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Construction and Decomposition of Planar Two-Cycle Resonant Graphs *

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Abstract

A connected graph G is said to be k-cycle resonant if, for $1 \leq t \leq k$, any t disjoint cycles in G are mutually resonant; that is there is a perfect matching M of G such that each of the t cycles is M-alternating cycle. Some necessary and sufficient conditions for a graph to be k-cycle resonant were given by Xiaofeng Guo and Fuji Zhang, and they also established some necessary and sufficient conditions for a planar graph to be 1-cycle resonant and 2-cycle resonant. In this paper, we give a method for constructing any planar 2-cycle resonant graph from smaller 2-cycle resonant graphs and simple planar 1-cycle resonant graphs.

1 Introduction.

The concept of k-cycle resonant graphs was introduced by Xiaofeng Guo and Fuji Zhang [1]. It was motivated by k-resonant (or k-coverable) hexagonal systems. In the topological theory of benzenoid hydrocarbons, a *hexagonal system* (or *benenoid system*)

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denotes the carbon atom skeleton graph of a benzenoid hydrocarbon, which is a 2-connected plane graph whose every interior face is bounded by a regular hexagon. A *Kekulé structure* K of a hexagonal system H is also a perfect matching of H. A hexagonal system is said to be *normal* if each of its edges lies in a Kekule structure K of it. A cycle (or circuit) C in H is said to be *conjugated* or *resonant* if there is a Kekulé structure K of H such that C is a K-alternating cycle. In the conjugated circuit model [2-25], conjugated circuit with different sizes have different resonant energies. On the other hand, from a purely standpoint, Clar found that various electronic properties of polycyclic aromatic hydrocarbons can be predicted by appropriately defining an aromatic sextet for their Kekulé structure [26-36]. According to Clar's aromatic sextet theory, the Clar formula of a hexagons is a set of mutually resonant sextets with the maximum cardinal number, where sextets mean resonant hexagons and a set of mutually resonant sextets means a set of disjoint hexagons for which there is a Kekulé structure K so that all of the the disjoint hexagons are K-alternating hexagons.

For a hexagonal system H with Clar number c (the number of sextets), Clar formula of H may be not unique, and, for $1 \leq k \leq c$, any k disjoint hexagons of H are not certainly mutually resonant. An interesting problem is that under how conditions any k disjoint hexagons of a hexagonal system H are mutually resonant? If a hexagonal system H satisfies such property, that is, for a positive integer k and $1 \leq t \leq k$, any disjoint hexagons of H are mutually resonant, it is said to be *k-resonant* or *k-coverable*.

The concept of *cover* was originally introduced by Gutman [37]. 1-coverable hexagonal systems were first investigated by Fuji zhang and Rongsi Chen [38]. Some necessary and sufficient conditions for a hexagonal systems to be *1-resonant* were established. Maolin Zheng [39] extended the concept of 1-coverable hexagonal systems to k-coverable hexagonal systems and some fair necessary and sufficient conditions for a hexagonal systems to be *k-resonant* were given.

As a generalization of k-coverable hexagonal systems, Guo Xiaofeng and Fuji Zhang [1] introduced *k-cycle resonant* graphs. Some properties of k-cycle resonant graphs and some necessary and sufficient conditions for a graph to be k-cycle resonant were given. Recently, Guo Xiaofeng and Fuji Zhang [40] further investigated general planar k-cycle resonant graphs with $k = 1, 2$. Some new necessary and sufficient conditions for a graph to be planar 1-cycle resonant graphs or planar 2-cycle resonant graphs were given. Zhixia Xu and Xiaofeng Guo [41] investigated the construction and the recognition of planar 1-cycle resonant graphs.

In the present paper, we investigate the construction of planar 2-cycle resonant graphs, in the light of the necessary and sufficient conditions for a graph to be 2-cycle resonant graphs. A recursive method for constructing any planar 2-cycle resonant graph from smaller planar 2-cycle resonant graphs and simple planar 1-cycle resonant graphs is given.

In this paper, for the basic terminology, we refer to the volumes Bondy and Murty [42] and L. Lovász and M. D. Plummer [43].

2 Some related results of k-cycle resonant graphs

Definition: A connected graph is said to be k-cycle resonant if, for $1 \leq t \leq k$, any t disjoint cycles in G are mutually resonant, that is, there is a perfect matching M of G such that each of the t cycles is an M-alternating cycle.

From the definition of k-cycle resonant graph, we can see that if G is a k-cycle resonant graph, then G is also (k-1)-cycle resonant.

