3 are homomeric so as to give a homomer set that is assigned to the enantiomeric molecule. This part corresponds to the coset  $\sigma_{d(1)}\mathbf{D}_2$ . (4) Molecules  $\bar{\mathbf{4a}}$  to  $\bar{\mathbf{4d}}$  appearing the bottom-right part of Fig. 3 are homomeric so as to give a homomer set that is assigned to the enantiomeric molecule. This part corresponds to the coset  $\bar{I}\mathbf{D}_2$ .

The four cosets correspond to different molecules. The homomer set concerning **3a** corresponds to the coset  $\mathbf{D}_2$ . This means that **3a** is chiral, *RS*-stereogenic, and scleral.

□

## 2.3 Stereoisograms of Five Types

Since stereoisograms of five types have been described in Part I [13], a minimum set of relevant items is described here. Let us select the first molecule as a representative from each part of Fig. 3: the original molecule, its *RS*-diastereomer, its enantiomer, and its holantimer, where a function,  $f(1) = A$ ,  $f(2) = B$ ,  $f(3) = C$ , and  $f(4) = p$ , is applied according to the numbering shown in Fig. 3. Thereby, we can construct a diagram, which is shown as Type III in Fig. 4. This diagram is called *stereoisogram*.

The other stereoisograms shown in Fig. 4 are obtained similarly by using the following functions:  $f(1) = A$ ,  $f(2) = B$ ,  $f(3) = C$ , and  $f(4) = D$  for Type I;  $f(1) = p$ ,  $f(2) = A$ ,  $f(3) = A$ , and  $f(4) = p$  for Type II;  $f(1) = B$ ,  $f(2) = A$ ,  $f(3) = A$ , and  $f(4) = B$  for Type IV; and  $f(1) = p$ ,  $f(2) = B$ ,  $f(3) = A$ , and  $f(4) = \bar{p}$  for Type V.

Each stereoisogram contains three distinct relationships: *RS*-diastereomeric (horizontal double-headed arrows or equality symbols), enantiomeric (vertical double-headed arrows or equality symbols), and holantimeric relationships (diagonal double-headed arrows or equality symbols).

## 2.4 Existence of Five Types of Subgroups

### 2.4.1 Factor Groups

Equations 1, 2, and 3 show that  $\mathbf{G}_C$  is a normal subgroup of each of  $\mathbf{G}_{C\sigma}$ ,  $\mathbf{G}_{C\bar{\sigma}}$ , and  $\mathbf{G}_{C\bar{I}}$ . It follows that  $\mathbf{G}_C$  is a normal subgroup of  $\mathbf{G}_{C\sigma\bar{\sigma}\bar{I}}$ , as proved in Theorem 1. Moreover, we can see  $\mathbf{G}_C\sigma = \sigma\mathbf{G}_C$  for eq. 1,  $\mathbf{G}_C\bar{\sigma} = \bar{\sigma}\mathbf{G}_C$  for eq. 2, and  $\mathbf{G}_C\bar{I} = \bar{I}\mathbf{G}_C$  for eq. 3. Hence, the four cosets of eq. 4 construct a group of cosets:

$$\mathbf{F} = \mathbf{G}_{C\sigma\bar{\sigma}\bar{I}}/\mathbf{G}_C = \{\mathbf{G}_C, \sigma\mathbf{G}_C, \bar{\sigma}\mathbf{G}_C, \bar{I}\mathbf{G}_C\} \quad (6)$$

This is a factor group of order 4. Each element (coset) of the factor group  $\mathbf{F}$  is assigned to a respective *RS*-stereoisomer as described above (Def. 2): i.e., the coset  $\mathbf{G}_C (= \mathbf{G}_C I)$  to the original molecule; the coset  $\sigma\mathbf{G}_C$  to its enantiomer; the coset  $\bar{\sigma}\mathbf{G}_C$  to its *RS*-diastereomer; and the coset  $\bar{I}\mathbf{G}_C$  to its holantimer.

**Theorem 3** *The factor group  $\mathbf{F}$  has five subgroups only as follows:*

$$\mathbf{F}_1 = \{\mathbf{G}_C\} \quad (7)$$

$$\mathbf{F}_2 = \{\mathbf{G}_C, \bar{\sigma}\mathbf{G}_C\} \quad (8)$$

$$\mathbf{F}_3 = \{\mathbf{G}_C, \bar{I}\mathbf{G}_C\} \quad (9)$$

$$\mathbf{F}_4 = \{\mathbf{G}_C, \sigma\mathbf{G}_C\} \quad (10)$$

$$\mathbf{F}_5 = \mathbf{F} = \mathbf{G}_{C\sigma\bar{\sigma}\bar{I}}/\mathbf{G}_C = \{\mathbf{G}_C, \sigma\mathbf{G}_C, \bar{\sigma}\mathbf{G}_C, \bar{I}\mathbf{G}_C\} \quad (11)$$

*Proof:* It is easy to prove that each set listed above is a group by considering the factor groups derived from eqs. 1, 2, 3, and 4. Any combination of three cosets appearing in

