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Distinction between modifications of Wiener indices

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Abstract

The Wiener index of a tree T is $W(T) = \frac{1}{2}\sum_{i,j} d(i,j) = \sum_e n_1(e) \cdot n_2(e)$, where $n_1(e)$ and $n_2(e)$ are the numbers of vertices on the two sides of the edge e. Recently, two modifications of this index were put forward: ${}^mW_\lambda(T) = \sum_e (n_1(e) \cdot n_2)^\lambda$ and $W_\lambda(T) = \sum_{i,j}(d(i,j))^\lambda$. We show that for each $\lambda_1, \lambda_2 > 0$, the indices ${}^mW_{\lambda_1}$ and ${}^mW_{\lambda_2}$ are essentially different, more precisely we show there are two chemical trees that are differently ordered by the indices ${}^mW_{\lambda_1}$ and ${}^mW_{\lambda_2}$. We also show that for each $\lambda_1, \lambda_2 > 0$, the indices W_{λ_1} and W_{λ_2} are also essentially different.

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1 Introduction

The molecular-graph-based quantity, known under the name Wiener index or Wiener number, is one of the most useful and most thoroughly studied molecular-structure-descriptors [1-3] and it is still a topic of current research. Details of the theory of Wiener numbers and an exhaustive bibliography is given in the recent reviews [4,5].

A large number of modifications and extensions of the Wiener index was considered in the chemical literature; an extensive list of references on this matter can be found in the reviews [6,7].

The definition of the Wiener index is given by

$$W(G) = \sum_{\{i,j\}} d(i,j)$$

where the summation goes over all unordered pairs of vertices and $d(i,j)$ denotes the distance between vertices i and j. An immediate generalization of the Wiener number is

$$W_\lambda(G) = \sum_{\{i,j\}} (d(i,j))^\lambda$$

where λ is some real number. In an explicit form of this Wiener-type invariant was first put forward in the works [8] and [9].

It is shown in [2] and [3] that the Wiener index of a tree (= a connected acyclic graph) satisfies the relation

$$W(G) = \sum_e n_1(e) \cdot n_2(e)$$

where $n_1(e)$ and $n_2(e)$ are the number of vertices of T lying on the two sides of the edge e and where the summation goes over all edges of T.

This formula inspired another modification

$${}^mW_\lambda(G) = \sum_e (n_1(e) \cdot n_2(e))^\lambda$$

first considered in [10].

It has been shown [8, 10] that for $\lambda > 0$, both modification of the Wiener index are suitable measures of branching. In [10] it is demonstrated that if all trees are ordered with regard to ${}^mW_\lambda(G)$, then in the general case this ordering is different for different λ. This difference in ordering was demonstrated for pairs of trees with different number of vertices and maximal vertex degree much greater than 4, hence for trees that do not pertain to isomers, and that are not molecular graphs.

In this paper, we prove the analogous claim but for pairs of chemical trees with the same number of vertices. (Recall that a chemical tree is a tree in which no vertex has degree greater than 4; such chemical trees are the graph representations of alkanes [2, 3].) Thus, the results communicated in this paper are much stronger than those in [10] and have an immediate chemical relevance. In addition to this, we also show that the analogous statement holds for indices W_λ.

More precisely, we prove:

Theorem 1 *Let $\lambda_1, \lambda_2 \in \mathbb{R}^+$, such that $\lambda_1 \neq \lambda_2$. There are chemical trees G_1 and G_2 with the same number of vertices, such that,*

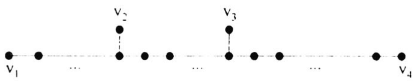
$$\begin{aligned} {}^{m}W_{\lambda_1}(G_1) &> {}^{m}W_{\lambda_1}(G_2) \\ {}^{m}W_{\lambda_2}(G_1) &< {}^{m}W_{\lambda_2}(G_2). \end{aligned}$$

Theorem 2 *Let $\lambda_1, \lambda_2 \in \mathbb{R}^+$, such that $\lambda_1 \neq \lambda_2$. There are chemical trees G_1 and G_2 with the same number of vertices, such that,*

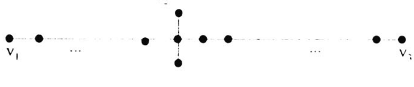
$$\begin{aligned} W_{\lambda_1}(G_1) &> W_{\lambda_1}(G_2) \\ W_{\lambda_2}(G_1) &< W_{\lambda_2}(G_2). \end{aligned}$$

1.1 Proof of the Theorem 1

Let $a, b, c \in \langle 0, \frac{1}{2} \rangle$ be any numbers, let q be the greatest common divisor of their nominators, and let n be an arbitrary natural number. Denote by $G(n, q, a, b)$ the graph whose structure is shown on the following diagram:



where $d(v_1, v_2) = nqa$, $d(v_1, v_3) = nqc$ and $d(v_1, v_4) = nq - 1$. Also, denote by $G(n, q, c)$ the graph given on the following diagram:



where $d(v_1, v_2) = nqb$ and $d(v_1, v_3) = nq - 1$.

We start with a few auxiliary results

Lemma 3 *Let $\lambda \in \mathbb{R}^+$ and let $a, b, c \in \langle 0, \frac{1}{2} \rangle \cap \mathbb{Q}$ such that $a < b < c$ and*

$$2\left(b - b^2\right)^\lambda - \left(a - a^2\right)^\lambda - \left(c - c^2\right)^\lambda \neq 0.$$

There is a sufficiently large $n_0 = n_0(a, b, c, \lambda)$, such that for each natural number $n > n_0$, we have

$$\begin{aligned} &\operatorname{sgn}\left[{}^{m}W_\lambda(G(n, q, a, c)) - {}^{m}W_\lambda(G(n, q, b))\right] = \\ &= \operatorname{sgn}\left[2\left(b - b^2\right)^\lambda - \left(a - a^2\right)^\lambda - \left(c - c^2\right)^\lambda\right]. \end{aligned}$$

Proof. We have,

$$\left|{}^{m}W_\lambda\left(G(n,q,a,c)\right) - {}^{m}W_\lambda\left(G(n,q,b)\right)\right| =$$

$$= \sum_{i=aqn}^{cqn-1} \left[(i+1)(qn+1-i)\right]^\lambda - \sum_{i=aqn}^{bqn-1} \left[i(qn+2-i)\right]^\lambda - \sum_{i=bqn}^{cqn-1} \left[(i+2)(qn-i)\right]^\lambda$$

$$= \sum_{i=aqn}^{cqn-1} \left[iqn - i^2 + qn + 1\right]^\lambda - \sum_{i=aqn}^{bqn-1} \left[iqn - i^2 + 2i\right]^\lambda - \sum_{i=bqn}^{cqn-1} \left[\begin{array}{c} iqn - i^2 + \\ +2qn - 2i \end{array}\right]^\lambda$$

Note that

$$\lim_{n\to\infty} \frac{\sum\limits_{i=aqn}^{cqn-1} \left[iqn - i^2 + qn + 1\right]^\lambda}{\sum\limits_{i=aqn}^{cqn-1} \left[iqn - i^2 + qn\right]^\lambda} = 1.$$

It follows that

$$\lim_{n\to\infty} \frac{\sum\limits_{i=aqn}^{cqn-1} \left[\begin{array}{c} iqn - i^2 + \\ qn+1 \end{array}\right]^\lambda - \sum\limits_{i=aqn}^{bqn-1} \left[iqn - i^2 + 2i\right]^\lambda - \sum\limits_{i=bqn}^{cqn-1} \left[\begin{array}{c} iqn - i^2 + \\ +2qn - 2i \end{array}\right]^\lambda}{\sum\limits_{i=aqn}^{cqn-1} \left[\begin{array}{c} iqn - i^2 \\ +qn \end{array}\right]^\lambda - \sum\limits_{i=aqn}^{bqn-1} \left[iqn - i^2 + 2i\right]^\lambda - \sum\limits_{i=bqn}^{cqn-1} \left[\begin{array}{c} iqn - i^2 + \\ +2qn - 2i \end{array}\right]^\lambda} = 1, \qquad (1)$$

so that there is a sufficiently large n_1, such that for each $n > n_1$,

$$\operatorname{sgn}\left(\sum_{i=aqn}^{cqn-1} \left[\begin{array}{c} iqn - i^2 \\ +qn+1 \end{array}\right]^\lambda - \sum_{i=aqn}^{bqn-1} \left[\begin{array}{c} iqn- \\ i^2 + 2i \end{array}\right]^\lambda - \sum_{i=bqn}^{cqn-1} \left[\begin{array}{c} iqn - i^2 + \\ 2qn - 2i \end{array}\right]^\lambda\right) =$$

$$= \operatorname{sgn}\left(\begin{array}{c} \sum\limits_{i=aqn}^{cqn-1} \left[\begin{array}{c} iqn - i^2 + \\ qn \end{array}\right]^\lambda - \sum\limits_{i=aqn}^{bqn-1} \left[\begin{array}{c} iqn - i^2 \\ +2i \end{array}\right]^\lambda - \\ \sum\limits_{i=bqn}^{cqn-1} \left[\begin{array}{c} iqn - i^2 + \\ 2qn - 2i \end{array}\right]^\lambda \end{array}\right) \qquad (1')$$

or

$$\lim_{n\to\infty} \left(\begin{array}{c} \sum\limits_{i=aqn}^{cqn-1} \left[iqn - i^2 + qn + 1\right]^\lambda - \sum\limits_{i=aqn}^{bqn-1} \left[iqn - i^2 + 2i\right]^\lambda - \\ - \sum\limits_{i=bqn}^{cqn-1} \left[iqn - i^2 + 2qn - 2i\right]^\lambda \end{array}\right) = 0 \qquad (1'')$$

or

$$\lim_{n\to\infty} \left(\begin{array}{c} \sum\limits_{i=aqn}^{cqn-1} \left[iqn - i^2 + qn\right]^\lambda - \sum\limits_{i=aqn}^{bqn-1} \left[iqn - i^2 + 2i\right]^\lambda - \\ - \sum\limits_{i=bqn}^{cqn-1} \left[iqn - i^2 + 2qn - 2i\right]^\lambda \end{array}\right) = 0. \qquad (1''')$$

From (1), it follows that if one of the conditions (1'') and (1''') is fulfilled, then both conditions (1'') and (1''') are fulfilled.