Theorem A [40]. A 2-connected graph with at least k disjoint cycles is k-cycle resonant if and only if G is a bipartite and, for $1 \leq t \leq k$ and any t disjoint cycles $C_1, C_2, \cdots, C_t$ in G, $G - \bigcup_{i=1}^{t} V(C_i)$ contains no odd component. ■

The above theorem is a revision of Theorem 3.1 [1], in which the condition "2-connected" is neglected, however, the condition is implicitly used in the proof of Theorem 3.1 [1].

Theorem A [40]. A connected graph with at least k disjoint cycles is k-cycle resonant if and only if G is a bipartite graph with perfect matchings and, for $1 \leq t \leq k$ and any t disjoint cycles $C_1, C_2, \cdots, C_t$ in G, $G - \bigcup_{i=1}^{t} V(C_i)$ contains no odd component. ■

A *block* of a connected graph G is either a maximal 2-connected subgraph of G or a cut edge of G.

Theorem B [40]. Let G be a k-cycle resonant graph, then

(i)for a 2-connected block G' of G with the maximum number k^* of disjoint cycles, if $k^* < k$, G' is k^*-cycle resonant, otherwise G' is k-cycle resonant;

(ii) the forest induced by all the vertices of G not in any 2-connected block of G has a unique perfect matching. ■

From the theorem B, we know that a non-2-connected k-cycle resonant graph can be constructed from some disjoint 2-connected k-cycle (or k^*-cycle if $k^* < k$, where k^* is the maximum number of disjoint cycles) resonant graphs and a forest with perfect matching by adding some edges between the 2-connected graphs and the forest so that the resultant graph is connected and the added edges are cut edges. Hence we need only to consider 2-connected k-cycle resonant graphs.

Let G be a connected graph, and H a subgraph of G. A vertex in H is said to be an attachment vertex of H if it is incident with an edge in $G - E(H)$. A bridge B of H in

G is either an edge in $G - E(H)$ with two end vertices being in H, or a subgraph of G induced by all the edges in a connected component B' of $G - V(H)$ together with all the edges with an end vertex in B' and the other in H. The vertices in $V(B) \cap V(H)$ are also attachment vertices of B to H. A bridge with k attachment vertices is called a k-bridge.

The attachment vertices of a k-bridge B of a cycle C in G divide C into k edge-disjoint paths, called the *segments* of B. Two bridges of C avoid one another if all the attachment vertices of one bridge lie in a single segment of the other bridge, otherwise they overlap.

For a bipartite graph, we always color its vertices black and white so that adjacent vertices have different colors.

Additional necessary and sufficient conditions for a graph to be planar 1- and 2-cycle resonant was given by Xiaofeng Guo and Fuji Zhang [40]:

Theorem C. [40] A 2-connected graph G is planar 1-cycle resonant if and only if G is bipartite and, for any cycle C in G, any bridge of C has exactly two attachment vertices which have different colors. ■

Theorem D. [40] A 2-connected graph G is planar 1-cycle resonant if and only if G is bipartite and, for any cycle C in G, any two bridges of C avoid one another and, for any 2-connected subgraph B of G with exactly two attachment vertices, the attachment vertices of B have different colors. ■

On the basis of these necessary and sufficient conditions, Zhixia Xu and Xiaofeng Guo [41] given a modification of that as the following.

A planar embedding of a planar graph is called a *plane graph*. We call the boundary of the exterior face of a 2-connected plane graph G the *outer cycle* of G.

Theorem E. [41] Let G be a 2-connected plane bipartite graph, then G is 1-cycle resonant if and only if any bridge of the outer cycle C of G has exactly two different colored attachment vertices and for any maximal 2-connected subgraph H of any bridge B of C, the following conditions are satisfied:

(1) H is 1-cycle resonant;

(2) H has exactly two different-colored attachment vertices u and v;

(3) u and v avoid any bridge of the outer cycle of H. ■

According to Theorem E, Zhixia Xu and Xiaofeng Guo [41] provided a recursive method for constructing any planar 1-cycle resonant graph from smaller planar 1-cycle resonant graphs. Furthermore, a linear algorithm for determining a plane graph to be 1-cycle resonant was established.

Following, we shall give some terminology and notions relating to 2-cycle resonant graphs.

For a 2-connected subgraph B in G with exactly two attachment vertices, we call $G[E(G) - E(B)]$ the complement of B in G, denoted by $\overline{B}$.

A path P in a graph G is said to be a chain if all internal vertex of P are of degree 2 in G and the degree of any end vertex of P is not equal to 2 in G. The set of internal vertices of a chain P in G is denoted by $V_I(P)$.