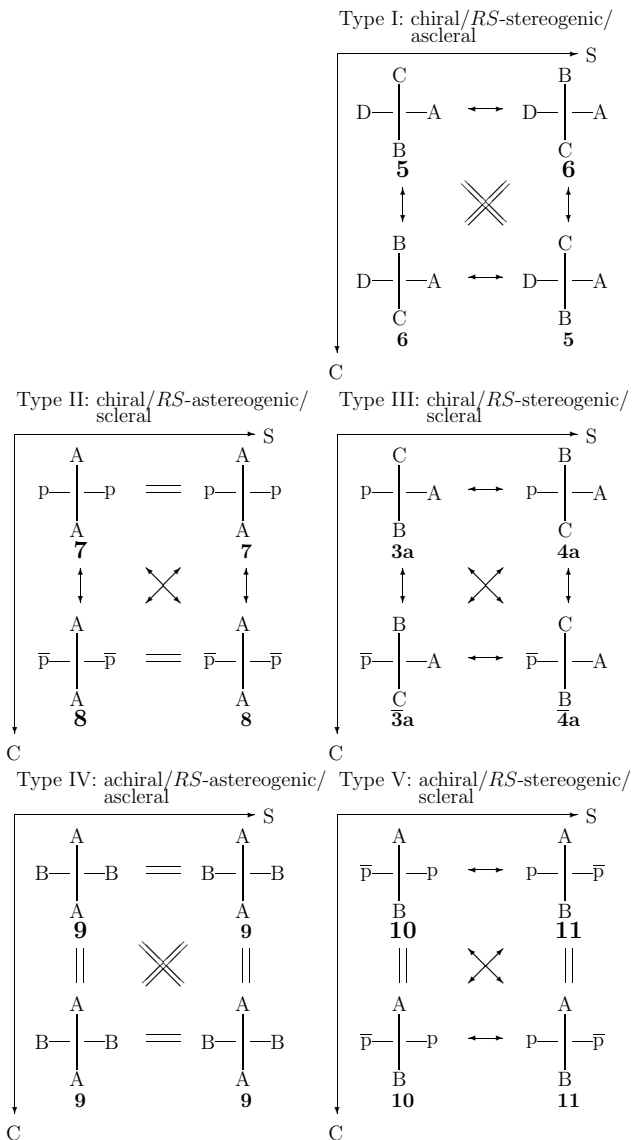


Figure 4: Stereoisograms of five types for allene derivatives.



eq. 6 does not construct a group. For example, consider  $\mathbf{F}' = \{\mathbf{G}_C, \tilde{\sigma}\mathbf{G}_C, \tilde{I}\mathbf{G}_C\}$ . Then we have  $(\tilde{\sigma}\mathbf{G}_C)(\tilde{I}\mathbf{G}_C) = \tilde{\sigma}\tilde{I}\mathbf{G}_C\mathbf{G}_C = \sigma\mathbf{G}_C \notin \mathbf{F}'$ . This means that the  $\mathbf{F}'$  is not so closed as to construct a group.  $\square$

Obviously, these subgroups correspond to the subgroups of  $\mathbf{G}_{C\sigma\tilde{\sigma}\tilde{I}}$ . Thus, we can assign  $\mathbf{F}_1$  to  $\mathbf{G}_C$ ;  $\mathbf{F}_2$  to  $\mathbf{G}_{C\tilde{\sigma}}$  (eq. 3);  $\mathbf{F}_3$  to  $\mathbf{G}_{C\tilde{I}}$  (eq. 2);  $\mathbf{F}_4$  to  $\mathbf{G}_{C\sigma}$  (eq. 1); and  $\mathbf{F}_5$  to  $\mathbf{G}_{C\sigma\tilde{\sigma}\tilde{I}}$  (eq. 4).

**Theorem 4** *Let  $\mathbf{H}_C$  be a subgroup of the  $\mathbf{G}_C$ . Then, according to the factor group  $\mathbf{F}$  shown in eq. 6, the group  $\mathbf{F}'$  defined by*

$$\mathbf{F}' = \mathbf{H}_{C\sigma\tilde{\sigma}\tilde{I}}/\mathbf{H}_C = \{\mathbf{H}_C, \sigma\mathbf{H}_C, \tilde{\sigma}\mathbf{H}_C, \tilde{I}\mathbf{H}_C\} \quad (12)$$

*is a factor group of order 4. Note that the representatives  $\sigma(\in \sigma\mathbf{G}_C)$  and  $\tilde{\sigma}(\in \tilde{\sigma}\mathbf{G}_C)$  are selected to satisfy  $\sigma\sigma \in \mathbf{H}_C$  and  $\tilde{\sigma}\tilde{\sigma} \in \mathbf{H}_C$ .*