We have

$$\left|\lim_{n\to\infty}\left[\frac{\int\limits_{i=aqn}^{cqn}\left[iqn-i^2+qn\right]^{\lambda}di-\left(\sum\limits_{i=aqn}^{cqn-1}\left[iqn-i^2+qn\right]^{\lambda}\right)}{n^{2\lambda+1}}\right]\right|\leq$$

$$\leq\lim_{n\to\infty}\frac{(cqn-aqn)\cdot\max_{aqn\leq i\leq cqn-1}\left\{\left|\begin{array}{c}\left[iqn-i^2+qn\right]^{\lambda}-\\ \int\limits_{i}^{i+1}\left[xqn-x^2+qn\right]^{\lambda}dx\end{array}\right|\right\}}{n^{2\lambda+1}}$$

$$=\lim_{n\to\infty}\left\{(cq-aq)\cdot\max_{aqn\leq i\leq cqn-1}\left\{\left|\int\limits_{i}^{i+1}\frac{\left(\begin{array}{c}\left[iqn-i^2+qn\right]^{\lambda}-\\ \left[xqn-x^2+qn\right]^{\lambda}\end{array}\right)}{n^{2\lambda}}dx\right|\right\}\right\}$$

$$\leq\lim_{n\to\infty}\left\{(cq-aq)\cdot\max_{aqn\leq i\leq cqn-1}\left\{\max_{i\leq y\leq i+1}\left\{\left|\frac{\left(\begin{array}{c}\left[yqn-y^2+qn\right]^{\lambda}-\\ \left[iqn-i^2+qn\right]^{\lambda}\end{array}\right)}{n^{2\lambda}}\right|\right\}\right\}\right\}.$$

Using the Lagrange theorem of medium value, we get

$$\frac{\left[yqn-y^2+qn\right]^{\lambda}-\left[iqn-i^2+qn\right]^{\lambda}}{y-i}=\lambda\cdot\left[zqn-z^2+qn\right]^{\lambda-1}\cdot(qn-2z),$$

where $z=z(n)$ is some number from the interval $\langle i,y\rangle$. It follows that

$$\lim_{n\to\infty}\frac{\left[yqn-y^2+qn\right]^{\lambda}-\left[iqn-i^2+qn\right]^{\lambda}}{n^{2\lambda}}=$$

$$=\lim_{n\to\infty}\frac{\lambda\cdot\left[zqn-z^2+qn\right]^{\lambda-1}\cdot(qn-2z)\cdot(y-i)}{n^{2\lambda}}=0.$$

Therefore,

$$\left|\lim_{n\to\infty}\left[\frac{\left(\sum\limits_{i=aqn}^{cqn-1}\left[iqn-i^2+qn\right]^{\lambda}\right)-\int\limits_{i=aqn}^{cqn}\left[iqn-i^2+qn\right]^{\lambda}di}{n^{2\lambda+1}}\right]\right|\leq 0,$$

i. e.,

$$\lim_{n\to\infty}\left[\frac{\left(\sum\limits_{i=aqn}^{cqn-1}\left[iqn-i^2+qn\right]^{\lambda}\right)-\int\limits_{i=aqn}^{cqn}\left[iqn-i^2+qn\right]^{\lambda}di}{n^{2\lambda+1}}\right]=0. \tag{2}$$

Denote

$$l_1=\lim_{n\to\infty}\left[\frac{\sum\limits_{i=aqn}^{cqn-1}\left[iqn-i^2+qn\right]^{\lambda}}{n^{2\lambda+1}}\right].$$

We have

$$l_1 = \lim_{n\to\infty}\left[\frac{\sum_{i=aqn}^{cqn-1}\left[iqn - i^2 + qn\right]^{\lambda}}{n^{2\lambda+1}}\right] >$$

$$> \lim_{n\to\infty}\frac{\sum_{i=aqn}^{cqn-1}\left[i\cdot(qn-i)\right]^{\lambda}}{n^{2\lambda+1}}$$

$$\geqslant \lim_{n\to\infty}\frac{(cqn-1-aqn)\cdot\left[aqn\cdot(qn-(cqn-1))\right]^{\lambda}}{n^{2\lambda+1}}$$

$$> 0. \tag{3}$$

From (2) and (3), it follows that

$$\lim_{n\to\infty}\frac{\sum_{i=aqn}^{cqn-1}\left[iqn-i^2+qn\right]^{\lambda}}{\int_{i=aqn}^{cqn}\left[iqn-i^2+qn\right]^{\lambda}di} =$$

$$= \lim_{n\to\infty}\frac{\frac{\sum_{i=aqn}^{cqn-1}\left[iqn-i^2+qn\right]^{\lambda}}{n^{2\lambda+1}}}{\frac{\int_{i=aqn}^{cqn}\left[iqn-i^2+qn\right]^{\lambda}di-\left(\sum_{i=aqn}^{cqn-1}\left[iqn-i^2+qn\right]^{\lambda}\right)}{n^{2\lambda+1}}+\frac{\sum_{i=aqn}^{cqn-1}\left[iqn-i^2+qn\right]^{\lambda}}{n^{2\lambda+1}}}$$

$$= \frac{l_1}{l_1+0} = 1. \tag{4}$$

Analogously, we get

$$\lim_{n\to\infty}\frac{\sum_{i=aqn}^{bqn-1}\left[iqn-i^2+2i\right]^{\lambda}}{\int_{aqn}^{bqn}\left[iqn-i^2+2i\right]^{\lambda}di} = 1 \tag{5}$$

and

$$\frac{\sum_{i=bqn}^{cqn-1}\left[iqn-i^2+2qn-2i\right]}{\int_{bqn}^{cqn}\left[iqn-i^2+2qn-2i\right]^{\lambda}di} = 1.\ . \tag{6}$$

Combining (4), (5) and (6), we get

$$\lim_{n\to\infty}\frac{\sum_{i=aqn}^{cqn-1}\begin{bmatrix} iqn-i^2\\ +qn\end{bmatrix}^{\lambda}-\sum_{i=aqn}^{bqn-1}\begin{bmatrix} iqn-i^2\\ +2i\end{bmatrix}^{\lambda}-\sum_{i=bqn}^{cqn-1}\begin{bmatrix} iqn-i^2+\\ 2qn-2i\end{bmatrix}^{\lambda}}{\left(\int_{aqn}^{cqn}\begin{bmatrix} iqn-i^2\\ +qn\end{bmatrix}^{\lambda}di-\int_{aqn}^{bqn}\begin{bmatrix} iqn-i^2\\ +2i\end{bmatrix}^{\lambda}di-\int_{bqn}^{cqn}\begin{bmatrix} iqn-i^2\\ +2qn-2i\end{bmatrix}^{\lambda}di\right)} = 1, \tag{7}$$

so that there is a sufficiently large n_2, such that for each $n > n_2$,

$$\operatorname{sgn}\left(\sum_{i=aqn}^{cqn-1}\begin{bmatrix} iqn-i^2\\ +qn\end{bmatrix}^{\lambda}-\sum_{i=aqn}^{bqn-1}\begin{bmatrix} iqn-i^2\\ +2i\end{bmatrix}^{\lambda}-\sum_{i=bqn}^{cqn-1}\begin{bmatrix} iqn-i^2+\\ 2qn-2i\end{bmatrix}^{\lambda}\right) =$$

$$= \operatorname{sgn}\left(\int_{aqn}^{cqn}\begin{bmatrix} iqn-i^2\\ +qn\end{bmatrix}^{\lambda}di-\int_{aqn}^{bqn}\begin{bmatrix} iqn-i^2\\ +2i\end{bmatrix}^{\lambda}di-\int_{bqn}^{cqn}\begin{bmatrix} iqn-i^2+\\ 2qn-2i\end{bmatrix}^{\lambda}di\right) \tag{7'}$$

or

$$\lim_{n\to\infty}\left(\begin{array}{c}\sum_{i=aqn}^{cqn-1}\left[iqn-i^2+qn\right]^{\lambda}-\sum_{i=aqn}^{bqn-1}\left[iqn-i^2+2i\right]^{\lambda}-\\-\sum_{i=bqn}^{cqn-1}\left[iqn-i^2+2qn-2i\right]^{\lambda}\end{array}\right)=0 \qquad (7'')$$

or

$$\lim_{n\to\infty}\left(\begin{array}{c}\int_{aqn}^{cqn}\left[iqn-i^2+qn\right]^{\lambda}di-\int_{aqn}^{bqn}\left[iqn-i^2+2i\right]^{\lambda}di-\\-\int_{bqn}^{cqn}\left[iqn-i^2+2qn-2i\right]^{\lambda}di\end{array}\right)=0\,. \qquad (7''')$$

From (7), it follows that if one of the conditions (7'') and (7''') is fulfilled, then both conditions (7'') and (7''') are fulfilled.