A vertex u of a graph G is said to be *cycle-related* to another vertex v of G, denoted by $u \Rightarrow v$, if u is contained in a 2-connected block of G and any cycle containing u must also contain v. If v is also cycle-related to u, then u and v are *mutually cycle-related*, denoted by $u \Leftrightarrow v$.

Theorem F. [40] A 2-connected graph G is planar 2-cycle resonant if and only if G is planar 1-cycle resonant, and

(i) for a chain P with even length and end vertices v_1 and v_2, $G - V_I(P)$ has exactly two blocks each of which is 2-connected, and v_1 and v_2 are cycle-related to the common vertex w of the two blocks.

(ii) for a chain P with odd length and end vertices v_1 and v_2 such that $G - V_I(P)$ is not 2-connected, either (a) $G - V_I(P)$ has exactly three blocks, each of which is a 2-connected, and each of v_1 and v_2 is cycle-related to the other attachment vertex of the block containing it, and the attachment vertices of the third block are mutually cycle-related in the third block, or (b) any two 2-connected blocks of $G - V_I(P)$ are disjoint,

(iii) for a 2-connected subgraph B_1 of G with exactly two attachment vertices, if B_1 is not 2-connected and every block of B_1 is 2-connected, then $\overline{B_1}$ has exactly three blocks, say B_2, B_3, B_4, and the attachment vertices of each of B_1, B_2, B_3, B_4 are mutually cycle-related in the block. ■

We also make use of the following results in this paper.

Lemma G. [41]: Suppose G_1 and G_2 are two 2-connected planar 1-cycle resonant graphs. Let u and v be two differently colored vertices on a chain P of G and P_1 be the path on P between u and v. Assume that u' and v' are differently colored vertices on a cycle C_0 of G_2 that avoid any bridge of C_0. If we replace P_1 with G_1 and identify u and u', and v and v', then the resultant graph is 1-cycle resonant. ■

Let G be a graph, u and v are two distinct vertices in G. Let $P^\bullet$ be a path with end vertices u and v, and$V(G) \cap V(P^\bullet) = \{u, v\}$. Let $(G + P^\bullet)_{(u,v)}$ denote the graph $G \cup P$.

Theorem H. [44] Let G be a 2-connected planar 1-cycle resonant graph, and $P^\bullet$ a path disjoint from G. Then $(G + P^\bullet)_{(u,v)}$ is planar1-cycle resonant if and only if (i) $P^\bullet$ is of odd length, (ii) u and v have different colors in G, (iii) either u and v are adjacent in G or $\{u, v\}$ is a vertex cut of G. ■

Corollary I. [40] Let G be a 2-connected planar 1-cycle resonant graph, C a cycle of G, and B a bridge of C. Then (i) B is not 2-connected, (ii) every block of B has exactly two attachment vertices, and the block graph of B is a path with odd length. ■

3 Fundamental Theorem.

We first give a useful Theorem.

Theorem 1. Let G be a 2-connected planar 1-cycle resonant graph, B a 2-connected subgraph of G with exactly two differently colored attachment vertices u and v, then

(1) B is 1-cycle resonant.

(2) u and v avoid any bridge of a cycle in B containing u and v.

Proof: (1) Let C be any cycle in B. By Theorem C, in order to prove that B is 1-cycle resonant, we need only to show that any bridge of C in B has exactly two attachment vertices which have different colors. we distinguish two cases as following:

Case 1: Suppose that C contains both u and v. Then any bridge of C in B is also a bridge of C in G. So every bridge of C in B has precisely two differently colored attachment vertices.

Case 2: Suppose that C contains at most one of u and v, say u. Let B_1, $B_2, \cdots, B_r$ be all bridges of C in G. Since B is connected, so u, v, B are contained in a same bridge of C in G, say B_1. Then in B there is a $u-v$ path whose internal vertices are not on C. Otherwise, the bridge of C in B containing v have at least two attachment vertices other than u since B is 2-connected. However, the bridge of C in G containing v and B would have at least three attachment vertices, contradicting that G is planar 1-cycle resonant. Hence $B_1 \cap B$ is a bridge of C in B, which has the same attachment vertices with the bridge of C in G and so has exactly two differently colored attachment vertices. The other bridges $B_2, \cdots, B_r$ of C in B are also bridges of C in G, and so has precisely two differently colored attachment vertices.