*Proof:* Presume that  $\sigma\sigma \notin \mathbf{H}_C$  ( $\forall \sigma \in \sigma\mathbf{G}_C$ ). Since the possible number of different  $\sigma$ 's is equal to  $|\mathbf{G}_C|$ , the possible number of  $\sigma\sigma$  ( $\notin \mathbf{H}_C$  but  $\in \mathbf{G}_C$ ) is also equal to  $|\mathbf{G}_C|$ . This means that  $\mathbf{H}_C$  is a null set. The first presumption turns out false. Hence, there exists at least one  $\sigma$  satisfying  $\sigma\sigma \in \mathbf{H}_C$ . Once we have selected  $\sigma$  satisfying  $\sigma\sigma \in \mathbf{H}_C$ , we can select  $\tilde{\sigma}$  satisfying  $\tilde{\sigma} = \tilde{I}\sigma$ . Hence, we have  $\tilde{\sigma}\tilde{\sigma} = (\tilde{I}\sigma)(\tilde{I}\sigma) = \tilde{I}\tilde{I}\sigma\sigma = \sigma\sigma \in \mathbf{H}_C$ . It should be added that we can always obtain  $\tilde{I}\tilde{I} = I \in \mathbf{H}_C$ . This means that we are able to select  $\tilde{I}$  as a representative of the coset  $\tilde{I}\mathbf{H}_C$ .  $\square$

In agreement with Theorem 4, a set  $\mathbf{H}_{C\sigma\tilde{\sigma}\tilde{I}}$  is obtained as a group:

$$\mathbf{H}_{C\sigma\tilde{\sigma}\tilde{I}} = \mathbf{H}_C + \sigma\mathbf{H}_C + \tilde{\sigma}\mathbf{H}_C + \tilde{I}\mathbf{H}_C, \quad (13)$$

which is a subgroup of  $\mathbf{G}_{C\sigma\tilde{\sigma}\tilde{I}}$ . In a parallel fashion to Theorem 3, we arrive at the following theorem:

**Theorem 5** *The factor group  $\mathbf{F}'$  has five subgroups only as follows:*

$$\mathbf{F}'_1 = \{\mathbf{H}_C\} \quad (14)$$

$$\mathbf{F}'_2 = \{\mathbf{H}_C, \tilde{\sigma}\mathbf{H}_C\} \quad (15)$$

$$\mathbf{F}'_3 = \{\mathbf{H}_C, \tilde{I}\mathbf{H}_C\} \quad (16)$$

$$\mathbf{F}'_4 = \{\mathbf{H}_C, \sigma\mathbf{H}_C\} \quad (17)$$

$$\mathbf{F}'_5 = \mathbf{F}' = \mathbf{H}_{C\sigma\tilde{\sigma}\tilde{I}}/\mathbf{H}_C = \{\mathbf{H}_C, \sigma\mathbf{H}_C, \tilde{\sigma}\mathbf{H}_C, \tilde{I}\mathbf{H}_C\}. \quad (18)$$

#### 2.4.2 Subgroups of RS-Stereoisomeric Groups

The results described in Theorems 4 and 5 can be translated to specify the group-subgroup relationship between  $\mathbf{G}_{C\sigma\tilde{\sigma}\tilde{I}}$  (eq. 4) and  $\mathbf{H}_{C\sigma\tilde{\sigma}\tilde{I}}$  (eq. 13). Let  $\mathbf{H}_{C\sigma}$  be a subgroup of  $\mathbf{G}_{C\sigma}$  and satisfy the following coset decompositions:

$$\mathbf{H}_{C\sigma} = \mathbf{H}_C + \sigma\mathbf{H}_C, \quad (19)$$

where we place  $\sigma\sigma \in \mathbf{H}_C$  and the group  $\mathbf{H}_C$  is a subgroup of  $\mathbf{G}_C$ . Note that the  $\sigma$  of eq. 1 can be selected to be the same as the present representative  $\sigma$ .

Let us next construct  $\mathbf{H}_{C\tilde{I}}$  as follows:

$$\mathbf{H}_{C\tilde{I}} = \mathbf{H}_C + \tilde{I}\mathbf{H}_C \quad (20)$$

Then,  $\mathbf{H}_{C\tilde{I}}$  is a subgroup of  $\mathbf{G}_{C\tilde{I}}$ , because we have  $\tilde{I}\tilde{I} = I \in \mathbf{H}_C$  and

$$\begin{aligned} hh' &\in \mathbf{H}_C \tilde{\mathbf{I}}\mathbf{H}_C = \tilde{\mathbf{I}}\mathbf{H}_C \mathbf{H}_C = \tilde{\mathbf{I}}\mathbf{H}_C \text{ and} \\ h'h' &\in \tilde{\mathbf{I}}\mathbf{H}_C \tilde{\mathbf{I}}\mathbf{H}_C = \tilde{\mathbf{I}}\tilde{\mathbf{I}}\mathbf{H}_C \mathbf{H}_C = \mathbf{H}_C \end{aligned}$$

for  $h \in \mathbf{H}_C$  and  $h' \in \tilde{\mathbf{I}}\mathbf{H}_C$ .