Using Taylor's theorem of medium value, we get

$$\begin{aligned}\left(iqn-i^2+qn\right)^{\lambda} &= \left(\begin{array}{c}\left(iqn-i^2\right)^{\lambda}+\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot qn+\\+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_1qn\right)^{\lambda-2}}{2}\cdot(qn)^2\end{array}\right)\\
\left(iqn-i^2+2i\right)^{\lambda} &= \left(\begin{array}{c}\left(iqn-i^2\right)^{\lambda}+\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot 2i+\\+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+2r_2i\right)^{\lambda-2}}{2}\cdot(2i)^2\end{array}\right)\\
\left(iqn-i^2+2qn-2i\right)^{\lambda} &= \left(\begin{array}{c}\left(iqn-i^2\right)^{\lambda}+\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot(2qn-2i)+\\+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_3(2qn-2i)\right)^{\lambda-2}}{2}\cdot(2qn-2i)^2\end{array}\right);\end{aligned}$$

where $r_1=r_1(n)$, $r_2=r_2(n)$ and $r_3=r_3(n)$ are some numbers from the interval $\langle 0,1\rangle$. Then

$$\int_{aqn}^{cqn}\left[\begin{array}{c}iqn-i^2\\+qn\end{array}\right]^{\lambda}di-\int_{aqn}^{bqn}\left[\begin{array}{c}iqn-i^2\\+2i\end{array}\right]^{\lambda}di-\int_{bqn}^{cqn}\left[\begin{array}{c}iqn-i^2\\+2qn-2i\end{array}\right]^{\lambda}di=$$

$$=\int_{aqn}^{cqn}\left(\begin{array}{c}\left(iqn-i^2\right)^{\lambda}+\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot qn+\\+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_1qn\right)^{\lambda-2}}{2}\cdot(qn)^2\end{array}\right)di-$$

$$-\int_{aqn}^{bqn}\left(\begin{array}{c}\left(iqn-i^2\right)^{\lambda}+\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot 2i+\\+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+2r_2i\right)^{\lambda-2}}{2}\cdot(2i)^2\end{array}\right)di-$$

$$-\int_{bqn}^{cqn}\left(\begin{array}{c}\left(iqn-i^2\right)^{\lambda}+\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot(2qn-2i)+\\+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_3(2qn-2i)\right)^{\lambda-2}}{2}\cdot(2qn-2i)^2\end{array}\right)di$$

$$=\int_{aqn}^{cqn}\left(\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot qn+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_1qn\right)^{\lambda-2}}{2}\cdot(qn)^2\right)di-$$

$$-\int_{aqn}^{bqn}\left(\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot 2i+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+2r_2i\right)^{\lambda-2}}{2}\cdot(2i)^2\right)di-$$

$$-\int_{bqn}^{cqn}\left(\begin{array}{c}\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot(2qn-2i)+\\+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_3(2qn-2i)\right)^{\lambda-2}}{2}\cdot(2qn-2i)^2\end{array}\right)di.$$

Now, note that

$$\lim_{n\to\infty} \frac{\left| \int\limits_{aqn}^{cqn} \left(\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_1qn\right)^{\lambda-2}}{2} \cdot (qn)^2 \right) di \right|}{n^{2\lambda}} \leq$$

$$\leq (cq-aq) \cdot \max_{i\in\langle aqn,cqn\rangle} \left\{ \left| \frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_1qn\right)^{\lambda-2}}{2n^{2\lambda-1}} \cdot (qn)^2 \right| \right\} = 0.$$

Denote

$$l_2 = \lim_{n\to\infty} \frac{\int\limits_{aqn}^{cqn} \lambda\cdot\left(iqn-i^2\right)^{\lambda-1} \cdot qn \cdot di}{n^{2\lambda}}.$$

Then we have

$$\begin{aligned} l_2 &= \lim_{n\to\infty} \frac{\int\limits_{aqn}^{cqn} \lambda\cdot\left(iqn-i^2\right)^{\lambda-1} \cdot qn \cdot di}{n^{2\lambda}} \geqslant \\ &\geqslant \lim_{n\to\infty} \frac{(cqn-aqn)\cdot\lambda\cdot aqn\cdot \min\left\{ \begin{array}{c} \left(aq^2n^2-a^2q^2n^2\right)^{\lambda-1}, \\ \left(cq^2n^2-c^2q^2n^2\right)^{\lambda-1} \end{array} \right\}}{n^{2\lambda}} > 0, \end{aligned}$$

so that

$$\begin{aligned} &\lim_{n\to\infty} \frac{\int\limits_{aqn}^{cqn} \left(\lambda\cdot\left(iqn-i^2\right)^{\lambda-1} \cdot qn + \frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_1qn\right)^{\lambda-2}}{2} \cdot (qn)^2 \right) di}{\int\limits_{aqn}^{cqn} \lambda\cdot\left(iqn-i^2\right)^{\lambda-1} \cdot qn \cdot di} = \\ &= \lim_{n\to\infty} \frac{\frac{\int_{aqn}^{cqn} \lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot qn\, di}{n^{2\lambda}} + \frac{\int_{aqn}^{cqn} \frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_1qn\right)^{\lambda-2}}{2}\cdot(qn)^2 di}{n^{2\lambda}}}{\frac{\int_{aqn}^{cqn} \lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot qn\cdot di}{n^{2\lambda}}} \\ &= \frac{l_2+0}{l_2} = 1. \end{aligned} \tag{8}$$

Analogously, we get

$$\lim_{n\to\infty} \frac{\int\limits_{aqn}^{bqn} \left(\lambda\cdot\left(iqn-i^2\right)^{\lambda-1} \cdot 2i + \frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_2i\right)^{\lambda-2}}{2} \cdot (2i)^2 \right) di}{\int\limits_{aqn}^{bqn} \lambda\cdot\left(iqn-i^2\right)^{\lambda-1} \cdot 2i \cdot di} = 1 \tag{9}$$

and

$$\lim_{n\to\infty} \frac{\int\limits_{bqn}^{cqn} \left(\begin{array}{c} \lambda\cdot\left(iqn-i^2\right)^{\lambda-1} \cdot (2qn-2i) + \\ + \frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_3(2qn-2i)\right)^{\lambda-2}}{2} \cdot (2qn-2i)^2 \end{array} \right) di}{-\int\limits_{bqn}^{cqn} \lambda\cdot\left(iqn-i^2\right)^{\lambda-1} \cdot (2qn-2i) \cdot di} = 1. \tag{10}$$

From (8), (9) and (10), it follows

$$\lim_{n\to\infty}\frac{\left(\begin{array}{l}\int\limits_{aqn}^{cqn}\left(\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot qn+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_1qn\right)^{\lambda-2}}{2}\cdot(qn)^2\right)di-\\ -\int\limits_{aqn}^{bqn}\left(\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot 2i+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+2r_2i\right)^{\lambda-2}}{2}\cdot(2i)^2\right)di-\\ -\int\limits_{bqn}^{cqn}\left(\begin{array}{c}\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot(2qn-2i)+\\ +\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_3(2qn-2i)\right)^{\lambda-2}}{2}\cdot(2qn-2i)^2\end{array}\right)di\end{array}\right)}{\left(\begin{array}{c}\int\limits_{aqn}^{cqn}\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot qn\cdot di-\int\limits_{aqn}^{bqn}\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot 2i\cdot di-\\ -\int\limits_{bqn}^{cqn}\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot(2qn-2i)\cdot di\end{array}\right)}=1, \qquad (11)$$

so that there is a sufficiently large n_3, such that for each $n>n_3$:

$$\operatorname{sgn}\left(\begin{array}{l}\int\limits_{aqn}^{cqn}\left(\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot qn+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_1qn\right)^{\lambda-2}}{2}\cdot(qn)^2\right)di-\\ -\int\limits_{aqn}^{bqn}\left(\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot 2i+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+2r_2i\right)^{\lambda-2}}{2}\cdot(2i)^2\right)di-\\ -\int\limits_{bqn}^{cqn}\left(\begin{array}{c}\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot(2qn-2i)+\\ +\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_3(2qn-2i)\right)^{\lambda-2}}{2}\cdot(2qn-2i)^2\end{array}\right)di\end{array}\right)=$$

$$\operatorname{sgn}\left(\begin{array}{c}\int\limits_{aqn}^{cqn}\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot qn\cdot di-\int\limits_{aqn}^{bqn}\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot 2i\cdot di-\\ -\int\limits_{bqn}^{cqn}\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot(2qn-2i)\cdot di\end{array}\right) \qquad (11')$$

or

$$\lim_{n\to\infty}\left(\begin{array}{l}\int\limits_{aqn}^{cqn}\left(\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot qn+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_1qn\right)^{\lambda-2}}{2}\cdot(qn)^2\right)di-\\ -\int\limits_{aqn}^{bqn}\left(\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot 2i+\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+2r_2i\right)^{\lambda-2}}{2}\cdot(2i)^2\right)di-\\ -\int\limits_{bqn}^{cqn}\left(\begin{array}{c}\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot(2qn-2i)+\\ +\frac{\lambda\cdot(\lambda-1)\cdot\left(iqn-i^2+r_3(2qn-2i)\right)^{\lambda-2}}{2}\cdot(2qn-2i)^2\end{array}\right)di\end{array}\right)=0 \qquad (11'')$$

or

$$\lim_{n\to\infty}\left(\begin{array}{c}\int\limits_{aqn}^{cqn}\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot qn\cdot di-\int\limits_{aqn}^{bqn}\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot 2i\cdot di-\\ -\int\limits_{bqn}^{cqn}\lambda\cdot\left(iqn-i^2\right)^{\lambda-1}\cdot(2qn-2i)\cdot di\end{array}\right)=0. \qquad (11''')$$

From (11), it follows that if one of the conditions (11″) and (11‴) is fulfilled, then both conditions (11″) and (11‴) are fulfilled.