(2) Let C be a cycle in B containing u and v. Then any bridge of C in B is also a bridge of C in G. Meantime, B contains a bridge of C in G whose two attachment vertices is exactly u and v. By Theorem D, u and v avoid any bridge of C in B. □

Corollary 1. Let G be a 2-connected planar 1-cycle resonant graph, B a 2-connected subgraph of G with exactly two differently colored attachment vertices u and v, if a cycle C in B contains neither u and v, then there exists a path in $B - V(C)$ with end vertices u and v.

Proof: Let C be a cycle C in B contains neither u and v. From the Case 2 in proof of Theorem 1, we know that u and v are in a same bridge of C in B, so there is path in the bridge of C in B with end vertices u and v. □

Let G be a 1-cycle resonant graph. If G has no disjoint cycles, we call it *simple* 1-cycle resonant graph. Obviously, 1-cycle resonant graphs with the cyclomatic number $\nu(G) = 1, 2$ are simple 1-cycle resonant graph.

Theorem 2. A 2-connected planar graph G is simple 1-cycle resonant if and only if G is bipartite, and in the vertices with degree greater than 2 there is a vertex v such that $G - v$ is a tree and the color of v is different from all the other vertices with degree greater than 2.

Proof. Necessity.

Let G is a simple planar 1-cycle resonant graph. By Theorem C, for any cycle C of G, any bridge of C has exactly two attachment vertices which have different colors.

Assume that, for any vertex w with $deg_G(w) > 2$, $G - w$ is not a tree. Then there is a cycle C_1 in $G - w$ since G is 2-connected. Let B be the bridge of C_1 in G containing w whose attachment vertices are u_1 and v_1. Then w is contained in a 2-connected block B_1 of B since $deg_G(w) > 2$. B_1 has exactly two attachment vertices, one of which must be on C_1 (that is, it is one of u_1 and v_1, say u_1), and the other is not on C_1 (since B is not 2-connected by Corollary I (i) [40]). Otherwise, in B there is a cycle C_2 disjoint with C_1, contradicting that G is simple planar 1-cycle resonant. By the same reason, $B_1 - u_1$ has no cycle. Then $B - u_1$ has no cycle too. Otherwise let C' be such a cycle. If either C' does not contain w or C' contains w and does not contain v_1, then it is disjoint with either a cycle in B_1 or C_1, a contradiction. If C' contains both w and v_1, there would be a 2-connected block in B with the attachment vertices w and v_1 which the same color since w and v_1 have different color from u_1. This also contradicts Theorem D. However, $G - u_1$ is not a tree too, so it has a cycle, say C_2, disjoint with B_1. Let C_3 be a cycle in B_1. Then C_2 and C_3 are disjoint cycles in G, again contradicting that G is simple 1-cycle resonant.

Hence there is a vertex v with $deg_G(v) > 2$ such that $G - v$ is a tree.

Assume that in the vertices with degree greater than 2, there is a vertex u which has the same color with v. Since $deg_G(u) > 2$ and $deg_G(v) > 2$, there are three $u - v$ paths P_1, P_2, P_3 which are internal disjoint. Let C be the cycle consisting of $P_1 \bigcup P_2$. Then bridge of C containing P_3 has u and v as its attachment vertices which have the same color. This also contradicts that G is planar 1-cycle resonant.

Sufficiency.

Since $G - v$ a tree, any cycle in G must contain v. So G has no disjoint cycles.

Let C be any cycle in G, and B a bridge of C. Then B has exactly two attachment vertices one of which must be v. Otherwise, there would be a cycle in G not containing v, contradicting that $G-v$ is a tree. By the condition of the Theorem, the two attachment vertices of B have different colors. Now it follows from Theorem C that G is simple planar 1-cycle resonant. □

The above Theorem give a complete characterization of simple 1-cycle resonant graphs. To recognize whether or not a 2-connected graph G to be simple planar 1-cycle resonant, we need only to check the vertices of degree greater than 2 to see that whether or not there is a vertex v having different color with all the other vertices of degree greater than 2, and whether or not $G-v$ is a tree. Conversely, we can construct any simple planar 1-cycle resonant graph from a tree T with the vertices of degree not equal to 2 having the same color by adding a new vertex v and the edges connecting v to every vertex of degree 1 in T.

From Theorem 1 (i), we know that any 2-connected subgraph of a planar 1-cycle resonant graph G with exactly two attachment vertices is also 1-cycle resonant. For a planar 2-cycle resonant graph G, a 2-connected subgraph of G with exactly two attachment vertices is not certainly 2-cycle resonant. However, we have the following theorem.

