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ON RESISTANCE MATRICES

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Abstract

According to Klein and Randić, the resistance distance r_{ij} between two vertices v_i and v_j of a (connected) graph G is equal to the resistance between the respective two vertices of an electrical network, constructed so as to correspond to G, such that the resistance between any two adjacent vertices is unity. The matrix $R = ||r_{ij}||$ is the resistance matrix of G. Let, respectively, n, m, A, and D denote the number of vertices, the number of edges, the adjacency matrix, and the vertex-degree matrix of G. Let I and J be the unit matrix and the square matrix whose all elements are unity, both of order n. In the study of R the matrices $X = \left(D - A + \frac{1}{n} J\right)^{-1}$, $Y = \left(D - A + \frac{1}{2m} D J\right)^{-1}$, and $Z = \left(I - D^{-1} A + \frac{1}{2m} J D\right)^{-1}$ are encountered. We establish some basic properties thereof, in particular determine their spectra and show that they are non-singular. Then some properties of R are deduced, in particular that for connected graphs with two or more vertices, R is non-singular too.

INTRODUCTION

Klein and Randić [1] introduced in 1993 the so-called *resistance distance* between the vertices of a (molecular) graph G, as the effective resistance between the respective two vertices of an electrical network, constructed so as to correspond to G, such that the resistance between any two adjacent vertices is unity. Such a resistance distance is then computed and studied by utilizing the methods of the theory of resistive electrical networks (based on Ohm's and Kirchhoff's laws). This theory is a well-elaborated part of electrical engineering sciences, on which a plethora of textbooks and monographs exists, many of them utilizing sophisticated, high-level and rigorous mathematics.

In this paper we outline some basic results on resistance distances, that may be known or well-known to electrical engineers, but that need not be fully familiar to colleagues active in mathematical chemistry. In addition, we offer a few results for which we believe that are new.

Let G be a (molecular) graph, possessing n vertices and m edges, $n \geq 2$, $m \geq 1$. Except when stated otherwise, it is assumed that G is connected. Let $v_1, v_2, \ldots, v_n$ be the vertices of G. For the considerations in this paper only connected graphs are of interest, because only for them the resistance-distance concept is meaningful. Nevertheless, some of our results apply also to disconnected graphs, as clearly specified in Theorems 1–3 and their corollaries. These are stated for the sake of completeness.

The *adjacency matrix* of G, denoted by A, is the square matrix of order n, whose (i,j)-entry, denoted by a_{ij}, is equal to unity if the vertices v_i and v_j are adjacent, and is zero otherwise.

The degree of the vertex v_i, denoted by d_i, is the number of first neighbors of v_i. Clearly,

$$d_i = \sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} .$$

The *vertex-degree matrix* of G, denoted by D, is the square matrix of order n, whose (i,i)-entry is equal to d_i and whose (i,j)-entry, $i \neq j$, is zero.

By I and J we denote, respectively, the unit matrix of order n and the square matrix of order n all of whose elements are unity. By $\mathbf{1}$ and $\mathbf{0}$ we denote the n-

dimensional column-vectors, whose all elements are unity and zero, respectively.

In what follows a square matrix whose diagonal elements are $q_1, q_2, \ldots, q_n$ and whose off-diagonal elements are zero is denoted by $diag[q_1, q_2, \ldots, q_n]$. Thus, for instance, $I = diag[1, 1, \ldots, 1]$ and $D = diag[d_1, d_2, \ldots, d_n]$.

The resistance distance between the vertices v_i and v_j of the graph G will be denoted by r_{ij} and the respective *resistance-distance matrix* by $R = ||r_{ij}||$. In the seminal paper [1] it was proven that r_{ij} is indeed a distance, namely that it satisfies the conditions $r_{ij} \geq 0$; $r_{ij} = 0 \Leftrightarrow i = j$; $r_{ij} = r_{ji}$; $r_{ij} + r_{jk} \geq r_{ik}$.

