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Enumeration Under Finite Group Action, Basic Tools, Results and Methods

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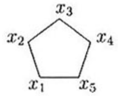
Abstract

This contribution to the special issue on applications of tables of marks in chemistry contains a review of the basics of enumeration theory under finite group action, including tables of marks. For more mathematical details the reader is referred to the article [7] and the book [5]. For applications to chemistry the articles in the present issue of MATCH are recommended as well as the books [3] and [2].

1 Finite Group Actions

Let G denote a multiplicative group and X a nonempty set. For example, X may be the set of sites of a molecular skeleton, while G means the symmetry group of the skeleton. Throughout the paper we shall consider the following very easy example: $X = \{x_1, \ldots, x_5\}$ is the set of vertices of a regular pentagon,



and G will be the group of rotations of it, in mathematical terms, $G = C_5$, the cyclic group of order 5.

An *action* of G on X (for example, the action of the symmetry group G on the set of sites X) is described by a mapping

$$G \times X \to X \colon (g, x) \mapsto gx,$$

such that, for each $x \in X$ and any $g, g' \in G$, the following holds:

1.1 $$g(g'x) = (gg')x, \text{ and } 1x = x,$$

for $x \in X, g, g' \in G$ and the identity element $1 \in G$. In the case of the action of the symmetry group on the set of sites, gx will be the site of the skeleton that replaces the site x (in space) after the application of the symmetry element g. We abbreviate this by saying that G *acts* on X or by writing

$$_GX,$$

since G acts from the *left* on X. It is clear that analogously actions of groups on sets *from the right* can be defined and applied. A second formulation of the conditions 1.1 is:

1.2 Lemma *The mapping*

$$\delta \colon G \to S_X \colon g \mapsto \bar{g},$$

from G to the symmetric group S_X on X, where $\bar{g} \colon x \mapsto gx$, is a homomorphism, a permutation representation *of G on X.*

We shall call $\bar{g}$ the permutation *induced by g on X* and $\bar{G} := \delta(G)$ the permutation group *induced by G on X*. The *kernel* of δ, i.e. the set of elements which are mapped onto the identity mapping id_X (the unit element of S_X), will be denoted as follows:

$$G_X := \ker(\delta) := \{g \mid \bar{g} = \delta(g) = \mathrm{id}_X\} = \{g \mid \forall\, x \in X \colon\ gx = x\}.$$

Being the kernel of a homomorphism, it is a *normal* subgroup of G, and the homomorphism theorem yields the *isomorphism*

1.3 $$\varphi \colon G/G_X \simeq \bar{G} \colon g \cdot G_X \mapsto \bar{g}.$$

An action of G on X has first of all the following property which is immediate from the two conditions 1.1 mentioned in its definition:

1.4 $$gx = x' \iff x = g^{-1}x'.$$

${}_GX$ induces several structures on X and G, and *it is the close arithmetic and algebraic connection between these structures which makes the concept of group action so efficient.* To begin with, the action induces the following equivalence relation on X:

$$x \sim_G x' \ :\iff \exists\, g \in G\colon\ x' = gx.$$

The equivalence classes $G(x) := \{gx \mid g \in G\}$ are called *orbits.* Please note that each element can be reached from every other element of this orbit, and so this orbit is in fact the orbit of every element of it under the action of G. As $\sim_G$ is an equivalence relation on X, a *transversal* T of the orbits, which means a complete set of representatives of the equivalence classes, yields a *set partition* of X, i.e. a complete dissection of X into the pairwise disjoint and nonempty subsets $G(t)$:

$$X = \dot{\bigcup}_{t\in T} G(t).$$

The *set of all orbits* will be denoted by

$$G\backslash\!\backslash X := \{G(t) \mid t \in T\} = \{G(x) \mid x \in X\},$$

and we shall denote the *set* of all their transversals by

$$\mathcal{T}(G\backslash\!\backslash X) := \{T \mid T \text{ is a transversal of } G\backslash\!\backslash X\}.$$

In the case when both G and X are finite, we call the action a *finite action.* We notice that, according to 1.3, for each finite G-set X, we may also assume without loss of generality that G is finite. If G has exactly one orbit on X, i.e. if and only if $G\backslash\!\backslash X = \{X\}$, then we say that the action is *transitive*, or that G acts *transitively* on the set X.