Now, we have

$$\begin{pmatrix} \int\limits_{aqn}^{cqn} \lambda \cdot \left(iqn - i^2\right)^{\lambda-1} \cdot qn \cdot di - \int\limits_{aqn}^{bqn} \lambda \cdot \left(iqn - i^2\right)^{\lambda-1} \cdot 2i \ \ di - \\ - \int\limits_{bqn}^{cqn} \lambda \cdot \left(iqn - i^2\right)^{\lambda-1} \cdot (2qn - 2i) \cdot di \end{pmatrix} =$$

$$= \int\limits_{aqn}^{bqn} \lambda \cdot \left(iqn - i^2\right)^{\lambda-1} \cdot (qn - 2i) \cdot di - \int\limits_{bqn}^{cqn} \lambda \cdot \left(iqn - i^2\right)^{\lambda-1} \cdot (qn - 2i) \cdot di$$

$$= \lambda \int\limits_{aq^2n^2 - a^2q^2n^2}^{bq^2n^2 - b^2q^2n^2} x^{\lambda-1} dx - \lambda \int\limits_{bq^2n^2 - b^2q^2n^2}^{cq^2n^2 - c^2q^2n^2} x^{\lambda-1} dx =$$

$$= 2\left(bq^2n^2 - b^2q^2n^2\right)^{\lambda} - \left(aq^2n^2 - a^2q^2n^2\right)^{\lambda} - \left(cq^2n^2 - c^2q^2n^2\right)^{\lambda}.$$

Note that

$$\operatorname{sgn}\left[2\left(bq^2n^2 - b^2q^2n^2\right)^{\lambda} - \left(aq^2n^2 - a^2q^2n^2\right)^{\lambda} - \left(cq^2n^2 - c^2q^2n^2\right)^{\lambda}\right] =$$

$$= \operatorname{sgn}\left[2\left(b - b^2\right)^{\lambda} - \left(a - a^2\right)^{\lambda} - \left(c - c^2\right)^{\lambda}\right]. \tag{12}$$

Since

$$2\left(b - b^2\right)^{\lambda} - \left(a - a^2\right)^{\lambda} - \left(c - c^2\right)^{\lambda} \neq 0,$$

it follows that

$$\begin{pmatrix} \int\limits_{aqn}^{cqn} \lambda \cdot \left(iqn - i^2\right)^{\lambda-1} \cdot qn \cdot di - \int\limits_{aqn}^{bqn} \lambda \cdot \left(iqn - i^2\right)^{\lambda-1} \cdot 2i \cdot di - \\ - \int\limits_{bqn}^{cqn} \lambda \cdot \left(iqn - i^2\right)^{\lambda-1} \cdot (2qn - 2i) \cdot di \end{pmatrix} \neq 0.$$

Therefore the condition $(11''')$ is not fulfilled, and then $(11'')$ is also not fulfilled. It follows that $(7''')$ is not fulfilled, but then $(7'')$ is also not fulfilled. This implies that $(1''')$ is not fulfilled. Therefore, $(1')$, $(7')$, $(11')$ and $(12')$ are fulfilled.

From $(1')$, $(7')$, $(11')$ and $(12')$, it follows that for each $n > \max\{n_1, n_2, n_3\}$, we have

$$\operatorname{sgn}\left[{}^{m}W_{\lambda}(n, q, a, c) - {}^{m}W_{\lambda}(n, q, b)\right] =$$

$$= \operatorname{sgn}\left[2\left(b - b^2\right)^{\lambda} - \left(a - a^2\right)^{\lambda} - \left(c - c^2\right)^{\lambda}\right],$$

so it is sufficient to take $n_0 = \max\{n_1, n_2, n_3\}$. ■

Lemma 4 *Let $\lambda_1, \lambda_2 \in \mathbb{R}^+$, such that $\lambda_1 \neq \lambda_2$. There exist numbers $a, b, c \in \left\langle 0, \frac{1}{2}\right\rangle \cap \mathbb{Q}$, such that $a < b < c$ and either*

$$\begin{aligned} 2\left(b - b^2\right)^{\lambda_1} - \left(a - a^2\right)^{\lambda_1} - \left(c - c^2\right)^{\lambda_1} &< 0 \\ 2\left(b - b^2\right)^{\lambda_2} - \left(a - a^2\right)^{\lambda_2} - \left(c - c^2\right)^{\lambda_2} &> 0 \end{aligned}$$

or

$$\begin{aligned} 2\left(b - b^2\right)^{\lambda_1} - \left(a - a^2\right)^{\lambda_1} - \left(c - c^2\right)^{\lambda_1} &> 0 \\ 2\left(b - b^2\right)^{\lambda_2} - \left(a - a^2\right)^{\lambda_2} - \left(c - c^2\right)^{\lambda_2} &< 0. \end{aligned}$$

Proof. Let a_0 and c_0 be any real numbers, such that $a_0 < c_0$. Note that the function $f_1 : \left\langle 0, \frac{1}{2}\right\rangle \to \mathbb{R}$, defined by $f_1(x) = 2\left(x - x^2\right)^{\lambda_1}$, is monotonically increasing. Then also the function $f_2 : \left\langle 0, \frac{1}{2}\right\rangle \to \mathbb{R}$, defined by

$$f_2(x) = 2\left(x - x\right)^{\lambda_2} - \left(a_0 - a_0^2\right)^{\lambda_2} - \left(c_0 - c_0^2\right)^{\lambda_2}$$

is monotonically increasing, and $f(a_0) < 0$ and $f(c_0) > 0$. It follows that there is a number $b_0 \in \langle a_0, c_0\rangle$ such that $f_2(b_0) = 0$.

Let us prove that

$$2\left(b_0 - b_0^2\right)^{\lambda_2} - \left(a_0 - a_0^2\right)^{\lambda_2} - \left(c_0 - c_0^2\right)^{\lambda_2} \neq 0.$$

Suppose to the contrary that

$$2\left(b_0 - b_0^2\right)^{\lambda_2} - \left(a_0 - a_0^2\right)^{\lambda_2} - \left(c_0 - c_0^2\right)^{\lambda_2} = 0.$$

It follows that

$$b_0 - b_0^2 = \sqrt[\lambda_2]{\frac{\left(a_0 - a_0^2\right)^{\lambda_2} + \left(c_0 - c_0^2\right)^{\lambda_2}}{2}}. \tag{1}$$

From $f_2(b_0) = 0$, it follows that

$$b_0 - b_0^2 = \sqrt[\lambda_1]{\frac{\left(a_0 - a_0^2\right)^{\lambda_1} + \left(c_0 - c_0^2\right)^{\lambda_1}}{2}}. \tag{2}$$

Relations (1) and (2) imply that

$$\sqrt[\lambda_1]{\frac{\left(a_0 - a_0^2\right)^{\lambda_1} + \left(c_0 - c_0^2\right)^{\lambda_1}}{2}} = \sqrt[\lambda_2]{\frac{\left(a_0 - a_0^2\right)^{\lambda_2} + \left(c_0 - c_0^2\right)^{\lambda_2}}{2}}.$$

Note that $a_0 - a_0^2 \neq c_0 - c_0^2$, but then the latter equation cannot hold. Therefore,

$$2\left(b_0 - b_0^2\right)^{\lambda_2} - \left(a_0 - a_0^2\right)^{\lambda_2} - \left(c_0 - c_0^2\right)^{\lambda_2} \neq 0.$$

Define the functions $f_3, f_4 : \langle 0, \frac{1}{2}\rangle \times \langle 0, \frac{1}{2}\rangle \times \langle 0, \frac{1}{2}\rangle \to \mathbb{R}$ by

$$\begin{aligned} f_3(x, y, z) &= \left(x - x^2\right)^{\lambda_1} + \left(y - y^2\right)^{\lambda_1} + \left(z - z^2\right)^{\lambda_1} \\ f_4(x, y, z) &= \left(x - x^2\right)^{\lambda_2} + \left(y - y^2\right)^{\lambda_2} + \left(z - z^2\right)^{\lambda_2}. \end{aligned}$$

Distinguish two cases:

1) CASE 1: $2\left(b_0 - b_0^2\right)^{\lambda_2} - \left(a_0 - a_0^2\right)^{\lambda_2} - \left(c_0 - c_0^2\right)^{\lambda_2} > 0$.