Theorem 3. Let G be a 2-connected planar 2-cycle resonant graph, B a 2-connected subgraph of G with exactly two attachment vertices u and v. Then $(B+P^*)_{(u,v)}$ is either 2-cycle resonant or simple 1-cycle resonant, for any path P^* of odd length with end vertices u and v, and $V(G)\cap V(P^*)=\{u,v\}$.

Proof : From Theorem 1 and Theorem H, it is easy to show that $(B+P^*)_{(u,v)}$ is 1-cycle resonant. If $(B+P^*)_{(u,v)}$ is not simple 1-cycle resonant, let C_1 and C_2 be any two disjoint cycles in $(B+P^*)_{(u,v)}$. We first prove that $(B+P^*)_{(u,v)}-V(C_1)\cup V(C_2)$ has no odd component.

Case 1. $|\{u,v\}\cap(V(C_1)\cup V(C_2))|=0$. Then the attachment vertices u and v of B belong to a same component B^* of $(B+P^*)_{(u,v)}-V(C_1)\cup V(C_2)$. $B^*\cup B-V_I(P^*)$ is an even component of $G-V(C_1)\cup V(C_2)$. Since $|V(B)|$ is even and $V(B^*)\cap V(B)=\{u,v\}$, so B^* is an even component of $(B+P^*)_{(u,v)}-V(C_1)\cup V(C_2)$. The other components of $(B+P^*)_{(u,v)}-V(C_1)\cup V(C_2)$ are also components of $G-V(C_1)\cup V(C_2)$, each of which is even.

Case 2. $|\{u,v\}\cap(V(C_1)\cup V(C_2))|=1$. Without loss of the generality, let $u\in V(C_1)$, that is, the cycle C_1 contains u but v, and the cycle C_2 contains no any one of u and v. Let B^* be the component containing v of $(B+P^*)_{(u,v)}-V(C_1)\cup V(C_2)$. Then $B^*\cup(B-u)-V_I(P^*)$ is an even component of $G-V(C_1)\cup V(C_2)$, and so $B^*\cup(B-u)$ is even. Since $|V(B-u)|$ is odd and $V(B^*)\cap V(B-u)=\{v\}$, so B^* is an even component of $(B+P^*)_{(u,v)}-V(C_1)\cup V(C_2)$. The other components of $(B+P^*)_{(u,v)}-V(C_1)\cup V(C_2)$ are also components of $G-V(C_1)\cup V(C_2)$, each of which is even.

Case 3. $|\{u,v\}\cap(V(C_1)\cup V(C_2))| = 2$. Then no component of $(B+P^*)_{(u,v)} - V(C_1)\cup V(C_2)$ contain u and v, any component of $(B+P^*)_{(u,v)} - V(C_1)\cup V(C_2)$ not containing $V_I(P^*)$ is also a component of $G - V(C_1)\cup V(C_2)$, each of which is even. If $V_I(P^*)$ are not on C_1 or C_2, the component of $(B+P^*)_{(u,v)} - V(C_1)\cup V(C_2)$ containing $V_I(P^*)$ is also even.

In any case, $(B+P^*)_{(u,v)} - V(C_1)\cup V(C_2)$ contains no odd component, and $(B+P^*)_{(u,v)}$ is 1-cycle resonant. By Theorem A, we have that $(B+P^*)_{(u,v)}$ is 2-cycle resonant. □

In Theorem F [40], Xiaofeng Guo and Fuji Zhang characterized structures of a 2-connected planar 2-cycle resonant graph G, in which the 2-connected subgraphs of G with exactly two attachment vertices play an important role.

Having the aid of Theorem F, we shall review the structures of a 2-connected planar 2-cycle resonant graph G, and get four structure models of a 2-connected planar 2-cycle resonant graph G as following:

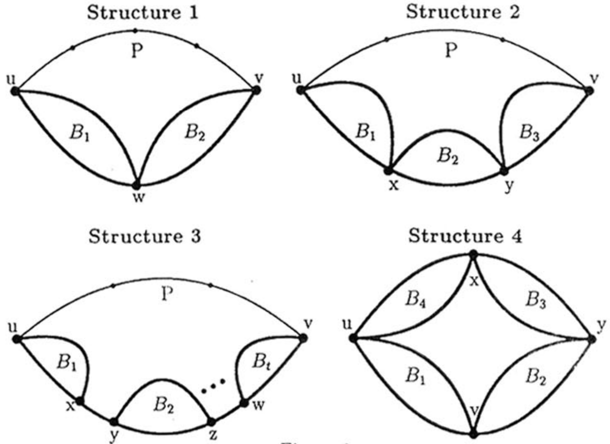

Figure 1.