As it was mentioned above, an action of G on X yields a partition of X. It is trivial but very important to note that also the converse is true: Each set partition of

X (and therefore each equivalence relation on X, too) gives rise to an action of a certain group G on X as follows. Let, for an index set I, X_i, $i \in I$, denote the blocks of the set partition (the classes of the equivalence relation) in question, i.e. the X_i are nonempty, pairwise disjoint, and their union is equal to X. Then the following subgroup of the symmetric group S_X acts in a natural way on X and it has the X_i as its orbits:

1.5 $$\bigoplus_i S_{X_i} := \{\pi \in S_X \mid \forall\, i \in I\colon\ \pi X_i = X_i\},$$

where $\pi X_i := \{\pi x \mid x \in X_i\}$. Summarizing our considerations in two sentences, we have obtained:

1.6 Corollary *An action of a group G on a set X is equivalent to a permutation representation of G on X and it yields a set partition of X into orbits. Conversely, each set partition of (or equivalence relation on) X corresponds in a natural way to an action of a certain subgroup of the symmetric group S_X which has the blocks of the partition (or the equivalence classes) as its orbits.*

The preceding result shows that

> **all the structures in mathematics and sciences that can be defined as equivalence classes on sets can be described as orbits of groups.**

Prominent examples are linear codes, designs, graphs, switching functions, physical states and chemical isomers.

To the orbits $G(x)$, which are *subsets* of X, there correspond certain *subgroups* of G. For each $x \in X$ we introduce its *stabilizer*:

$$G_x := \{g \in G \mid gx = x\},$$

the subgroup of elements of G that *stabilize* the *point* x.

The last one of the fundamental concepts induced by an action of G on X is that of *fixed points*. A point $x \in X$ is said to be *fixed* under $g \in G$ if and only if $gx = x$, and the set of all the fixed points of g is indicated by

$$X_g := \{x \in X \mid gx = x\}.$$

More generally, for any subset $S \subseteq G$, we put

$$X_S := \{x \in X \mid \forall\, g \in S\colon\ gx = x\}.$$

The following bunch of examples will show that various important group theoretical structures can be considered as orbits or stabilizers:

1.7 Application (conjugacy classes, centralizers, cosets in groups) If G denotes a group, then

- G acts on itself by *left multiplication*:

  $$G \times G \to G\colon (g, x) \mapsto g \cdot x.$$

  This action is called the *(left) regular representation* of G, it is obviously transitive, and all the stabilizers are equal to the identity subgroup $\{1\}$.

- If we restrict attention to the subgroup U, then we obtain the action

  $$U \times G \to G\colon (u, x) \mapsto u \cdot x$$

  of U on G and the orbits of the elements are the *right cosets* $U(g) = Ug$ of the subgroup U. The stabilizers are trivial again: $U_g = \{1\}$. Correspondingly, we obtain the *left cosets* gU of U as orbits, if we consider the *right* regular representation.

  This shows that *different right cosets as well as different left cosets are disjoint* and that both *the set of right cosets and the set of left cosets of U is a set partition of G*.

- G acts on itself by *conjugation*:

  $$G \times G \to G\colon (g, x) \mapsto g \cdot x \cdot g^{-1}.$$

The orbits of this action are the *conjugacy classes* of elements,

$$G(x) = C^G(x) := \{gxg^{-1} \mid g \in G\},$$

and the stabilizers are the *centralizers* of elements:

$$G_x = C_G(x) := \{g \in G \mid gxg^{-1} = x\}.$$

An immediate consequence is that *different conjugacy classes of elements are disjoint,* since they are orbits, *they also form a set partition of G.* Moreover, *centralizers of elements are subgroups,* they are stabilizers.

- If U denotes a subgroup of G (in short: $U \leq G$), then G acts on the set $G/U := \{xU \mid x \in G\}$ of its *left cosets* as follows:

  $$G \times G/U \to G/U \colon (g, xU) \mapsto gxU.$$

  This action is transitive, and the stabilizer of xU is the subgroup xUx^{-1} which is conjugate to U.

  Thus xUx^{-1} *is a subgroup,* too.