Since f_4 is continuos, there is a sufficiently small $\varepsilon > 0$, such that

$$\begin{aligned} 2\left((b_0 - \varepsilon) - (b_0 - \varepsilon)^2\right)^{\lambda_2} - \left(a_0 - a_0^2\right)^{\lambda_2} - \left(c_0 - c_0^2\right)^{\lambda_2} &> 0 \\ b_0 - \varepsilon &> a. \end{aligned}$$

We have

$$\begin{aligned} f_3(a_0, b_0 - \varepsilon, c_0) &< 0 \\ f_4(a_0, b_0 - \varepsilon, c_0) &> 0. \end{aligned}$$

Since f_3 and f_4 are continuos functions, there are rational numbers a, b and c sufficiently close to a_0, $b_0 - \varepsilon$ and c_0, such that

$$\begin{aligned} f_3(a, b, c) &< 0 \\ f_4(a, b, c) &> 0, \\ a, b, c &\in \left\langle 0, \frac{1}{2}\right\rangle \\ a &< b < c \end{aligned}$$

which proves the claim in this case.

1) CASE 2: $2\left(b_0 - b_0^2\right)^{\lambda_2} - \left(a_0 - a_0^2\right)^{\lambda_2} - \left(c_0 - c_0^2\right)^{\lambda_2} < 0$.

Since f_3 is continuous, there is a sufficiently small $\varepsilon > 0$ such that

$$\begin{aligned} 2\left(\left(b_0+\varepsilon\right)-\left(b_0+\varepsilon\right)^2\right)^{\lambda_2}-\left(a_0-a_0^2\right)^{\lambda_2}-\left(c_0-c_0^2\right)^{\lambda_2} &> 0 \\ b_0+\varepsilon &< c_0. \end{aligned}$$

We now have

$$\begin{aligned} f_3\left(a_0, b_0+\varepsilon, c_0\right) &> 0 \\ f_4\left(a_0, b_0+\varepsilon, c_0\right) &< 0. \end{aligned}$$

Since f_3 and f_4 are continuous functions, there exist rational numbers a, b and c, sufficiently close to $a_0, b_0+\varepsilon$ and c_0, such that

$$\begin{aligned} f_3(a,b,c) &> 0 \\ f_4(a,b,c) &< 0 \\ a,b,c &\in \left\langle 0, \frac{1}{2}\right\rangle \\ a &< b < c, \end{aligned}$$

which proves the claim in this case. ■

Now, we can prove Theorem 1. From Lemma 4, it follows that there are numbers $a, b, c \in \langle 0, \frac{1}{2}\rangle \cap \mathbb{Q}$, such that $a < b < c$ and

$$\begin{aligned} 2\left(b-b^2\right)^{\lambda_1}-\left(a-a^2\right)^{\lambda_1}-\left(c-c^2\right)^{\lambda_1} &< 0 \\ 2\left(b-b^2\right)^{\lambda_2}-\left(a-a^2\right)^{\lambda_2}-\left(c-c^2\right)^{\lambda_2} &> 0 \end{aligned}$$

or

$$\begin{aligned} 2\left(b-b^2\right)^{\lambda_1}-\left(a-a^2\right)^{\lambda_1}-\left(c-c^2\right)^{\lambda_1} &> 0 \\ 2\left(b-b^2\right)^{\lambda_2}-\left(a-a^2\right)^{\lambda_2}-\left(c-c^2\right)^{\lambda_2} &< 0. \end{aligned}$$

Without loss of generality, we may assume that

$$\begin{aligned} 2\left(b-b^2\right)^{\lambda_1}-\left(a-a^2\right)^{\lambda_1}-\left(c-c^2\right)^{\lambda_1} &> 0 \\ 2\left(b-b^2\right)^{\lambda_2}-\left(a-a^2\right)^{\lambda_2}-\left(c-c^2\right)^{\lambda_2} &< 0. \end{aligned}$$

From Lemma 3 it follows that there is a number $n_0(a,b,c,\lambda_1)$, such that for each $n > n_0(a,b,c,\lambda_1)$,

$$\begin{aligned} &\operatorname{sgn}\left[{}^mW_{\lambda_1}\left(G(n,q,a,c)\right)-{}^mW_{\lambda_1}\left(G(n,q,b)\right)\right]= \\ =\ &\operatorname{sgn}\left[2\left(b-b^2\right)^{\lambda_1}-\left(a-a^2\right)^{\lambda_1}-\left(c-c^2\right)^{\lambda_1}\right]. \end{aligned}$$

Therefore, for each $n > n_0(a,b,c,\lambda_1)$, we have

$${}^mW_{\lambda_1}(n,q,a,c)-{}^mW_{\lambda_1}(n,q,b) > 0 .$$

From the same Lemma, it follows that there is a number $n_0(a,b,c,\lambda_2)$, such that for each $n > n_0(a,b,c,\lambda_2)$,

$$\begin{aligned} &\operatorname{sgn}\left[{}^mW_{\lambda_2}\left(G(n,q,a,c)\right)-{}^mW_{\lambda_2}\left(G(n,q,b)\right)\right]= \\ =\ &\operatorname{sgn}\left[2\left(b-b^2\right)^{\lambda_2}-\left(a-a^2\right)^{\lambda_2}-\left(c-c^2\right)^{\lambda_2}\right]. \end{aligned}$$

Therefore, for each $n > n_0(a,b,c,\lambda_2)$,

$${}^mW_{\lambda_2}\left(G(n,q,a,c)\right)-{}^mW_{\lambda_2}\left(G(n,q,b)\right) < 0$$

and

$$\left(\begin{array}{c} {}^{m}W_{\lambda_1} \left(G \left(\max \left\{ n_0 \left(a, b, c, \lambda_1 \right), n_0 \left(a, b, c, \lambda_2 \right) \right\} + 1, q, a, c \right) \right) - \\ - {}^{m}W_{\lambda_1} \left(G \left(\max \left\{ n_0 \left(a, b, c, \lambda_1 \right), n_0 \left(a, b, c, \lambda_2 \right) \right\} + 1, q, b \right) \right) \end{array} \right) > 0$$

$$\left(\begin{array}{c} {}^{m}W_{\lambda_2} \left(G \left(\max \left\{ n_0 \left(a, b, c, \lambda_1 \right), n_0 \left(a, b, c, \lambda_2 \right) \right\} + 1, q, a, c \right) \right) - \\ - {}^{m}W_{\lambda_2} \left(G \left(\max \left\{ n_0 \left(a, b, c, \lambda_1 \right), n_0 \left(a, b, c, \lambda_2 \right) \right\} + 1, q, b \right) \right) \end{array} \right) < 0 .$$

In view of this, it is sufficient to take

$$\begin{aligned} G_1 &= G \left(\max \left\{ n_0 \left(a, b, c, \lambda_1 \right), n_0 \left(a, b, c, \lambda_2 \right) \right\} + 1, q, a, c \right) \\ G_2 &= G \left(\max \left\{ n_0 \left(a, b, c, \lambda_1 \right), n_0 \left(a, b, c, \lambda_2 \right) \right\} + 1, q, b \right) . \end{aligned}$$

2 Proof of the Theorem 2

We start with a few auxiliary results.

Lemma 5 *Let $p, q \in \mathbb{R}$, such that $p \neq q$ and $p, q > 1$, and let $c_1 \in \left\langle 0, \frac{1}{2} \right\rangle$. There is a number $a_1 \in \left\langle 0, \frac{1}{2} \right\rangle$, such that*

$$\frac{(1 - a_1)^{p-1} - a_1^{p-1}}{c_1^{p-1} - (1 - c_1)^{p-1}} \neq \frac{(1 - a_1)^{q-1} - a_1^{q-1}}{c_1^{q-1} - (1 - c_1)^{q-1}} .$$

Proof. Suppose, to the contrary, that for each $a_1 \in \left\langle 0, \frac{1}{2} \right\rangle$, we have

$$\frac{(1 - a_1)^{p-1} - a_1^{p-1}}{c_1^{p-1} - (1 - c_1)^{p-1}} \neq \frac{(1 - a_1)^{q-1} - a_1^{q-1}}{c_1^{q-1} - (1 - c_1)^{q-1}}$$

This is equivalent to the requirement that for each $a_1 \in \left\langle 0, \frac{1}{2} \right\rangle$,

$$\frac{a_1^{p-1} - (1 - a_1)^{p-1}}{a_1^{q-1} - (1 - a_1)^{q-1}} = \frac{c_1^{p-1} - (1 - c_1)^{p-1}}{c_1^{q-1} - (1 - c_1)^{q-1}} .$$