Eventually, the resistance-distance concept was much studied [2-12]. In analogy to the classical Wiener index, one introduced [1] the sum of resistance distances of all pairs of vertices of a molecular graph,

$$Kf = \sum_{i<j} r_{ij}$$

a structure-descriptor that eventually was named [2] the "*Kirchhoff index*".

In the case of acyclic graphs, the resistance distance coincides with the usual vertex distance. Then, of course, the Kirchhoff index is same as the Wiener index. Therefore the resistance distance and Kirchhoff index are worth studying only for cycle-containing graphs [2,7,8,10,12].

COMPUTING THE RESISTANCE MATRIX AND THE KIRCHHOFF INDEX

The Laplacian matrix of a graph G is defined as $L = D - A$. The spectral theory of this matrix is nowadays well elaborated (see, for instance, the reviews [13,14], as well as [15,16] for a survey of chemical applications). Let $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n$ be the eigenvalues of the Laplacian matrix, ordered in a non-increasing manner. Then for all graphs, $\mu_n = 0$, whereas $\mu_{n-1} > 0$ if and only if the underlying graph G is connected. Consequently, the Laplacian matrix of any graph is singular, and its inverse does not exist.

A singular matrix has no inverse, but has a so-called *Moore–Penrose generalized inverse* [17,18]; for a brief review of the properties of the Moore–Penrose generalized inverse of a singular matrix see [4].

Denote by $L^\dagger$ the Moore–Penrose generalized inverse of the Laplacian matrix. Within the theory of electrical networks the standard method to compute the resistance matrix is via $L^\dagger$:

$$r_{ij} = (L^\dagger)_{ii} - (L^\dagger)_{ij} - (L^\dagger)_{ji} + (L^\dagger)_{jj} \ . \tag{1}$$

Eq. (1) is found in [1] and, explicitly, in [3], but was, for sure, known much earlier.

Klein and Randić proved that [1]

$$Kf = n\, tr L^\dagger \tag{2}$$

whereas Mohar and one of the present authors [4] demonstrated that the Kirchhoff index can be obtained from the (non-zero) eigenvalues of the Laplacian matrix:

$$Kf = n \sum_{i=1}^{n-1} \frac{1}{\mu_i} \tag{3}$$

without knowing its Moore–Penrose inverse. Thus, Eq. (3) is from a computational point of view somewhat more appropriate than Eq. (2).

In view of this, efforts have been made to find some non-singular matrices, such that the resistance distances can be expressed in terms of their inverses. In the subsequent section we examine three such invertible matrices, and denote their inverses by $X = ||x_{ij}||$, $Y = ||y_{ij}||$, and $Z = ||z_{ij}||$. Then in terms of the matrix X:

$$r_{ij} = x_{ii} + x_{jj} - 2\, x_{ij} \tag{4}$$

i. e.,

$$R = diag\,[x_{11}, x_{22}, \ldots, x_{nn}]\; J + J\; diag\,[x_{11}, x_{22}, \ldots, x_{nn}] - 2\, X \tag{5}$$

in terms of the matrix Y:

$$r_{ij} = y_{ii} + y_{jj} - y_{ij} - y_{ji} \tag{6}$$

i. e.,

$$R = diag\,[y_{11}, y_{22}, \ldots, y_{nn}]\; J + J\; diag\,[y_{11}, y_{22}, \ldots, y_{nn}] - Y - Y^t \tag{7}$$

and in terms of the matrix Z:

$$r_{ij} = \frac{z_{ii}}{d_i} + \frac{z_{jj}}{d_j} - 2\, \frac{z_{ij}}{d_j} \tag{8}$$

i. e.,

$$R = diag\left[\frac{z_{11}}{d_1}, \frac{z_{22}}{d_2}, \ldots, \frac{z_{nn}}{d_n}\right] J + J\, diag\left[\frac{z_{11}}{d_1}, \frac{z_{22}}{d_2}, \ldots, \frac{z_{nn}}{d_n}\right] - 2\, Z\, D^{-1} .$$

Formula (4) was communicated by Babić et al. [12], without any explanation of how it was obtained. Formula (8) was deduced by Palacios [9], using previous results on electrical networks and Markov chain theory. Formula (6) is our own design.