By using the  $\sigma$  of eq. 19 and placing  $\tilde{\sigma} = \tilde{\mathbf{I}}\sigma = \sigma\tilde{\mathbf{I}}$ , a set  $\mathbf{H}_{C\tilde{\sigma}}$  is constructed as follows:

$$\mathbf{H}_{C\tilde{\sigma}} = \mathbf{H}_C + \tilde{\sigma}\mathbf{H}_C \quad (21)$$

Then, the set  $\mathbf{H}_{C\tilde{\sigma}}$  is a subgroup of  $\mathbf{G}_{C\tilde{\sigma}}$ , because we have

$$\begin{aligned} hh' &\in \mathbf{H}_C \tilde{\sigma}\mathbf{H}_C = \mathbf{H}_C \tilde{\mathbf{I}}\sigma\mathbf{H}_C = \tilde{\mathbf{I}}\mathbf{H}_C \mathbf{H}_C \sigma = \tilde{\mathbf{I}}\mathbf{H}_C \sigma = \tilde{\mathbf{I}}\sigma\mathbf{H}_C = \tilde{\sigma}\mathbf{H}_C \text{ and} \\ h'h' &\in \tilde{\sigma}\mathbf{H}_C \tilde{\sigma}\mathbf{H}_C = \tilde{\sigma}\tilde{\sigma}\mathbf{H}_C \mathbf{H}_C = \mathbf{H}_C \end{aligned}$$

for  $h \in \mathbf{H}_C$  and  $h' \in \tilde{\sigma}\mathbf{H}_C$ . Note that  $\tilde{\sigma}\tilde{\sigma} = \tilde{\mathbf{I}}\sigma\tilde{\mathbf{I}}\sigma = \tilde{\mathbf{I}}\tilde{\mathbf{I}}\sigma\sigma = \sigma\sigma \in \mathbf{H}_C$ .

The combination of eqs. 19, 20, and 21 produces the set  $\mathbf{H}_{C\sigma\tilde{\mathbf{I}}}$  represented by eq. 13. Because we have

$$\begin{aligned} \sigma\tilde{\sigma} &= \sigma\tilde{\mathbf{I}}\sigma = \tilde{\mathbf{I}}\sigma\sigma \in \tilde{\mathbf{I}}\mathbf{H}_C, \\ \sigma\tilde{\mathbf{I}} &= \tilde{\sigma} \in \tilde{\sigma}\mathbf{H}_C, \\ \tilde{\sigma}\tilde{\mathbf{I}} &= \sigma\tilde{\mathbf{I}}\tilde{\mathbf{I}} = \sigma \in \sigma\mathbf{H}_C, \end{aligned}$$

and so on, the set  $\mathbf{H}_{C\sigma\tilde{\mathbf{I}}}$  is concluded to be a group. This group is a subgroup of the *RS*-stereoisomeric group  $\mathbf{G}_{C\sigma\tilde{\mathbf{I}}}$  and eq. 13 represents a coset decomposition.

Equations 19, 20, and 21 show that  $\mathbf{H}_C$  is a normal subgroup of each of  $\mathbf{H}_{C\sigma}$ ,  $\mathbf{H}_{C\tilde{\sigma}}$ , and  $\mathbf{H}_{C\tilde{\mathbf{I}}}$ . It follows that  $\mathbf{H}_C$  is a normal subgroup of  $\mathbf{H}_{C\sigma\tilde{\mathbf{I}}}$  (eq. 13), which is in agreement with Theorems 4 and 5. Moreover, we can arrive at the following important theorem:

**Theorem 6** *There exist five types of subgroups in the  $\mathbf{H}_{C\sigma\tilde{\mathbf{I}}}$  (eq. 13):  $\mathbf{H}_C$  assigned to  $\mathbf{F}'_1$  of Theorem 5;  $\mathbf{H}_{C\tilde{\sigma}}$  (eq. 21) assigned to  $\mathbf{F}'_2$ ;  $\mathbf{H}_{C\tilde{\mathbf{I}}}$  (eq. 20) assigned to  $\mathbf{F}'_3$ ;  $\mathbf{H}_{C\sigma}$  (eq. 19) assigned to  $\mathbf{F}'_4$ ; and  $\mathbf{H}_{C\sigma\tilde{\mathbf{I}}}$  (eq. 13) assigned to  $\mathbf{F}'_5$ .*

## 3 Symmetries of *RS*-Stereoisomers

### 3.1 Stereoisograms and *RS*-Stereoisomeric Groups

Suppose that the  $n$ -positions of a skeleton characterized by a point group  $\mathbf{G}_{C\sigma}$  (eq. 1) accommodate a set of (pro)ligands to generate a derivative of  $\mathbf{H}_{C\sigma}$ -symmetry ( $\in \mathbf{G}_{C\sigma}$ ). The derivative is also considered to belong to  $\mathbf{H}_{C\sigma\tilde{\mathbf{I}}}$  (eq. 13), which is a subgroup of  $\mathbf{G}_{C\sigma\tilde{\mathbf{I}}}$  (eq. 4). Before we begin with a general treatment, we examine some examples of allene derivatives.