Note that the function $f_1 : \left\langle 0, \frac{1}{2} \right\rangle \to \mathbb{R}$, defined by

$$f_1 (a_1) = \frac{a_1^{p-1} - (1 - a_1)^{p-1}}{a_1^{q-1} - (1 - a_1)^{q-1}} .$$

is a constant. Therefore $f_1' (a_1) = 0$ for each $a_1 \in \left\langle 0, \frac{1}{2} \right\rangle$, and consequently,

$$\left[\begin{array}{c} \left[(p - 1) a_1^{p-2} + (p - 1)(1 - a_1)^{p-2} \right] \cdot \left[a_1^{q-1} - (1 - a_1)^{q-1} \right] - \\ - \left[(q - 1) a_1^{q-2} + (q - 1)(1 - a_1)^{q-2} \right] \cdot \left[a_1^{p-1} - (1 - a_1)^{p-1} \right] \end{array} \right] = 0$$

for each $a_1 \in \left\langle 0, \frac{1}{2} \right\rangle$, i. e.,

$$\begin{aligned} \frac{a_1^{p-2} + (1 - a_1)^{p-2}}{a_1^{q-2} + (1 - a_1)^{q-2}} &= \frac{q - 1}{p - 1} \cdot \frac{a_1^{p-1} - (1 - a_1)^{p-1}}{a_1^{q-1} - (1 - a_1)^{q-1}} \\ \frac{a_1^{p-2} + (1 - a_1)^{p-2}}{a_1^{q-2} + (1 - a_1)^{q-2}} &= \frac{q - 1}{p - 1} \cdot \frac{c_1^{p-1} - (1 - c_1)^{p-1}}{c_1^{q-1} - (1 - c_1)^{q-1}} . \end{aligned}$$

for each $a_1 \in \left\langle 0, \frac{1}{2} \right\rangle$. It follows that the function $f_2 : \left\langle 0, \frac{1}{2} \right\rangle \to \mathbb{R}$, defined by

$$f_2 (a_1) = \frac{a_1^{p-2} + (1 - a_1)^{p-2}}{a_1^{q-2} + (1 - a_1)^{q-2}}$$

is also a constant on the interval $\langle 0, \frac{1}{2} \rangle$, but

$$\lim_{a_1 \to 0} \frac{a_1^{p-2} + (1 - a_1)^{p-2}}{a_1^{q-2} + (1 - a_1)^{q-2}} = 1$$

and

$$\lim_{a_1 \to \frac{1}{2}} \frac{a_1^{p-2} + (1 - a_1)^{p-2}}{a_1^{q-2} + (1 - a_1)^{q-2}} = \frac{2 \cdot (\frac{1}{2})^{p-2}}{2 \cdot (\frac{1}{2})^{q-2}} = \left(\frac{1}{2}\right)^{p-q} \neq 1$$

which is a contradiction. ■

Lemma 6 *Let $p, q \in \mathbb{R}$, such that $p \neq q$ and $p, q > 1$. There are numbers $a, b, c \in \langle 0, \frac{1}{2} \rangle$, such that either*

$$\begin{aligned} a^p + (1-a)^p + (1-c)^p + c^p - 2b^p - 2(1-b)^p &> 0 \\ a^q + (1-a)^q + (1-c)^q + c^q - 2b^q - 2(1-b)^q &< 0 \end{aligned}$$

or

$$\begin{aligned} a^p + (1-a)^p + (1-c)^p + c^p - 2b^p - 2(1-b)^p &< 0 \\ a^q + (1-a)^q + (1-c)^q + c^q - 2b^q - 2(1-b)^q &> 0. \end{aligned}$$

Proof. Let a_1 and c_1 be the numbers that satisfy the conditions of the previous lemma. Consider the function $f_3 : \langle 0, \frac{1}{2} \rangle \to \mathbb{R}$, defined as

$$f_3(x) = x^p + (1-x)^p.$$

Note that

$$f_3'(x) = px^{p-1} - p(1-x)^{p-1} < 0,$$

which means that f_3 is a strictly decreasing function. Now, consider the function $f_4 : \langle 0, \frac{1}{2} \rangle \to \mathbb{R}$, defined by

$$f_4(b_1) = a_1^p + (1-a_1)^p + (1-c_1)^p + c_1^p - 2b_1^p - 2(1-b_1)^p.$$

It follows that f_4 is a strictly increasing function, that

$$\lim_{b_1 \to 0} f_4(b_1) < 0$$

and

$$\lim_{b_1 \to \frac{1}{2}} f_4(b_1) > 0.$$

Therefore there exists a number $b_2 \in \langle 0, \frac{1}{2} \rangle$, such that

$$f_4(b_2) = 0.$$

We have to distinguish three cases:
CASE 1: $a_1^q + (1-a_1)^q + (1-c_1)^q + c_1^q - 2b_2^q - 2(1-b_2)^q > 0$.
Denote by $f_5 : \langle 0, \frac{1}{2} \rangle \to \mathbb{R}$ the function defined by

$$f_5(x) = a_1^q + (1-a_1)^q + (1-c_1)^q + c_1^q - 2x^q - 2(1-x)^q.$$

Obviously, $f_5(b_2) > 0$. Since f_5 is a continuos function and f_4 is a strictly increasing function, for sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned} a_1^p + (1-a_1)^p + (1-c_1)^p + c_1^p - 2(b_2-\varepsilon)^p - 2(1-(b_2-\varepsilon))^p &< 0 \\ a_1^q + (1-a_1)^q + (1-c_1)^q + c_1^q - 2(b_2-\varepsilon)^q - 2(1-(b_2-\varepsilon))^q &> 0 \\ b_2 - \varepsilon &> 0. \end{aligned}$$

Taking $a = a_1, b = b_2 - \varepsilon$ and $c = c_1$, we prove the claim.
CASE 2: $a_1^q + (1-a_1)^q + (1-c_1)^q + c_1^q - 2b_2^q - 2(1-b_2)^q < 0$.

Denote by $f_5 : \langle 0, \frac{1}{2} \rangle \to \mathbb{R}$ the function defined by

$$f_5(x) = a_1^q + (1-a_1)^q + (1-c_1)^q + c_1^q - 2x^q - 2(1-x)^q .$$

Obviously, $f_5(b_2) < 0$. Since f_5 is continuos and f_4 is strictly increasing, for sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned} a_1^p + (1-a_1)^p + (1-c_1)^p + c_1^p - 2(b_2+\varepsilon)^p - 2(1-(b_2+\varepsilon))^p &> 0 \\ a_1^q + (1-a_1)^q + (1-c_1)^q + c_1^q - 2(b_2+\varepsilon)^q - 2(1-(b_2+\varepsilon))^q &< 0 \\ b_2 + \varepsilon &< \frac{1}{2}. \end{aligned}$$

Taking $a = a_1$, $b = b_2 + \varepsilon$, $c = c_1$, we prove the claim.
CASE 3: $a_1^q + (1-a_1)^q + (1-c_1)^q + c_1^q - 2b_2^q - 2(1-b_2)^q = 0$.
Distinguish 2 subcases:
SUBCASE 3.1: $\frac{(1-a_1)^{p-1} - a_1^{p-1}}{c_1^{p-1} - (1-c_1)^{p-1}} > \frac{(1-a_1)^{q-1} - a_1^{q-1}}{c_1^{q-1} - (1-c_1)^{q-1}}$.
There is a number k, such that

$$\frac{(1-a_1)^{p-1} - a_1^{p-1}}{c_1^{p-1} - (1-c_1)^{p-1}} > k > \frac{(1-a_1)^{q-1} - a_1^{q-1}}{c_1^{q-1} - (1-c_1)^{q-1}}.$$

Define the function

$$f_6 : \left[0, \min\left\{\frac{\frac{1}{2} - c_1}{k}, \frac{1}{2} - a_1\right\}\right\rangle \to \mathbb{R}$$

by

$$f_6(x) = (a_1 + x)^p + (1-(a_1+x))^p + (c_1 + kx)^p + (1-(c_1+kx))^p - 2b_2^p - 2(1-b_2)^p .$$

We have

$$f_6'(x) = p(a_1+x)^{p-1} - p(1-(a_1+x))^{p-1} + pk(c_1+kx)^{p-1} - pk(1-(c_1+kx))^{p-1},$$

and therefore

$$\begin{aligned} &\lim_{x \to 0^+} f_6'(x) = \\ &= pa_1^{p-1} - p(1-a_1)^{p-1} + \underbrace{\left[pc_1^{p-1} - p(1-c_1)^{p-1}\right]}_{<0} k \\ &> pa_1^{p-1} - p(1-a_1)^{p-1} + \left[pc_1^{p-1} - p(1-c_1)^{p-1}\right] \cdot \frac{(1-a_1)^{p-1} - a_1^{p-1}}{c_1^{p-1} - (1-c_1)^{p-1}} \\ &= 0 . \end{aligned}$$

Define the function

$$f_7 : \left[0, \min\left\{\frac{\frac{1}{2} - c_1}{k}, \frac{1}{2} - a_1\right\}\right\rangle \to \mathbb{R}$$

by

$$f_7(x) = (a_1 + x)^q + (1-(a_1+x))^q + (c_1 + kx)^q + (1-(c_1+kx))^q - 2b_2^q - 2(1-b_2)^q .$$

We have

$$f_7'(x) = q(a_1+x)^{q-1} - q(1-(a_1+x))^{q-1} + qk(c_1+kx)^{q-1} - qk(1-(c_1+kx))^{q-1},$$

implying

$$\begin{aligned} &\lim_{x \to 0^+} f_7'(x) \\ &= qa_1^{q-1} - q(1-a_1)^{q-1} + \underbrace{\left[qc_1^{q-1} - q(1-c_1)^{q-1}\right]}_{<0} k \\ &< qa_1^{q-1} - q(1-a_1)^{q-1} + \left[qc_1^{q-1} - q(1-c_1)^{q-1}\right] \cdot \frac{(1-a_1)^{q-1} - a_1^{q-1}}{c_1^{q-1} - (1-c_1)^{q-1}} \\ &= 0. \end{aligned}$$