PROPERTIES OF THE MATRIX Z

In this section we consider the matrix Z, defined as

$$Z = ||z_{ij}|| = \left(I - D^{-1} A + \frac{1}{2m} J\, D\right)^{-1} .$$

It is easily verified that

$$Z\, D^{-1} = \left(D - A + \frac{1}{2m} D\, J\, D\right)^{-1} .$$

Therefore, $Z\, D^{-1}$ is a symmetric matrix and so

$$d_i\, z_{ij} = d_j\, z_{ji} . \tag{9}$$

Note that Eq. (9) follows also from (8) because of $r_{ij} = r_{ji}$.

In view of Eq. (8) we obtain

$$Kf = \sum_{i<j}\left(\frac{z_{ii}}{d_i} + \frac{z_{jj}}{d_j} - \frac{2\, z_{ij}}{d_j}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{z_{jj} - z_{ij}}{d_j}$$

which gives

$$Kf = n\, tr\left(ZD^{-1}\right) - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{z_{ij}}{d_j} .$$

We now show that $\mathbf{1}$ is an eigenvector of the matrix Z, with eigenvalue 1. Observe first that.

$$Z^{-1}\,\mathbf{1} = \left(I - D^{-1} A + \frac{1}{2m} J\, D\right)\mathbf{1} = \mathbf{1} - D^{-1} A\,\mathbf{1} + \frac{1}{2m} J\, D\,\mathbf{1} .$$

It is immediately verified that $D^{-1} A\,\mathbf{1} = \mathbf{1}$ and $J\, D\,\mathbf{1} = \left(\sum_i d_i\right)\mathbf{1} = 2m\,\mathbf{1}$, which implies

$$Z^{-1}\,\mathbf{1} = \mathbf{1} .$$

Multiplying the above relation by Z we get

$$Z\,\mathbf{1} = \mathbf{1}$$

the immediate consequences of which are

$$\sum_{i=1}^{n} z_{ij} = 1 \tag{10}$$

and

$$Z\,J = J\ . \tag{11}$$

Since $Z^{-1} = I - D^{-1}\,A + \frac{1}{2m}\,J\,D$, from $Z^{-1}\,Z = I$ we infer

$$z_{ij} + \sum_{k=1}^{n}\left(\frac{d_k}{2m} - \frac{a_{ik}}{d_i}\right) z_{kj} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

and

$$z_{ij} - \sum_{k \in \Gamma(i)} \frac{z_{kj}}{d_i} + \sum_{k=1}^{n}\left(\frac{d_k}{2m}\right) z_{kj} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

where $\Gamma(i)$ is the set of neighbors of the vertex v_i. Application of (9) and (10) yields

$$\sum_{k=1}^{n}\left(\frac{d_k}{2m}\right) z_{kj} = \frac{1}{2m}\sum_{k=1}^{n} d_k\, z_{kj} = \frac{d_j}{2m}\sum_{k=1} z_{kj} = \frac{d_j}{2m}$$

and we finally obtain

$$z_{ij} - \sum_{k \in \Gamma(i)} \frac{z_{kj}}{d_i} + \frac{d_j}{2m} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \ .$$

* * * * *

Previously we established that $\mathbf{1}$ is an eigenvector of $D^{-1}\,A$, with eigenvalue 1. Because the matrices $D^{-1}\,A$ and $D^{1/2}\,(D^{-1}\,A)\,D^{-1/2}$ are similar, they have identical spectra. Because $D^{1/2}\,(D^{-1}\,A)\,D^{-1/2} = D^{-1/2}\,A\,D^{-1/2}$ is a symmetric matrix with real elements, all its eigenvalues are real-valued. Furthermore, because these matrix elements are non-negative, we may apply the Perron–Frobenius theorem (see, for instance, [19,20]).