*Example 2a:* The ABCp derivative of Example 2 (Fig. 3) is an example for stereoisograms of Type III. On the other hand, this derivative belongs to  $\mathbf{C}_1$ , which is ascribed to  $\mathbf{F}'_1$  of Theorem 6. Note that the original molecule (**3a**) is different from its *RS*-diastereomer (**4a**), from its enantiomer (**3a**), and from its holantimer (**4a**).  $\square$

*Example 3:* Homomer sets for ABp<sup>2</sup> under the *RS*-stereoisomeric group  $\mathbf{D}_{2d\sigma\tilde{\mathbf{I}}}$  (a special case of  $\mathbf{G}_{C\sigma\tilde{\mathbf{I}}}$ ) are shown in Fig. 5, where the function,  $f(1) = \text{A}$ ,  $f(2) = \text{p}$ ,  $f(3) = \text{p}$ , and  $f(4) = \text{B}$ , is applied. Thereby, a stereoisogram of Type II can be obtained. See

		$D_{2\sigma} \widetilde{=} (=S_9^{[4]})$							
$D_{2d}$	$I$	$B^4 \begin{array}{c} p \\ 3 \\ 1 \\ p \end{array} A$	$p^3 \begin{array}{c} B \\ 4 \\ 2 \\ p \end{array}$	$p^2 \begin{array}{c} A \\ 1 \\ 3 \\ p \end{array}$	$A^1 \begin{array}{c} p \\ 2 \\ 4 \\ p \end{array} B$	$B^4 \begin{array}{c} p \\ 2 \\ 1 \\ p \end{array} A$	$p^3 \begin{array}{c} A \\ 1 \\ 2 \\ p \end{array}$	$p^2 \begin{array}{c} B \\ 4 \\ 3 \\ p \end{array}$	$A^1 \begin{array}{c} p \\ 3 \\ 4 \\ p \end{array} B$
	$C_{2(1)} \sim$	$(1)(2)(3)(4)$	$(1)(2)(3)4$	$(1)(3)(2)4$	$(1)(4)(2)(3)$	$(1)(2)(3)(4)$	$(1)(2)(3)4$	$(1)(3)(2)4$	$(1)(4)(2)(3)$
	$C_{2(2)} \sim$	$(1)(2)(3)(4)$	$(1)(2)(3)4$	$(1)(3)(2)4$	$(1)(4)(2)(3)$	$(1)(2)(3)(4)$	$(1)(2)(3)4$	$(1)(3)(2)4$	$(1)(4)(2)(3)$
	$\tilde{I}$	$B^4 \begin{array}{c} \bar{p} \\ 3 \\ 1 \\ \bar{p} \end{array} A$	$\bar{p}^3 \begin{array}{c} A \\ 4 \\ 2 \\ \bar{p} \end{array}$	$\bar{p}^2 \begin{array}{c} B \\ 1 \\ 3 \\ \bar{p} \end{array}$	$A^1 \begin{array}{c} \bar{p} \\ 2 \\ 4 \\ \bar{p} \end{array} B$	$B^4 \begin{array}{c} \bar{p} \\ 2 \\ 1 \\ \bar{p} \end{array} A$	$\bar{p}^3 \begin{array}{c} A \\ 1 \\ 2 \\ \bar{p} \end{array}$	$\bar{p}^2 \begin{array}{c} B \\ 4 \\ 3 \\ \bar{p} \end{array}$	$A^1 \begin{array}{c} \bar{p} \\ 3 \\ 4 \\ \bar{p} \end{array} B$
	$\tilde{C}_{2(1)} \sim$	$(1)(2)(3)(4)$	$(1)(2)(3)4$	$(1)(3)(2)4$	$(1)(4)(2)(3)$	$(1)(2)(3)(4)$	$(1)(2)(3)4$	$(1)(3)(2)4$	$(1)(4)(2)(3)$
	$\tilde{C}_{2(2)} \sim$	$(1)(2)(3)(4)$	$(1)(2)(3)4$	$(1)(3)(2)4$	$(1)(4)(2)(3)$	$(1)(2)(3)(4)$	$(1)(2)(3)4$	$(1)(3)(2)4$	$(1)(4)(2)(3)$
	$\tilde{C}_{2(3)} \sim$	$(1)(2)(3)(4)$	$(1)(2)(3)4$	$(1)(3)(2)4$	$(1)(4)(2)(3)$	$(1)(2)(3)(4)$	$(1)(2)(3)4$	$(1)(3)(2)4$	$(1)(4)(2)(3)$

Figure 5: Homomer Sets for  $ABp^2$  under the  $RS$ -stereoisomeric group  $G_{C\sigma\sigma\tilde{I}}$ .

eq. 3 and  $F_2$  (eq. 8) for Type II. The same conclusion can be obtained by considering the subgroup  $C_{\sigma\sigma\tilde{I}} = \{I, \sigma_{d(1)}, \tilde{I}\}$  as a special case of  $H_{C\sigma\sigma\tilde{I}}$  (eq. 13). See eq. 21 and  $F'_2$  (eq. 15) for Type II.

By the inspection of Fig. 5, the molecule **12e** is identical with the original molecule **12a**. In other words, the permutation (1)(2)(3)(4) fixes the molecule **12a**. It follows that **12a** belongs to the group represented by

$$C_{1\tilde{\sigma}} = \{I, \tilde{\sigma}_{d(1)}\} = \{(1)(2)(3)(4), (1)(2)(3)(4)\}. \quad (22)$$

This is a special case represented by  $H_{C\tilde{\sigma}}$  (eq. 21), where  $H_C = C_1 = \{I\}$ . The corresponding factor group is represented by eq. 15, which shows  $F'_2$  for Type II. It should be noted that the point group of the molecule **12a** is  $C_1$ .  $\square$

By examining the stereoisograms of Fig. 4 by virtue of the subgroups shown in Theorem 3 (eqs. 7 to 11) and Theorem 5 (eqs. 14 to 18), we obtain the following correspondence:  $F_1$  or  $F'_1$  to Type III;  $F_2$  or  $F'_2$  to Type II;  $F_3$  or  $F'_3$  to Type I;  $F_4$  or  $F'_4$  to Type V; and  $F_5$  or  $F'_5$  to Type IV. Thus, we have proved the existence of five types of stereoisograms:

**Theorem 7** *There exist five types of stereoisograms only, as exemplified in Fig. 4,*

Theorem 7 is summarized in Table 1 together with five types of factor groups, where relevant subgroups are categorized in terms of chirality/achirality,  $RS$ -stereogenicity/ $RS$ -astereogenicity, and sclerality/asclerality.