Structure 1: If there is a chain P with even length and end vertices u and v, then $G - V_I(P)$ has exactly two 2-connected blocks B_1 and B_2 with a common vertex w, and $u \Rightarrow w$ in B_1 and $v \Rightarrow w$ in B_2, as shown in Figure 1.

Structure 2: For a chain P with odd length and end vertices u and v such that $G - V_I(P)$ is not 2-connected, $G - V_I(P)$ has exactly three blocks B_1, B_2 and B_3, each of which is a 2-connected, and each of u and v is cycle-related to the other attachment vertex of the block containing it, and the attachment vertices of the third block are mutually cycle-related in the third block, that is, $u \Rightarrow x$, $v \Rightarrow y$ and $x \Leftrightarrow y$, as shown in Figure 1.

Structure 3: For a chain P with odd length and end vertices u and v such that $G - V_I(P)$ is not 2-connected, any two 2-connected blocks of $G - V_I(P)$ are disjoint, in addition any chain P_i in $G - V_I(P)$ induced by non-2-connected blocks of $G - V_I(P)$ is of odd length, as shown in Figure 1, where B_i is 2-connected block, for $i = 1, 2, \cdots t$, $(t \geq 2)$.

Structure 4: For a 2-connected subgraph B_1 of G with exactly two attachment vertices, if B_1 is not 2-connected and every block of $\overline{B_1}$ is 2-connected, then B_1 has exactly three blocks, say B_2, B_3, B_4, and the attachment vertices of each of B_1, B_2, B_3, B_4 are mutually cycle-related in the block, as shown in Figure 1.

From these four structure models, we can see that a 2-connected planar 2-cycle resonant graph G consists of two kind of subgraphs: chains and 2-connected subgraphs with exactly two attachment vertices, based on some rules. We call the two kind of subgraphs as *structural-bricks* of G.

Let G be a 2-connected planar 2-cycle resonant graph with the cyclomatic number $\nu(G) \geq 3$, and let C_1 and C_2 be two disjoint cycles in G. Then C_2 must be contained in a 2-connected block B of a bridge of C_1 in G. By Corollary I, B is 2-connected subgraph of G with exactly two attachment vertices, and $\overline{B}$ is neither chain of G nor 2-connected. Then G must be one of four structure models, in which B is a 2-connected structure-brick.

To sum up the state above, we have get the following Lemma.

Lemma 1. Let G be a 2-connected planar 2-cycle resonant graph with the cyclomatic number $\nu(G) \geq 3$, then there exists a 2-connected subgraph with exactly two attachment vertices such that G is one of structure 1, 2, 3 and 4. □

Combining Lemma 1 with Theorem 3, we can get some new necessary and sufficient conditions for a graph to be planar 2-cycle resonant. In fact, it is a modification of Theorem F.

Theorem 4. A 2-connected graph G with the cyclomatic number $\nu(G) \geq 3$ is planar 2-cycle resonant if and only if there is a 2-connected subgraph B_1 in G with exactly two attachment vertices such that G is one of structure 1, 2, 3 and 4, and for every 2-connected structural-brick B_i of G with exactly two attachment vertices u_i and v_i, $(B_i + P^\bullet)_{(u_i,v_i)}$ is 2-cycle resonant or simple 1-cycle resonant, $i = 1, 2, \cdots, t$, $(t \geq 2)$. Moreover, let G be of structure 1, there is at least one 2-connected structural-brick B of G with exactly two attachment vertices u and v such that $(B + P^\bullet)_{(u,v)}$ is planar 2-cycle resonant.

Proof: Necessity. By Lemma 1, there exists a 2-connected subgraph B_1 with exactly two attachment vertices such that G is one of four structures which containing B_1 as a 2-connected structural-brick of G. From Theorem 3, for every 2-connected structural-brick B_i of G with exactly two attachment vertices u_i and v_i, $(B_i + P^\bullet)_{(u_i,v_i)}$ is 2-cycle resonant or simple 1-cycle resonant, $i = 1, 2, \cdots, t$, $(t \geq 2)$. Moreover, for each of structure 1, 2, 3 and 4, the attachment vertices of B_i satisfy some cycle-related conditions.