Let the eigenvalues of $D^{-1}\,A$ be $\lambda_1, \lambda_2, \ldots, \lambda_n$ labelled in a non-increasing order.

By the Perron–Frobenius theorem, because all components of the eigenvector $\mathbf{1}$ are positive, this eigenvector corresponds to the greatest eigenvalue of $D^{-1}\,A$ (which is λ_1), and $\lambda_1 \geq |\lambda_i|$ for all $i = 2, \ldots, n$. If the graph G is connected, then the algebraic multiplicity of the eigenvalue $\lambda = 1$ is one.

If G is not connected, then the spectrum of $D^{-1}A$ is the union of the spectra of the components of G. Consequently, the algebraic multiplicity of the eigenvalue $\lambda = 1$ is equal to the number of components of G.

More on the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, i. e., on the spectrum of of the matrix $D^{-1/2} A D^{-1/2}$ can be found elsewhere [20].

Of course, the above consideration is restricted to graphs without isolated vertices (i. e., without vertices of degree 0), because otherwise the matrices D^{-1}, $D^{-1/2}$, etc do not exist.

Theorem 1. *Let G be a graph possessing n vertices, m edges, and no isolated vertices. Let A and D be the adjacency and vertex-degree matrices corresponding to G. If $\lambda_1 = 1, \lambda_2, \lambda_3 \ldots, \lambda_n$ are the eigenvalues of the matrix $D^{-1}A$, then the eigenvalues of the matrix $I - D^{-1}A + \frac{1}{2m} J D$ are $1, 1-\lambda_2, 1-\lambda_3, \ldots, 1-\lambda_n$.*

Proof. (a) We already know that $\mathbf{1}$ is an eigenvector of $I - D^{-1}A + \frac{1}{2m} J D$, with eigenvalue 1.

(b) Let γ be an eigenvector of $D^{-1}A$, with eigenvalue λ. Then

$$\left(I - D^{-1}A + \frac{1}{2m} J D\right)\gamma = (1-\lambda)\gamma + \frac{1}{2m} J D \gamma .$$

Now,

$$J D \gamma = J A \gamma = J D \left(D^{-1}A\right)\gamma = J D \lambda \gamma$$

resulting in

$$J D \gamma = \lambda J D \gamma . \tag{12}$$

If $\lambda = 1$, then Eq. (12) is automatically satisfied. If $\lambda \neq 1$ then Eq. (12) implies that it must be $J D \gamma = \mathbf{0}$, and then

$$\left(I - D^{-1}A + \frac{1}{2m} J D\right)\gamma = (1-\lambda)\gamma .$$

We see that any eigenvector of $D^{-1}A$, whose eigenvalue λ is different from unity, is an eigenvector of $I - D^{-1}A + \frac{1}{2m} J D$, with eigenvalue $1-\lambda$. This implies that if G is connected (when $D^{-1}A$ has $n-1$ eigenvectors with eigenvalues different from unity), then the statement of Theorem 1 is satisfied.

(c) If the graph G is not connected, then we proceed as follows. Suppose G has p components, $G_1, G_2, \ldots, G_p$, none of which is an isolated vertex. We construct the graph G_w by introducing an edge between a vertex of G_1 and a vertex of G_i, $i = 2, \ldots, p$, a total of $p-1$ edges. Each of these edges is given a weight w. As long as $w > 0$, G_w is connected and, by arguments analogous to those outlined in point (b), Theorem 1 is applicable. The validity of Theorem 1 is then extended to disconnected graphs by considering the limit $w \to 0$.