It should be noted that ascleral cases correspond to Types I and IV, as found in Table 1 (the symbol  $\times$  in the ‘‘scleral’’ column). By the inspection of Fig. 4, the Types I and IV contain achiral (pro)ligands only. If we focus our attention on the Types I and IV, we can find by the data of Table 1 that chirality and  $RS$ -stereogenicity coincide with each other. Thus, the traditional stereochemistry implicitly omits or pays subsidiary attention to cases other than Type I and IV, although Table 1 teaches the further existence of Types II, III, and V. The implicit omission is concluded to be a reason for the previous confusion concerning chirality and stereogenicity [4, 5].

In previous discussions described in standard textbooks [4, 5], stereogenicity has been recognized to connote chirality, as stereogenic centers etc. involve chirality centers etc.

Table 1: Stereogenicity Types and Factor Group Types

stereogenicity type	factor group type	(sub)group type	properties of subgroups		
			chiral	<i>RS</i> -stereogenic	scleral
I	$\mathbf{F}_3$ or $\mathbf{F}'_3$	$\mathbf{G}_{C\tilde{I}}$ or $\mathbf{H}_{C\tilde{I}}$	○	○	×
II	$\mathbf{F}_2$ or $\mathbf{F}'_2$	$\mathbf{G}_{C\tilde{\sigma}}$ or $\mathbf{H}_{C\tilde{\sigma}}$	○	×	○
III	$\mathbf{F}_1$ or $\mathbf{F}'_1$	$\mathbf{G}_C$ or $\mathbf{H}_C$	○	○	○
IV	$\mathbf{F}_5$ or $\mathbf{F}'_5$	$\mathbf{G}_{C\sigma\tilde{\sigma}I}$ or $\mathbf{H}_{C\sigma\tilde{\sigma}I}$	×	×	×
V	$\mathbf{F}_4$ or $\mathbf{F}'_4$	$\mathbf{G}_{C\sigma}$ or $\mathbf{H}_{C\sigma}$	×	○	○

This holds true only if *RS*-stereogenic cases (Types I, III, and V) have been taken into consideration and the other cases (Types II and IV) are disregarded. On the other hand, Table 1 indicates that *RS*-stereogenicity and chirality are distinct from each other. Thus, Type II indicates the presence of chiral but *RS*-astereogenic cases.

### 3.2 Minimum Subgroup for Type IV

If the  $\sigma$  and the  $\tilde{\sigma}$  can be selected to satisfy  $\sigma\sigma = I$  and  $\tilde{\sigma}\tilde{\sigma} = I$ , the following relationships are obtained:

$$\begin{aligned}\tilde{\sigma}\sigma &= \tilde{I}\sigma\sigma = \tilde{I}I = \tilde{I}; & \sigma\tilde{\sigma} &= \sigma\tilde{I}\sigma = \tilde{I}\sigma\sigma = \tilde{I}I = \tilde{I}; \\ \tilde{I}\tilde{\sigma} &= \tilde{I}\tilde{I}\sigma = \sigma; & \tilde{\sigma}\tilde{I} &= \sigma\tilde{I}\tilde{I} = \sigma;\end{aligned}$$

This means that the transversal appearing in eq. 4 constructs a group  $\mathbf{H}_0$  as follows:

$$\mathbf{H}_0 = \mathbf{C}_{\sigma\tilde{\sigma}I} \approx \{I, \sigma, \tilde{\sigma}, \tilde{I}\} \quad (23)$$

Obviously,  $\mathbf{H}_0$  is a subgroup of  $\mathbf{G}_{C\sigma\tilde{\sigma}I}$ . The subgroup  $\mathbf{H}_0$  has only five subgroups represented by

$$\mathbf{H}_{01} = \mathbf{C}_1 = \{I\} \quad (24)$$

$$\mathbf{H}_{02} = \mathbf{C}_{1\tilde{\sigma}} = \{I, \tilde{\sigma}\} \quad (25)$$

$$\mathbf{H}_{03} = \mathbf{C}_{1\tilde{I}} = \{I, \tilde{I}\} \quad (26)$$

$$\mathbf{H}_{04} = \mathbf{C}_s = \{I, \sigma\} \quad (27)$$

$$\mathbf{H}_{05} = \mathbf{H}_0 = \mathbf{C}_{\sigma\tilde{\sigma}I} \approx \{I, \sigma, \tilde{\sigma}, \tilde{I}\} \quad (28)$$