- G acts on the set $L(G) := \{U \mid U \leq G\}$ of all its subgroups by *conjugation*:

  $$G \times L(G) \to L(G) \colon (g, U) \mapsto gUg^{-1}.$$

  The orbits of this action are the *conjugacy classes of subgroups*, and the stabilizers are the *normalizers* :

  $$G(U) = \widetilde{U} := \{\ gUg^{-1} \mid g \in G\},$$

  and

  $$G_U = N_G(U) := \{g \in G \mid gU = Ug\}.$$

  Hence *different conjugacy classes of subgroups are disjoint, and normalizers of subgroups are also subgroups.*

$\diamond$

Returning to the general case we first state the main (and obvious) properties of the stabilizers of elements belonging to the same orbit:

1.8 $$G_{gx} = gG_xg^{-1}, \widetilde{G_x} = \{gG_xg^{-1} \mid g \in G\} = \{G_{x'} \mid x' \in G(x)\}.$$

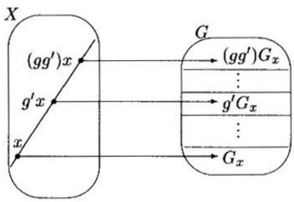
It means that the total set of stabilizers $G_{gx}, g \in G$, of the elements in the orbit $G(x)$ forms the conjugacy class $\widetilde{G_x}$ of G_x (recall the last item of the examples given in 1.7).

2 Orbits, Cosets and Double Cosets

The crucial point is the following natural bijection between the orbit of x and the set of left cosets of its stabilizer G_x :

2.1 The Fundamental Lemma *The mapping* $G(x) \to G/G_x: gx \mapsto gG_x$ *is a bijection between the orbit* $G(x)$ *and the set of left cosets*

$$G/G_x = \{gG_x \mid g \in G\}.$$



This bijection allows us to replace the elements of the orbit by subsets of the acting group. The group is, of course, usually much bigger, but it carries a very helpful algebraic structure that can be used both for enumerative and for constructive purposes.

2.2 Corollary *The length of the orbit is the index of the stabilizer:*

$$|G(x)| = |G/G_x|.$$

In particular, if $|G|$ is finite, then $|G(x)| = |G|/|G_x|$, and so the length of each orbit is a divisor of the group order in this finite case.

2.3 Application (divisibilities in groups) An application to the examples given in 1.7 yields: If G is finite, $g \in G$, and $U \leq G$, then the order of U divides the order of G :

$$|U| \text{ divides } |G|,$$

and the orders of the conjugacy classes of elements and of subgroups satisfy the following equations:

$$|C^G(g)| = |G|/|C_G(g)| \ , \text{ and } \ |\widetilde{U}| = |G|/|N_G(U)|.$$

◇

Besides these divisibility results we note that the mapping

$$\varphi: G(x) \to G/G_x: gx \mapsto gG_x$$

commutes with the action of G, $\varphi(gx) = g\varphi(x)$. This shows that in fact the restricted action of G on the orbit $G(x)$,

$$G \times G(x) \to G(x): (g, hx) \mapsto ghx,$$

is essentially the same — in a sense which we are going to describe next — as the action of G on G/G_x via left multiplication of the cosets:

2.4 $$G \times G/G_x \to G/G_x: (g, hG_x) \mapsto ghG_x.$$

This leads to the question of a suitable concept of morphism between actions of groups. Two actions $G \times X \to X$ and $G \times Y \to Y$ will be called *similar* iff they differ only by a bijection $\theta: X \to Y$ between the sets which satisfies $g\theta(x) = \theta(gx)$. We indicate this in the following way:

$$_GX \approx {}_GY.$$

An important special case follows directly from 2.1:

2.5 Corollary *If ${}_GX$ is an action then, for any $x \in X$, the action of G on the orbit $G(x)$ is similar to the action of G on the set of left cosets of the stabilizer G_x,*

$$ {}_GG(x) \approx {}_G(G/G_x). $$

¿From a given action we can derive various other actions in a natural way, e.g. ${}_GX$ yields ${}_{\bar{G}}X$, $\bar{G}$ being the homomorphic image of G in S_X, which was already mentioned. We also obtain the *subactions* ${}_GM$ on subsets $M \subseteq X$ which are nonempty unions of orbits. Furthermore there are the *restrictions* ${}_UX$ to the subgroups U of G. As the orbits of ${}_GX$ are unions of orbits of ${}_UX$, the comparison of subactions and restrictions is a suitable way of generalizing or specializing structures if they can be defined as orbits. The following example will show what is meant by this.