Therefore there is a sufficiently small ε, such that for each $x \in \langle 0, \varepsilon \rangle$,

$$\begin{aligned} f_6'(x) &> 0 \\ f_7'(x) &< 0. \end{aligned}$$

Since

$$f_6(0) = f_7(0) = 0,$$

for each $x_0 \in \langle 0, \varepsilon \rangle$,

$$\begin{aligned} f_6(x_0) &> 0 \\ f_7(x_0) &< 0, \end{aligned}$$

i. e.,

$$\begin{aligned} \left(\begin{array}{c} (a_1 + x_0)^p + (1 - (a_1 + x_0))^p + (c_1 + kx_0)^p + \\ (1 - (c_1 + kx_0))^p - 2b_2^p - 2(1 - b_2)^p \end{array} \right) &> 0 \\ \left(\begin{array}{c} (a_1 + x_0)^q + (1 - (a_1 + x_0))^q + (c_1 + kx_0)^q + \\ (1 - (c_1 + kx_0))^q - 2b_2^q - 2(1 - b_2)^q \end{array} \right) &< 0, \end{aligned}$$

so by taking $a = a_1 + x_0$, $b = b_2$ and $c = c_1 + kx_0$ for any $x_0 \in \langle 0, \varepsilon \rangle$ we prove the claim.

SUBCASE 3.2: $\frac{(1-a_1)^{p-1} - a_1^{p-1}}{c_1^{p-1} - (1-c_1)^{p-1}} < \frac{(1-a_1)^{q-1} - a_1^{q-1}}{c_1^{q-1} - (1-c_1)^{q-1}}$.

There is a number k such that

$$\frac{(1 - a_1)^{p-1} - a_1^{p-1}}{c_1^{p-1} - (1 - c_1)^{p-1}} < k < \frac{(1 - a_1)^{q-1} - a_1^{q-1}}{c_1^{q-1} - (1 - c_1)^{q-1}}.$$

Define the function

$$f_6 : \left[0, \min\left\{\frac{\frac{1}{2} - c_1}{k}, \frac{1}{2} - a_1\right\}\right\rangle \to \mathbb{R}$$

by

$$f_6(x) = (a_1 + x)^p + (1 - (a_1 + x))^p + (c_1 + kx)^p + (1 - (c_1 + kx))^p - 2b_2^p - 2(1 - b_2)^p.$$

We have

$$f_6'(x) = p(a_1 + x)^{p-1} - p(1 - (a_1 + x))^{p-1} + pk(c_1 + kx)^{p-1} - pk(1 - (c_1 + kx))^{p-1},$$

and thus

$$\begin{aligned} &\lim_{x \to 0^+} f_6'(x) = \\ &= pa_1^{p-1} - p(1 - a_1)^{p-1} + \underbrace{\left[pc_1^{p-1} - p(1 - c_1)^{p-1}\right]}_{<0} k \\ &< pa_1^{p-1} - p(1 - a_1)^{p-1} + \left[pc_1^{p-1} - p(1 - c_1)^{p-1}\right] \frac{(1 - a_1)^{p-1} - a_1^{p-1}}{c_1^{p-1} - (1 - c_1)^{p-1}} \\ &= 0. \end{aligned}$$

Define the function

$$f_7 : \left[0, \min\left\{\frac{\frac{1}{2} - c_1}{k}, \frac{1}{2} - a_1\right\}\right\rangle \to \mathbb{R}$$

as

$$f_7(x) = (a_1 + x)^q + (1 - (a_1 + x))^q + (c_1 + kx)^q + (1 - (c_1 + kx))^q - 2b_2^q - 2(1 - b_2)^q.$$

We then have

$$f_7'(x) = q(a_1 + x)^{q-1} - q(1 - (a_1 + x))^{q-1} + qk(c_1 + kx)^{q-1} - qk(1 - (c_1 + kx))^{q-1}.$$

and thus

$$\begin{aligned}
&\lim_{x\to 0^+} f_7'(x) \\
&= q a_1^{q-1} - q(1-a_1)^{q-1} + \underbrace{\left[q c_1^{q-1} - q(1-c_1)^{q-1}\right]}_{<0} k \\
&> q a_1^{q-1} - q(1-a_1)^{q-1} + \left[q c_1^{q-1} - q(1-c_1)^{q-1}\right] \cdot \frac{(1-a_1)^{q-1} - a_1^{q-1}}{c_1^{q-1} - (1-c_1)^{q-1}} \\
&= 0.
\end{aligned}$$

Therefore, there is a sufficiently small ε, such that for each $x \in \langle 0, \varepsilon \rangle$,

$$\begin{aligned}
f_6'(x) &< 0 \\
f_7'(x) &> 0.
\end{aligned}$$

Since

$$f_6(0) = f_7(0) = 0,$$

for each $x_0 \in \langle 0, \varepsilon \rangle$, we have

$$\begin{aligned}
f_6(x_0) &< 0 \\
f_7(x_0) &> 0,
\end{aligned}$$

i. e.,

$$\begin{aligned}
\left(\begin{array}{c} (a_1 + x_0)^p + (1 - (a_1 + x_0))^p + (c_1 + kx_0)^p + \\ (1 - (c_1 + kx_0))^p - 2b_2^p - 2(1 - b_2)^p \end{array} \right) &< 0 \\
\left(\begin{array}{c} (a_1 + x_0)^q + (1 - (a_1 + x_0))^q + (c_1 + kx_0)^q + \\ (1 - (c_1 + kx_0))^q - 2b_2^q - 2(1 - b_2)^q \end{array} \right) &> 0,
\end{aligned}$$

so by taking $a = a_1 + x_0$, $b = b_2$ and $c = c_1 + kx_0$ for any $x_0 \in \langle 0, \varepsilon \rangle$, we prove the claim. ■

Because the functions $f_7, f_8 : \langle 0, \frac{1}{2} \rangle \times \langle 0, \frac{1}{2} \rangle \times \langle 0, \frac{1}{2} \rangle$, defined by

$$\begin{aligned}
f_7(a, b, c) &= a^p + (1-a)^p + (1-c)^p + c^p - 2b^p - 2(1-b)^p \\
f_8(a, b, c) &= a^q + (1-a)^q + (1-c)^q + c^q - 2b^q - 2(1-b)^q
\end{aligned}$$

are continuos, from the last lemma it directly follows:

Lemma 7 *Let $p, q \in \mathbb{R}$, such that $p \neq q$ and $p, q > 1$. Then there are are numbers $a, b, c \in \langle 0, \frac{1}{2} \rangle \cap \mathbb{Q}$, such that either*

$$\begin{aligned}
a^p + (1-a)^p + (1-c)^p + c^p - 2b^p - 2(1-b)^p &> 0 \\
a^q + (1-a)^q + (1-c)^q + c^q - 2b^q - 2(1-b)^q &< 0
\end{aligned}$$

or

$$\begin{aligned}
a^p + (1-a)^p + (1-c)^p + c^p - 2b^p - 2(1-b)^p &< 0 \\
a^q + (1-a)^q + (1-c)^q + c^q - 2b^q - 2(1-b)^q &> 0.
\end{aligned}$$

Denote the graphs $G(n, q, a, b)$ and $G(n, q, c)$ as in the previous section.

Now, we can prove Theorem 2. Replacing $p = \lambda_1 + 1$ and $q = \lambda_2 + 1$ in the previous Lemma, we get that there exist numbers $a, b, c \in \langle 0, \frac{1}{2} \rangle \cap \mathbb{Q}$, such that either

$$\begin{aligned}
a^{\lambda_1+1} + (1-a)^{\lambda_1+1} + (1-c)^{\lambda_1+1} + c^{\lambda_1+1} - 2b^{\lambda_1+1} - 2(1-b)^{\lambda_1+1} &> 0 \\
a^{\lambda_2+1} + (1-a)^{\lambda_2+1} + (1-c)^{\lambda_2+1} + c^{\lambda_2+1} - 2b^{\lambda_2+1} - 2(1-b)^{\lambda_2+1} &< 0
\end{aligned}$$

or

$$\begin{aligned} a^{\lambda_1+1}+(1-a)^{\lambda_1+1}+(1-c)^{\lambda_1+1}+c^{\lambda_1+1}-2b^{\lambda_1+1}-2(1-b)^{\lambda_1+1} &< 0 \\ a^{\lambda_2+1}+(1-a)^{\lambda_2+1}+(1-c)^{\lambda_2+1}+c^{\lambda_2+1}-2b^{\lambda_2+1}-2(1-b)^{\lambda_2+1} &> 0. \end{aligned}$$

Without loss of generality, we may assume that there are numbers $a, b, c \in \left\langle 0, \frac{1}{2} \right\rangle \cap \mathbb{Q}$, such that

$$\begin{aligned} a^{\lambda_1+1}+(1-a)^{\lambda_1+1}+(1-c)^{\lambda_1+1}+c^{\lambda_1+1}-2b^{\lambda_1+1}-2(1-b)^{\lambda_1+1} &> 0 \\ a^{\lambda_2+1}+(1-a)^{\lambda_2+1}+(1-c)^{\lambda_2+1}+c^{\lambda_2+1}-2b^{\lambda_2+1}-2(1-b)^{\lambda_2+1} &< 0. \end{aligned}$$