*Example 4:* Homomer sets for  $A^2BC$  under the *RS*-stereoisomeric group  $\mathbf{D}_{2d\sigma\tilde{\sigma}I}$  (a special case of  $\mathbf{G}_{C\sigma\tilde{\sigma}I}$ ) are shown in Fig. 6, where the function,  $f(1) = B$ ,  $f(2) = A$ ,  $f(3) = A$ , and  $f(4) = C$ , is applied. Thereby, a stereoisogram of Type IV can be obtained. See eq. 4 and  $\mathbf{F}_5$  (eq. 10) for Type IV. The same conclusion can be obtained by considering the subgroup  $\mathbf{C}_{\sigma\tilde{\sigma}I}$  (eq. 23), which is in turn regarded as a special case of  $\mathbf{H}_{C\sigma\tilde{\sigma}I}$  (eq. 13). See also eq. 13 and  $\mathbf{F}'_5$  (eq. 18) for Type IV.

By the inspection of Fig. 6, the molecules **13e**, **13i**, and **13m** are identical with the original molecule **13a**. In other words, the permutations,  $(1)(2\ 3)(4)$ ,  $(\bar{1})(2\ 3)(4)$ , and  $(\bar{1})(2)(3)(4)$ , fix the molecule **13a** invariant. It follows that **13a** belongs to the group  $\mathbf{C}_{\sigma\tilde{\sigma}I}$  (eq. 23), which corresponds to Type IV. It should be noted that the point group of the molecule **13a** is  $\mathbf{C}_s$ .  $\square$

		$D_{2\sigma}^- (=S_9^{[4]})$							
$D_{2d}$		$C^4 \begin{array}{c} A \\   \\ 3 \\   \\ B \\   \\ A \end{array} \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array}$	$A^3 \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array}$	$A^2 \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array}$	$B^1 \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array}$	$C^4 \begin{array}{c} A \\   \\ 2 \\   \\ B \\   \\ A \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A^3 \begin{array}{c} B \\   \\ 2 \\   \\ A \\   \\ C \end{array} \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array} \begin{array}{c} A^2 \begin{array}{c} C \\   \\ 3 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array}$	$A^3 \begin{array}{c} B \\   \\ 2 \\   \\ A \\   \\ C \end{array} \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array} \begin{array}{c} A^2 \begin{array}{c} C \\   \\ 3 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array}$	$A^2 \begin{array}{c} C \\   \\ 3 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array}$	$B^1 \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array}$
		$I$	$C_{2(1)} \sim$	$C_{2(2)} \sim$	$C_{2(3)} \sim$	$\tilde{\sigma}_{d(1)} \sim$	$\tilde{S}_4 \sim$	$\tilde{S}_3^3 \sim$	$\tilde{\sigma}_{d(2)} \sim$
		$(1)(2)(3)(4)$	$(1\ 2)(3\ 4)$	$(1\ 3)(2\ 4)$	$(1\ 4)(2\ 3)$	$(1)(2\ 3)(4)$	$(1\ 2\ 4\ 3)$	$(1\ 3\ 4\ 2)$	$(1\ 4)(2)(3)$
	$C^4 \begin{array}{c} A \\   \\ 2 \\   \\ B \\   \\ A \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A^3 \begin{array}{c} B \\   \\ 2 \\   \\ A \\   \\ C \end{array} \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array} \begin{array}{c} A^2 \begin{array}{c} C \\   \\ 3 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array}$	$A^3 \begin{array}{c} B \\   \\ 2 \\   \\ A \\   \\ C \end{array} \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array} \begin{array}{c} A^2 \begin{array}{c} C \\   \\ 3 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array}$	$A^2 \begin{array}{c} C \\   \\ 3 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array}$	$B^1 \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array}$	$C^4 \begin{array}{c} A \\   \\ 2 \\   \\ B \\   \\ A \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A^3 \begin{array}{c} B \\   \\ 2 \\   \\ A \\   \\ C \end{array} \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array} \begin{array}{c} A^2 \begin{array}{c} C \\   \\ 3 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array}$	$A^3 \begin{array}{c} B \\   \\ 2 \\   \\ A \\   \\ C \end{array} \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array} \begin{array}{c} A^2 \begin{array}{c} C \\   \\ 3 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array}$	$A^2 \begin{array}{c} C \\   \\ 3 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array}$	$B^1 \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array} \begin{array}{c} C \\   \\ 2 \\   \\ A \\   \\ B \end{array} \begin{array}{c} B \\   \\ 1 \\   \\ A \\   \\ C \end{array} \begin{array}{c} A \\   \\ 2 \\   \\ C \\   \\ A \end{array}$	
	$\sigma_{d(1)} \sim$	$S_4 \sim$	$S_3^3 \sim$	$\sigma_{d(2)} \sim$	$\tilde{I}$	$\tilde{C}_{2(1)} \sim$	$\tilde{C}_{2(2)} \sim$	$\tilde{C}_{2(3)} \sim$	
	$(1)(2)(3)(4)$	$(1\ 2\ 4\ 3)$	$(1\ 3\ 4\ 2)$	$(1\ 4)(2)(3)$	$(1)(2)(3)(4)$	$(1\ 2)(3\ 4)$	$(1\ 3)(2\ 4)$	$(1\ 4)(2\ 3)$	

Figure 6: Homomer Sets for  $A^2BC$  under the  $RS$ -stereoisomeric group  $\mathbf{G}_{C\sigma\tilde{\sigma}I}$ .