2.6 Applications (bilateral classes, cosets and double cosets) Let U denote a subgroup of the direct product $G \times G$. Then U acts on G as follows:

$$ U \times G \to G: ((a,b), g) \mapsto agb^{-1}. $$

The orbits $U(g) = \{agb^{-1} \mid (a,b) \in U\}$ of this action are called the *bilateral classes* of G with respect to U (this notion was introduced by Hässelbarth, Ruch, Klein and Seligman, see [4] and its motivation [11]). We note that therefore *different bilateral classes, being orbits, are disjoint.* By specializing U we obtain various interesting group theoretical structures some of which have already been mentioned:

- If A is a subgroup of G, then both $A \times \{1\}$ and $\{1\} \times A$ are subgroups of $G \times G$. Their orbits are the subsets

  $$ (A \times \{1\})(g) = Ag, $$

  the *right cosets* of A in G, and

  $$ (\{1\} \times A)(g) = gA, $$

  the *left cosets* of A in G.

- If B denotes a second subgroup of G, then we can put U equal to the subgroup $A \times B$, obtaining as orbits the (A, B)–*double cosets* of G:

$$(A \times B)(g) = AgB := \{agb \mid a \in A, b \in B\}.$$

- Another subgroup of $G \times G$ is its *diagonal* subgroup

$$\Delta(G \times G) := \{(g, g) \mid g \in G\}.$$

  Its orbits are the conjugacy classes:

$$\Delta(G \times G)(g) = \{g'gg'^{-1} \mid g' \in G\} = C^G(g).$$

Hence left and right cosets, double cosets and conjugacy classes turn out to be special cases of bilateral classes. Being orbits, two left cosets, two right cosets, two double cosets and two conjugacy classes are either equal or disjoint and the stabilizer of an element is a subgroup of the group in question. Moreover, in the finite case, the order of each of these orbits is equal to the index of the stabilizer. We have mentioned this in connection with conjugacy classes and centralizers of elements already, here is the consequence for double cosets: Since the stabilizer of $g \in G$ in $A \times B$ is

$$(A \times B)_g = \{(gbg^{-1}, b) \mid b \in B, gbg^{-1} \in A\} = A \cap gBg^{-1},$$

we obtain, for finite groups G with subgroups A and B,

2.7 $$|AgB| = \frac{|A||B|}{|A \cap gBg^{-1}|},$$

and if D denotes a transversal of the set $A \backslash G / B$ of (A, B)-double cosets, then

2.8 $$|G| = \sum_{g \in D} |AgB| = \sum_{g \in D} \frac{|A||B|}{|A \cap gBg^{-1}|}.$$

◇

Having double cosets now at hand, we can formulate another very interesting and useful consequence of 2.1 and 2.5:

2.9 Breaking of Symmetry *If ${}_GX$ is an action and U a subgroup of G, then for each $x \in X$ we have the following bijection between the orbits of U on $G(x)$ and the set of (U, G_x)-double cosets:*

$$\psi_x : U \backslash\backslash G(x) \to U \backslash G / G_x : U(gx) \mapsto UgG_x,$$

with respect to the action

$$U \times G(x) \to G(x) : (u, gx) \mapsto ugx.$$

We shall return to that later, since it is of enormous importance, in particular for constructive purposes. It was intensively used for the construction of unlabeled graphs, representatives of isometry classes of linear codes, isomorphism types of designs, chemical isomers, physical states, and it can be applied to very many other cases, too. The method used is the following application of the Breaking of Symmetry 2.9.

2.10 Application (a method for the construction of finite unlabeled structures)

- Define the desired set of finite unlabeled structures as the set of classes of an equivalence relation on the set X of labeled structures.

- Replace the equivalence relation by a suitable action of a group U on X, so that the structures in question correspond to the set of orbits $U \backslash\backslash X$.

- In order to reduce complexity by restricting attention to a suitable subset of $U \backslash\backslash X$, introduce a suitable bigger group $G \supset U$ and an action ${}_GX$ for which ${}_UX$ is the restriction of ${}_GX$ to U.