This completes the proof of Theorem 1. □

Corollary 1.1. *If the graph G is connected and has at least two vertices, then the matrix* $I - D^{-1} A + \frac{1}{2m} J D$ *is non-singular, implying that* $Z = \left(I - D^{-1} A + \frac{1}{2m} J D\right)^{-1}$ *does exist.*

Corollary 1.2. *If the graph G is disconnected, possessing p components none of which is an isolated vertex, then in the spectrum of the matrix* $I - D^{-1} A + \frac{1}{2m} J D$ *there are* $p-1$ *zeros, implying that* $Z = \left(I - D^{-1} A + \frac{1}{2m} J D\right)^{-1}$ *does not exist.*

Corollary 1.3. *If the matrix Z exists, then its eigenvalues are* 1 *and* $1/(1-\lambda_i)$, $i = 2, 3, \ldots, n$.

PROPERTIES OF THE MATRIX Y

In this section we consider the matrix Y, defined as

$$Y = ||y_{ij}|| = \left(D - A + \frac{1}{2m} D J\right)^{-1} .$$

We first state the following:

Theorem 2. *Let G be a graph on n vertices and let* $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n = 0$ *be its Laplacian eigenvalues. Then the eigenvalues of the matrix* $D - A + \frac{1}{2m} D J$ *are* $1, \mu_1, \mu_2, \ldots, \mu_{n-1}$.

Proof. Let $\mathbf{c}$ be any n-dimensional column-vector, and let $s(\mathbf{c})$ be the sum of its components. Then

$$J\,\mathbf{c} = s(\mathbf{c})\,\mathbf{1} . \tag{13}$$

Because of $(D - A)\,\mathbf{1} = \mathbf{0}$, the vector $\mathbf{1}$ is an eigenvector of the Laplacian matrix of any graph G, corresponding to the eigenvalue zero, namely to the eigenvalue μ_n.

Let γ_i be an eigenvector of the Laplacian matrix of G, corresponding to an eigenvalue μ_i other than μ_n (not necessarily different from μ_n). Then γ_i is orthogonal to $\mathbf{1}$ and therefore the sum of the components of γ_i is zero. As a special case of (13) we have

$$J\,\gamma_i = \mathbf{0}\ . \tag{14}$$

Taking into account the relation (14), and that γ_i is an eigenvector of $D - A$, we get

$$\left(D - A + \frac{1}{2m}\,D\,J\right)\,\gamma_i = \mu_i\,\gamma_i$$

which holds for all $i = 1, 2, \ldots, n-1$.

Hence $\mu_1, \mu_2, \ldots, \mu_{n-1}$ are eigenvalues of $D - A + \frac{1}{2m}\,D\,J$.

In order to find the remaining eigenvalue of $D - A + \frac{1}{2m}\,D\,J$, denoted by μ, note that the respective eigenvector, denoted by γ, is a linear combination of γ_i, $i = 1, 2, \ldots, n-1$, and $\mathbf{1}$. Because $s(\gamma_i) = 0$, the sum of components of γ must be non-zero.

Now, starting with

$$\left(D - A + \frac{1}{2m}\,D\,J\right)\,\gamma = \mu\,\gamma$$

and

$$J\left(D - A + \frac{1}{2m}\,D\,J\right)\,\gamma = \mu\,J\,\gamma \tag{15}$$

and bearing in mind the identities $J\,D = J\,A$ and $J\,D\,J = 2m\,J$, we obtain for the left-hand side of (15)

$$\left(J\,D - J\,A + \frac{1}{2m}\,J\,D\,J\right)\,\gamma = \left(\frac{1}{2m}\,2m\,J\right)\,\gamma = J\,\gamma = s(\gamma)\,\mathbf{1}$$

whereas for its right-hand side

$$\mu\,J\,\gamma = \mu\,s(\gamma)\,\mathbf{1}\ .$$

Because $s(\gamma) \neq 0$, from $s(\gamma)\,\mathbf{1} = \mu\,s(\gamma)\,\mathbf{1}$ follows $\mu = 1$.