## 4 Skeletons

If a reference molecule derived from a skeleton with at least one ligand pattern produces its holantimer that is different from the reference molecule, the skeleton is capable of generating molecules of Types I to V. Such a skeleton is called *RS-distinguishable skeleton*, where  $RS$ -stereogenic molecules (Types I, III, and V) as well as  $RS$ -astereogenic molecules (Types II and IV) can be generated. For example, the allene skeleton discussed as an example belongs to this category of skeletons. According to the present approach, the  $RS$ -stereoisomeric group  $\mathbf{G}_{C\sigma\tilde{\sigma}I}$  (eq. 4) contains three normal subgroups that are different from each other, i.e.,  $\mathbf{G}_{C\sigma}$  (eq. 1);  $\mathbf{G}_{C\tilde{I}}$  (eq. 2); and  $\mathbf{G}_{C\tilde{\sigma}}$  (eq. 3). As for the elements  $\sigma$  (eq. 1) and  $\tilde{I}$  (eq. 2), this case is realized by placing  $\sigma \neq \tilde{I}$  ( $\forall \sigma \in \sigma\mathbf{G}_C$ ).

On the other hand, if a skeleton generates a reference molecule and its holantimer that always coincide with each other for any ligand pattern, the skeleton is capable of generating molecules of Types II and IV but incapable of generating molecules of Type I, III, and V. Such a skeleton is called *RS-indistinguishable skeleton*, where  $RS$ -astereogenic molecules (Types II and IV) can be generated, while  $RS$ -stereogenic molecules (Types I, III, and V) cannot be generated intrinsically. For example, an ethylene skeleton belongs to this category of skeletons. As for the elements  $\sigma$  (eq. 1) and  $\tilde{I}$  (eq. 2), this case is realized by placing  $\sigma = \tilde{I}$  ( $\exists \sigma \in \sigma\mathbf{G}_C$ ). Thus, the point group  $\mathbf{G}_{C\sigma}$  (eq. 1) and the inversion group  $\mathbf{G}_{C\tilde{I}}$  (eq. 2) coincide with each other. In other words, the corresponding  $RS$ -stereoisomeric group  $\mathbf{G}_{C\sigma\tilde{\sigma}I}$  (eq. 4) becomes degenerate to produce the point group  $\mathbf{G}_{C\sigma}$  (eq. 1).

In both of the two cases,  $RS$ -stereoisomeric groups are keys for comprehending the hierarchy of groups that specifies the stereochemistry and stereoisomerism of molecules derived from a given skeleton. Such a molecule belongs to an  $RS$ -stereoisomeric group that is a subgroup of the  $RS$ -stereoisomeric group of the skeleton. The  $RS$ -stereoisomeric group assigned to the molecule, which is classified into Type I, II, III, IV, or V, contains the corresponding point group as a normal subgroup.

## 5 Conclusion

The existence of five stereoisograms or of five stereogenicity types is proved generally on the basis of the five factor groups derived from *RS*-stereoisomeric groups. Moreover, the concepts of holantimers and *RS*-diastereomers, which have once been proposed in an intuitive fashion, gain a sound mathematical basis. As a result, the concepts formulated in the present series of papers (Parts 1 to 3) provide us with a succinct logical framework for restructuring stereochemistry.

## References

- [1] Prelog, V. *Science* **1976**, *193*, 17–24.
- [2] Prelog, V.; Helmchen, G. *Angew. Chem. Int. Ed. Eng.* **1982**, *21*, 567–583.
- [3] Mislow, K.; Siegel, J. *J. Am. Chem. Soc.* **1984**, *106*, 3319–3328.
- [4] Eliel, E.; Wilen, S. H. *Stereochemistry of Organic Compounds*; John Wiley & Sons: New York, 1994.
- [5] Buxton, S. R.; Roberts, S. M. *Guide to Organic Stereochemistry*; Addison Wesley Longman: New York, 1996.
- [6] Fujita, S. *J. Am. Chem. Soc.* **1990**, *112*, 3390–3397.
- [7] Fujita, S. *J. Org. Chem.* **2002**, *67*, 6055–6063.
- [8] Fujita, S. *Symmetry and Combinatorial Enumeration in Chemistry*; Springer-Verlag: Berlin-Heidelberg, 1991.
- [9] Fujita, S. *Bull. Chem. Soc. Jpn.* **2001**, *74*, 1585–1603.
- [10] Fujita, S. *J. Math. Chem.* **2003**, *33*, 113–143.
- [11] Fujita, S. *J. Math. Chem.* **2004**, *35*, 261–283.
- [12] Fujita, S. *J. Org. Chem.* **2004**, *69*, 3158–3165.
- [13] Fujita, S. *MATCH Commun. Math. Comput. Chem.* **2004**, *52*, 3–18.
- [14] Fujita, S. *MATCH Commun. Math. Comput. Chem.* **2005**, *53*, 147–159.
- [15] Fujita, S. *Tetrahedron* **1991**, *47*, 31–46.