If G is of structure 1. Since G is 2-cycle resonant, there are two disjoint cycles in G, one of which, say C, does not contain the common vertex w of B_1 and B_2, and so C also does not contain the other attachment vertex of the 2-connected structure-brick containing C, say B_1. By Corollary 1, There is a $u-w$ path in $B_1 - V(C)$. Then $(B_1 + P^*)_{(u,w)}$ contains two disjoint cycles. By Theorem 3, $(B_1 + P^*)_{(u,w)}$ is 2-cycle resonant.

Sufficiency. Let G be one of structure 1, 2, 3 and 4.

Suppose that G is of structure 1, as shown in Figure 1. Let B_1 be the 2-connected subgraphs in G with exactly two attachment vertices u and w, and $u \Rightarrow w$ in B_1. Let B_2 be the 2-connected subgraphs in G with exactly two attachment vertices w and v, and $v \Rightarrow w$ in B_2. Let P be the chain of even length in G with end vertices u and v. $(B_1 + P^*)_{(u,w)}$ is either 2-cycle resonant or simple 1-cycle resonant, and so does $(B_2 + P^*)_{(w,v)}$

At first, making use of Lemma G, we shall show that G is 1-cycle resonant. $(B_1 + P^*)_{(u,w)}$ is 1-cycle resonant, without loss the generality, let $P^* = P \cup vw$. Then G is the graph obtained from $(B_1 + P^*)_{(u,w)}$ by replacing vw with B_2. Since $(B_2 + P^*)_{(w,v)}$ is also 1-cycle resonant, by Theorem 1, then w and v avoid every bridge of a cycle of B_2 containing w and v. So G is 1-cycle resonant.

Let C_1 and C_2 be two any disjoint cycles in G. We now show that $G - V(C_1) \cup V(C_2)$ has a perfect matching. We distinguish the following cases:

Case 1. Either C_1 or C_2, say C_2, contains chain P. Then C_2 must contains the vertex w. Let $B_1 \cap C_2 = P_1$ and $B_2 \cap C_2 = P_2$. Without loss of the generality, let C_1 be in B_1. If we take that $P^* = P \cup P_2$ for $(B_1 + P^*)_{(u,w)}$, then $(B_1 + P^*)_{(u,w)} = (B_1 + (P \cup P_2))_{(u,w)}$ is 2-cycle resonant, and C_1 and C_2 are two disjoint cycles in $(B_1 + (P \cup P_2))_{(u,w)}$. So $B_1 - V(P_1) \cup V(C_1) = (B_1 + (P \cup P_2))_{(u,w)} - V(C_1) \cup V(C_2)$ has a perfect matching M_1.

Similarly, we take that $P^* = P \cup P_1$ for $(B_2 + P^*)_{(v,w)}$, then $(B_2 + P^*)_{(v,w)} = (B_2 + (P \cup P_1))_{(v,w)}$, which has a cycle C_2, is either 2-cycle resonant or simple 1-cycle resonant. So $B_2 - V(P_2) = (B_2 + (P \cup P_1))_{(u,v)} - V(C_2)$ has a perfect matching M_2. Thus $G - V(C_1) \cup V(C_2) = (B_1 - V(P_1) \cup V(C_1)) \cup (B_2 - V(P_2))$ has a perfect matching $M_1 \cup M_2$.

Case 2. Neither of C_1 and C_2 contain chain P.

Subcase 2.1: Both C_1 and C_2 are in B_1 or B_2, say B_1. Since B_2 is connected, there is a path P_2 in B_2 with end vertices v and w. Taking $P^* = P \cup P_2$, then $(B_1 + P^*)_{(u,w)} = (B_1 + (P \cup P_2))_{(u,w)}$ is 2-cycle resonant, and $(B_1 + (P \cup P_2))_{(u,w)} - V(C_1) \cup V(C_2)$ has a perfect matching M_1. Since $(B_2 + P^*)_{(v,w)}$ is 1-cycle resonant, in which $P^* \cup P_2$ is a cycle, then $B_2 - V(P_2) = (B_2 + P^*)_{(v,w)} - V(P^* \cup P_2)$ has a perfect matching M_2. Then $G - V(C_1) \cup V(C_2) = ((B_1 + V(P \cup P_2))_{(v,w)} - V(C_1) \cup V(C_2)) \cup (B_2 - V(P_2))$ has a perfect matching $M_1 \cup M_2$.

Subcase 2.2: C_1 and C_2 are separately in B_1 and B_2, say C_1 in B_1, C_2 in B_2.