Note that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{W_{\lambda_1}(G(n,q,a,b)) - W_{\lambda_1}(G(n,q,c))}{\begin{pmatrix} a^{\lambda_1+1}+(1-a)^{\lambda_1+1}+(1-c)^{\lambda_1+1}+ \\ +c^{\lambda_1+1}-2b^{\lambda_1+1}-2(1-b)^{\lambda_1+1} \end{pmatrix}\cdot (nq)^{\lambda_1+1}} = \\
&= \lim_{n\to\infty} \frac{\begin{pmatrix} \sum\limits_{i=1}^{nqa} i^{\lambda_1} + \sum\limits_{i=1}^{nq-(nqa-1)} i^{\lambda_1} + \sum\limits_{i=1}^{nqc} i^{\lambda_1} + \sum\limits_{i=1}^{nq-(nqc-1)} i^{\lambda_1} \\ -2\sum\limits_{i=1}^{nqb} i^{\lambda_1} - 2\sum\limits_{i=1}^{nq-(nqb-1)} i^{\lambda_1} \end{pmatrix}}{\begin{pmatrix} a^{\lambda_1+1}+(1-a)^{\lambda_1+1}+(1-c)^{\lambda_1+1}+ \\ +c^{\lambda_1+1}-2b^{\lambda_1+1}-2(1-b)^{\lambda_1+1} \end{pmatrix}\cdot (nq)^{\lambda_1+1}} \\
&> \lim_{n\to\infty} \frac{\begin{pmatrix} \int\limits_0^{nqa} x^{\lambda_1}dx + \int\limits_0^{nq-nqa+1} x^{\lambda_1}dx + \int\limits_0^{nqc} x^{\lambda_1}dx + \\ + \int\limits_0^{nq-nqc+1} x^{\lambda_1}dx - 2\int\limits_1^{nqb+1} x^{\lambda_1}dx - 2\int\limits_1^{nq-nqb+2} x^{\lambda_1}dx \end{pmatrix}}{\begin{pmatrix} a^{\lambda_1+1}+(1-a)^{\lambda_1+1}+(1-c)^{\lambda_1+1}+ \\ +c^{\lambda_1+1}-2b^{\lambda_1+1}-2(1-b)^{\lambda_1+1} \end{pmatrix}\cdot (nq)^{\lambda_1+1}} \\
&= \lim_{n\to\infty} \frac{\begin{pmatrix} \frac{(nqa)^{\lambda_1+1}}{\lambda_1+1} + \frac{(nq-nqa+1)^{\lambda_1+1}}{\lambda_1+1} + \frac{(nqc)^{\lambda_1+1}}{\lambda_1+1} + \frac{(nq-nqc+1)^{\lambda_1+1}}{\lambda_1+1} \\ -2\frac{(nqb+1)^{\lambda_1+1}}{\lambda_1+1} + \frac{2}{\lambda_1+1} - 2\frac{(nq-nqb+2)^{\lambda_1+1}}{\lambda_1+1} + \frac{2}{\lambda_1+1} \end{pmatrix}}{\begin{pmatrix} a^{\lambda_1+1}+(1-a)^{\lambda_1+1}++c^{\lambda_1+1}+ \\ +(1-c)^{\lambda_1+1}-2b^{\lambda_1+1}-2(1-b)^{\lambda_1+1} \end{pmatrix}\cdot (nq)^{\lambda_1+1}} \\
&= \frac{1}{\lambda_1+1}.
\end{aligned}$$

Thus, there exists a sufficiently large $n_1 \in \mathbb{N}$, such that for each $n > n_1$,

$$\frac{W_{\lambda_1}(G(n,q,a,b)) - W_{\lambda_1}(G(n,q,c))}{\begin{pmatrix} a^{\lambda_1+1}+(1-a)^{\lambda_1+1}+(1-c)^{\lambda_1+1}+ \\ c^{\lambda_1+1}-2b^{\lambda_1+1}-2(1-b)^{\lambda_1+1} \end{pmatrix}\cdot (nq)^{\lambda_1+1}} > 0.$$

It follows that for each $n > n_1$,

$$W_{\lambda_1}(G(n,q,a,b)) - W_{\lambda_1}(G(n,q,c)) > 0.$$

Also, note that

$$\lim_{n\to\infty} \frac{W_{\lambda_2}(G(n,q,a,b)) - W_{\lambda_2}(G(n,q,c))}{\left(\begin{array}{c} a^{\lambda_2+1} + (1-a)^{\lambda_2+1} + (1-c)^{\lambda_2+1} + \\ +c^{\lambda_2+1} - 2b^{\lambda_2+1} - 2(1-b)^{\lambda_2+1} \end{array}\right) \cdot (nq)^{\lambda_2+1}} =$$

$$= \lim_{n\to\infty} \frac{\left(\begin{array}{c} \sum_{i=1}^{nqa} i^{\lambda_2} + \sum_{i=1}^{nq-(nqa-1)} i^{\lambda_2} + \sum_{i=1}^{nqc} i^{\lambda_2} + \\ \sum_{i=1}^{nq-(nqc-1)} i^{\lambda_2} - 2\sum_{i=1}^{nqb} i^{\lambda_2} - 2\sum_{i=1}^{nq-(nqb-1)} i^{\lambda_2} \end{array}\right)}{\left(\begin{array}{c} a^{\lambda_2+1} + (1-a)^{\lambda_2+1} + (1-c)^{\lambda_2+1} + \\ c^{\lambda_2+1} - 2b^{\lambda_2+1} - 2(1-b)^{\lambda_2+1} \end{array}\right) \cdot (nq)^{\lambda_2+1}}$$

$$> \lim_{n\to\infty} \frac{\left(\begin{array}{c} \int_1^{nqa+1} x^{\lambda_2}dx + \int_1^{nq-nqa+2} x^{\lambda_2}dx + \int_1^{nqc+1} x^{\lambda_2}dx + \\ \int_1^{nq-nqc+2} x^{\lambda_2}dx - 2\int_0^{nqb} x^{\lambda_2}dx - 2\int_0^{nq-nqb+1} x^{\lambda_2}dx \end{array}\right)}{\left(\begin{array}{c} a^{\lambda_2+1} + (1-a)^{\lambda_2+1} + (1-c)^{\lambda_2+1} + \\ c^{\lambda_2+1} - 2b^{\lambda_2+1} - 2(1-b)^{\lambda_2+1} \end{array}\right) \cdot (nq)^{\lambda_2+1}}$$

$$= \lim_{n\to\infty} \frac{\left(\begin{array}{c} \frac{(nqa+1)^{\lambda_2+1}}{\lambda_2+1} - \frac{1}{\lambda_2+1} + \frac{(nq-nqa+2)^{\lambda_2+1}}{\lambda_2+1} \\ -\frac{1}{\lambda_2+1} + \frac{(nqc+1)^{\lambda_2+1}}{\lambda_2+1} - \frac{1}{\lambda_2+1} + \\ +\frac{(nq-nqc+2)^{\lambda_2+1}}{\lambda_2+1} - \frac{1}{\lambda_2+1} - 2\frac{(nqb)^{\lambda_2+1}}{\lambda_2+1} - 2\frac{(nq-nqb+1)^{\lambda_2+1}}{\lambda_2+1} \end{array}\right)}{\left(\begin{array}{c} a^{\lambda_2+1} + (1-a)^{\lambda_2+1} + +c^{\lambda_2+1} + \\ (1-c)^{\lambda_2+1} - 2b^{\lambda_2+1} - 2(1-b)^{\lambda_2+1} \end{array}\right) \cdot (nq)^{\lambda_2+1}}$$

$$= \frac{1}{\lambda_2+1},$$

so that there is a sufficiently large $n_2 \in \mathbb{N}$, such that for each $n > n_2$, we have

$$\frac{W_{\lambda_1}(G(n,q,a,b)) - W_{\lambda_1}(G(n,q,c))}{\left(\begin{array}{c} a^{\lambda_2+1} + (1-a)^{\lambda_2+1} + (1-c)^{\lambda_2+1} + \\ c^{\lambda_2+1} - 2b^{\lambda_2+1} - 2(1-b)^{\lambda_2+1} \end{array}\right) \cdot (nq)^{\lambda_2+1}} > 0.$$

It follows that for each $n > n_2$,

$$W_{\lambda_2}(G(n,q,a,b)) - W_{\lambda_2}(G(n,q,c)) < 0.$$

Therefore,

$$\begin{array}{rcl} W_{\lambda_1}(G(\max\{n_1,n_2\}+1,q,a,b)) & > & W_{\lambda_1}(G(\max\{n_1,n_2\}+1,q,c)) \\ W_{\lambda_2}(G(\max\{n_1,n_2\}+1,q,a,b)) & < & W_{\lambda_2}(G(\max\{n_1,n_2\}+1,q,c)). \end{array}$$

so that it is sufficient to take

$$\begin{array}{rcl} G_1 & = & W_{\lambda_1}(G(\max\{n_1,n_2\}+1,q,a,b)) \\ G_2 & = & W_{\lambda_2}(G(\max\{n_1,n_2\}+1,q,c)). \end{array}$$

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