This completes the proof of Theorem 2. □

Corollary 2.1. *If the graph G is connected and has at least two vertices, then the matrix $D - A + \frac{1}{2m}\,D\,J$ is non-singular, implying that $Y = \left(D - A + \frac{1}{2m}\,D\,J\right)^{-1}$ does exist.*

Corollary 2.2. *If the graph G is disconnected, then because of $\mu_{n-1} = 0$, the matrix $D - A + \frac{1}{2m} D J$ is singular, implying that $Y = \left(D - A + \frac{1}{2m} D J\right)^{-1}$ does not exist.*

Corollary 2.3. *If the matrix Y exists, then its eigenvalues are 1 and $1/\mu_i$, $i = 1, 2, \ldots, n-1$.*

It is worth noting that **1** is not an eigenvector of Y, but is an eigenvector (with eigenvalue 1) of the transpose of Y. From $Y^t \mathbf{1} = \mathbf{1}$ we get

$$\sum_{i=1}^{n} y_{ij} = 1 \tag{16}$$

a relation analogous to (10) and (19).

We now consider the relations between the matrices Z and Y. It is easily verified that

$$Y^{-1} = D\, Z^{-1} \left[I - \frac{1}{2m} J\,(D - I)\right]$$

which implies

$$Z\, D^{-1} = \left[I - \frac{1}{2m} J(D - I)\right] Y\,. \tag{17}$$

From (17) we obtain

$$\frac{z_{ij}}{d_j} = y_{ij} - \frac{1}{2m} \sum_{k=1}^{n} (d_k - 1)\, y_{kj}$$

which combined with (8) and (16) yields

$$\begin{aligned} r_{ij} &= y_{ii} - \frac{1}{2m} \sum_{k=1}^{n} (d_k - 1)\, y_{ki} + y_{jj} - \frac{1}{2m} \sum_{k=1}^{n} (d_k - 1)\, y_{kj} \\ &- 2\, y_{ij} + \frac{1}{m} \sum_{k=1}^{n} (d_k - 1)\, y_{kj} \\ &= y_{ii} + y_{jj} - 2\, y_{ij} + \frac{1}{2m} \sum_{k=1}^{n} (d_k - 1)(y_{kj} - y_{ki})\,. \end{aligned}$$

By (9) we have

$$\frac{1}{2m} \sum_{k=1}^{n} (d_k - 1)(y_{kj} - y_{ki}) = y_{ij} - y_{ji}$$

from which Eqs. (6) and (7) follow.

Using Eq. (6) and Theorem 2 we can offer a simple proof of formula (3). Indeed,

$$Kf = \sum_{i<j} r_{ij} = \sum_{i<j} (y_{ii} + y_{jj} - y_{ij} - y_{ji}) = n \sum_{i=1}^{n} y_{ii} - \sum_{i=1}^{n} \sum_{j=1}^{n} y_{ij}$$

which bearing in mind (16) is simplified to

$$Kf = n \sum_{i=1}^{n} y_{ii} - n \ . \tag{18}$$

The term $\sum_{i=1}^{n} y_{ii}$ is just the trace of the matrix Y, which according to Corollary 2.3 is equal to

$$\sum_{i=1}^{n} y_{ii} = 1 + \sum_{i=1}^{n-1} \frac{1}{\mu_i} \ .$$

Substituting the above relation back into (18) readily yields formula (3).

PROPERTIES OF THE MATRIX X

The matrix X is encountered in the work of Babić et al. [12], where formula (4) is given, but no other property of X specified. This matrix is defined as

$$X = ||x_{ij}|| = \left(D - A + \frac{1}{n} J\right)^{-1} \ .$$

As in the case of matrix Y we first determine its eigenvalues.

Theorem 3. *Let G be a graph on n vertices and let $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n = 0$ be its Laplacian eigenvalues. Then the eigenvalues of the matrix $D - A + \frac{1}{n} J$ are $1, \mu_1, \mu_2, \ldots, \mu_{n-1}$.*

The analogy between Theorems 2 and 3 is evident. The proof of Theorem 3 is similar to the proof of Theorem 2, but is somewhat simpler.