If C_1 and C_2 do not contain the vertex w, then neither of them contain u and v, since $u \Rightarrow w$ in B_1 and $v \Rightarrow w$ in B_2. From Corollary 1, we know that there is a path P_1 in $B_1 - V(C_1)$ with end vertices u and w, and a path P_2 in $B_2 - V(C_2)$ with end vertices v and w, and then $P \cup P_1 \cup P_2$ is a cycle C in $(B_1 + (P \cup P_2))_{(u,w)}$ and $(B_2 + (P \cup P_1))_{(v,w)}$. Then $(B_1 + (P \cup P_2))_{(u,w)}$ is 2-cycle resonant, and so does $(B_2 + (P \cup P_1))_{(v,w)}$. Let M_1 be a perfect matching of $(B_1 + (P \cup P_2))_{(u,w)} - V(C_1) \cup V(C) = B_1 - V(C_1) \cup V(P_1)$, and M_2 a perfect matching of $(B_2 + (P \cup P_1))_{(v,w)} - V(C_2) \cup V(C) = B_2 - V(C_2) \cup V(P_2)$, M_3 a perfect matching of C. So $G - V(C_1) \cup V(C_2) = (B_1 - V(C_1) \cup V(P_1)) \cup (B_2 - V(C_2) \cup V(P_2)) \cup C$ has a perfect matching $M_1 \cup M_2 \cup M_3$.

If either C_1 or C_2 contains the vertex v, say C_1, then C_2 does not contain v and w, since $w \Rightarrow v$ in B_2. Similarly, from Corollary 1, there is a path P_2 in $B_2 - V(C_2)$ with end vertices v and w, so $P^* \cup P_2$ is a cycle disjoint from C_2 in $(B_2 + P^*)_{(v,w)}$. Then $(B_2 + P^*)_{(v,w)}$ is 2-cycle resonant, and $(B_2 + P^*)_{(v,w)} - V(C_2) \cup V(P^* \cup P_2) = B_2 - V(C_2) \cup V(P_2)$ has a perfect matching M_1. Taking $P^* = P \cup P_2$, then $(B + P^*)_{(u,w)} = (B_1 + (P \cup P_2))_{(u,w)}$ is 1-cycle resonant, and $(B_1 + (P \cup P_2))_{(u,w)} - V(C_1)$ has a perfect matching M_2. So $M_1 \cup M_2$ is a perfect matching of $G - V(C_1) \cup V(C_2)$.

To sum up the cases above, we have that G is 2-cycle resonant.

If G is of other structures, making use of the method similar to that above, we can prove that G is 2-cycle resonant. Here we omit it.

The proof is thus completed. □

4 Construction And Decomposition of 2-Connected Planar 2-Cycle Resonant Graphs

Decomposition: Let G be a 2-connected planar 2-cycle resonant graph. From Theorem 4, there must be a 2-connected subgraph B_1 with exactly two differently colored attachment vertices such that G is one of four structure models. Let B_i be a 2-connected structure-brick G with exactly two attachment vertices u_i and v_i, for $i = 1, 2, \cdots, t$, $(t \geq 2)$. Then every $(B_i + P^*)_{(u_i,v_i)}$ is either planar 2-cycle resonant or planar simple 1-cycle resonant. Thus we can decompose G into smaller 2-connected planar 2-cycle resonant graphs or simple 1-cycle resonant graphs. If some of $(B_i + P^*)_{(u_i,v_i)}$'s are also planar 2-cycle resonant, we can further decompose them, until G is decomposed into some planar simple 1-cycle resonant graphs.

In reverse, we can construct any 2-connected planar 2-cycle resonant graph G from some planar simple 1-cycle resonant graphs as follows:

Construction: In reverse order of decomposing, we can construct any required 2-connected planar 2-cycle resonant graph as follows:

1. Let $SB = \{B_i\}$ be a set of some structure-bricks such that $(B_i + P^*)_{(u_i,v_i)}$ are simple planar 1-cycle resonant graphs. Let S be a set of 2-connected planar 2-cycle resonant graphs.

2. Take some structure-bricks from SB and assemble the structure-bricks into a 2-connected planar 2-cycle resonant graph G_i according to the four structure models as shown in Figure 1.

3. If some G_i is the required 2-connected planar 2-cycle resonant graph, go to step 6.

4. Add G_i into S, and find a chain P^* of G_i such that $B_i = G_i - V_I(P^*)$ satisfies some conditions of structure-bricks given in structure 1, 2, 3 and 4, and add the new structure-brick B_i into SB.

5. Go to step 2.

6. Stop.

Remark: Based on the construction and decomposition above, an efficient algorithm for determining whether or not a 2-connected graph to planar 2-cycle resonant can be developed. We shall establish such an algorithm in further works.

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