Proof. In the case of the matrix $D - A + \frac{1}{n} J$ the vector $\mathbf{1}$ is an eigenvector, with eigenvalue 1. To see this note that $(D - A)\,\mathbf{1} = \mathbf{0}$ and $J\,\mathbf{1} = n\,\mathbf{1}$.

If γ_i is any other eigenvector of the Laplacian matrix, then it is also eigenvector of $D - A + \frac{1}{n} J$, with the same eigenvalue:

$$\left(D - A + \frac{1}{n} J\right) \gamma_i = (D - A)\,\gamma_i + \frac{1}{n} J\,\gamma_i = \mu_i\,\gamma_i$$

where relation (14) has been taken into account. □

Corollary 3.1. *If the graph G is connected, then the matrix $D - A + \frac{1}{n} J$ is non-singular, implying that $X = \left(D - A + \frac{1}{n} J\right)^{-1}$ does exist.*

Corollary 3.2. *If the graph G is disconnected, then because of $\mu_{n-1} = 0$, the matrix $D - A + \frac{1}{n} J$ is singular, implying that $X = \left(D - A + \frac{1}{2m} J\right)^{-1}$ does not exist.*

Corollary 3.3. *If the matrix X exists, then its eigenvalues are 1 and $1/\mu_i$, $i = 1, 2, \ldots, n-1$.*

As a direct consequence of $X\,\mathbf{1} = \mathbf{1}$ we have

$$\sum_{i=1}^{n} x_{ij} = 1 \ . \tag{19}$$

Directly from the definition of the matrices X and Y follows

$$X^{-1} = D\,Z^{-1}\left[I + \frac{1}{n}\,Z\,D^{-1}\,J - \frac{1}{2m}\,Z\,J\,D\right]$$

which in view of (11) becomes

$$X^{-1} = D\,Z^{-1}\left[I + \frac{1}{n}\,Z\,D^{-1}\,J - \frac{1}{2m}\,J\,D\right]$$

or

$$Z\,D^{-1} = \left[I + \frac{1}{n}\,Z\,D^{-1}\,J - \frac{1}{2m}\,J\,D\right] X \ . \tag{20}$$

By (20) we obtain

$$\frac{z_{ij}}{d_j} = x_{ij} + \sum_{k=1}^{n} \frac{z_{ik}}{n\,d_k} - \frac{1}{2m} \sum_{k=1}^{n} d_k\,x_{kj}$$

which combined with (8) and by taking into account relation (19) gives:

$$\begin{aligned} r_{ij} &= x_{ii} + \sum_{k=1}^{n} \frac{z_{ik}}{n\,d_k} - \frac{1}{2m} \sum_{k=1}^{n} d_k\,x_{ki} + x_{jj} + \sum_{k=1}^{n} \frac{z_{jk}}{n\,d_k} - \frac{1}{2m} \sum_{k=1}^{n} d_k\,x_{kj} \\ &- 2\,x_{ij} - \sum_{k=1}^{n} \frac{2\,z_{ik}}{n\,d_k} + \frac{1}{m} \sum_{k=1}^{n} d_k\,x_{kj} \\ &= x_{ii} + x_{jj} - 2\,x_{ij} + \sum_{k=1}^{n} \frac{1}{n\,d_k}\,(z_{jk} - z_{ik}) + \frac{1}{2m} \sum_{k=1}^{n} d_k\,(x_{kj} - x_{ki}) \ . \end{aligned}$$

Because of (9)

$$\sum_{k=1}^{n} \frac{1}{n\,d_k}\,(z_{jk} - z_{ik}) + \frac{1}{2m} \sum_{k=1}^{n} d_k\,(x_{kj} - x_{ki}) = 0$$

and thus we arrive at formulas (4) and (5).

Theorem 4. *The resistance matrix R of any connected graph on at least two vertices is non-singular.*

Proof. Supposing that R is singular we get a contradiction.

Suppose thus that there is a non-zero column–vector $\mathbf{g}$, such that $R\,\mathbf{g} = \mathbf{0}$. Let $\mathbf{g} = (g_1, g_2, \ldots, g_n)^t$. Further, let $\mathbf{x} = (x_{11}, x_{22}, \ldots, x_{nn})^t$. Then by (5),

$$\left(\sum_{i=1}^{n} g_i\right) \mathbf{x} + \left(\sum_{i=1}^{n} x_{ii}\, g_i\right) \mathbf{1} - 2\,X\,\mathbf{g} = \mathbf{0} \,. \tag{21}$$

which gives

$$\mathbf{g}^t\, X\, \mathbf{g} = \left(\sum_{i=1}^{n} g_i\right) \left(\sum_{i=1}^{n} x_{ii}\, g_i\right) \,.$$

Since by Theorem 2 the matrix X is positive definite and symmetric, and $\mathbf{g} \neq \mathbf{0}$, it must be

$$\left(\sum_{i=1}^{n} g_i\right) \left(\sum_{i=1}^{n} x_{ii}\, g_i\right) \; > \; 0 \,. \tag{22}$$

From (21) we have for $k = 1, 2, \ldots, n$,

$$2 \sum_{j=1}^{n} x_{kj}\, g_j = \sum_{i=1}^{n} g_i\, x_{kk} + \sum_{i=1}^{n} x_{ii}\, g_i \tag{23}$$

which summed over the index k and in view of (19) yields

$$n \sum_{i=1}^{n} x_{ii}\, g_i = \left(2 - \sum_{i=1}^{n} x_{ii}\right) \sum_{j=1}^{n} g_j \,. \tag{24}$$

Combining (22)–(24) we obtain for $k = 1, 2, \ldots, n$,

$$2 \left(\sum_{j=1}^{n} g_j\right)^{-1} \sum_{i=1}^{n} x_{ki}\, g_i = x_{kk} + \frac{1}{n} \left(2 - \sum_{i=1}^{n} x_{ii}\right) \,.$$

In addition,

$$\begin{aligned}
\left(\sum_{j=1}^{n} g_j\right)^{-1} \mathbf{g} &= \frac{1}{2} X^{-1} \left[\mathbf{x} + \frac{1}{n}\left(2 - \sum_{i=1}^{n} x_{ii}\right) \mathbf{1}\right] \\
&= \frac{1}{2}(D - A)\,\mathbf{x} + \frac{1}{2n}\left(\sum_{i=1}^{n} x_{ii}\right) \mathbf{1} + \frac{1}{2n}\left(2 - \sum_{i=1}^{n} x_{ii}\right) \mathbf{1} \\
&= \frac{1}{2}(D - A)\,\mathbf{x} + \frac{1}{n}\,\mathbf{1}
\end{aligned}$$

and

$$\left(\sum_{j=1}^{n} g_j\right)^{-1} \sum_{i=1}^{n} x_{ii}\, g_i = \frac{1}{2} \mathbf{x}^t (D-A)\, \mathbf{x} + \frac{1}{n} \sum_{i=1}^{n} x_{ii} \,. \tag{25}$$

Hence by (24) and (25) we arrive at

$$\mathbf{x}^t (D-A)\, \mathbf{x} = \frac{4}{n} \left(1 - \sum_{i=1}^{n} x_{ii}\right) \,. \tag{26}$$

The term $\sum_{i=1}^{n} x_{ii}$ is equal to the trace of the matrix X, which by Corollary 3.3 must be greater than unity.

Since the Laplacian matrix $L = D - A$ is semi-positive definite and symmetric, the left-hand side of (26) is greater than or equal to zero. On the other hand, the right-hand side of (26) is negative-valued, contradiction. □

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