- Restrict attention to the orbits $G(x)$. Evaluate a transversal T_G of $G \backslash\backslash X$, and for each $x \in T_G$ construct a transversal T_x of $U \backslash G / G_x$.

- Retranslate the elements of the T_x into elements of X and obtain this way *a transversal T_U of the equivalence classes $U \backslash\backslash X$, i.e. of the desired structures in* *question:*

$$T_U := \bigcup_{x \in T_G} \psi_x^{-1}(T_x).$$

◇

2.11 Example (Symmetry Classes of Mappings) Now we are going to introduce an action, derived from ${}_GX$, which forms our *paradigmatic example*, since it is most relevant for applications in chemistry and has many other applications, too. It is used in several other contributions to this special issue of MATCH.

In order to prepare this, we form the set of all the mappings from X into another set Y (the reader may think of X as the set of sites of a molecular skeleton, and as Y being a set of admissible types of substituents that can be attached to these sites):

$$Y^X := \{f \mid f : X \to Y\}.$$

Let us introduce the following action on Y^X induced by the given action of G (which will be the symmetry group of the skeleton, say) on this set of mappings:

$$G \times Y^X \to Y^X \colon (g, f) \mapsto f \circ \bar{g}^{-1},$$

i.e. (g, f) is mapped onto $\tilde{f}$, where

$$\tilde{f}(x) := \left(f \circ \bar{g}^{-1}\right)(x) = f(g^{-1}x).$$

The orbits of G on Y^X will be called *symmetry classes of mappings*. ◇

3 The Number of Orbits

The equation $|G(x)| = |G/G_x|$ is crucial in the proof of the following counting lemma which, together with later refinements, forms *the basic tool of the theory of enumeration under finite group action*[1]:

[1]This Lemma of Cauchy and Frobenius is quite often attributed to Burnside and called *Burnside's Lemma*, but it is older, and Burnside proved a much stronger result which I call Burnside's Lemma and which is given below. For the history of this attribution see [9] and [14]

3.1 The Cauchy–Frobenius Lemma *The number of orbits of a finite group G acting on a finite set X is equal to the average number of fixed points:*

$$|G \backslash\backslash X| = \frac{1}{|G|} \sum_{g \in G} |X_g|.$$

The next remark helps considerably to shorten the calculations necessary for applications of this lemma. It shows that we can replace the summation over all $g \in G$ by a summation over a *transversal* of the conjugacy classes, as the number of fixed points turns out to be constant on each such class:

3.2 Lemma *For each finite group action, the mapping*

$$X_{g'} \to X_{gg'g^{-1}} : x \mapsto gx$$

is a bijection between these two sets of fixed points, and hence

$$\chi : G \to \mathbb{N} : g \mapsto |X_g|$$

is a class function, *i.e. it is constant on the conjugacy classes of G. More formally, for any $g, g' \in G$, we have that $|X_{g'}| = |X_{gg'g^{-1}}|$.*

The mapping χ is called the *character* of ${}_GX$.

3.3 Corollary *Let ${}_GX$ be a finite action and let C denote a transversal of the conjugacy classes of G. Then*

$$|G \backslash\backslash X| = \frac{1}{|G|} \sum_{g \in C} |C^G(g)||X_g| =_{2.3} \sum_{g \in C} |C_G(g)|^{-1} |X_g|.$$

Another formulation of the Cauchy–Frobenius Lemma makes use of the permutation representation $g \mapsto \bar{g}$ defined by the action in question. The permutation group $\bar{G}$ which is the image of G under this representation, yields an action ${}_{\bar{G}}X$ of $\bar{G}$ on X,which has the same orbits, and so we also have:

3.4 Corollary *If ${}_GX$ denotes a finite action, then*

$$|G \backslash\backslash X| = \frac{1}{|\bar{G}|} \sum_{\bar{g} \in \bar{G}} |X_{\bar{g}}| = \frac{1}{|\bar{G}|} \sum_{\bar{g} \in \bar{C}} |C^{\bar{G}}(\bar{g})||X_{\bar{g}}|,$$

where $\bar{C}$ denotes a transversal of the conjugacy classes of $\bar{G}$.

3.5 Application (the numbers of bilateral classes and of double cosets) The Cauchy-Frobenius Lemma yields among many other cardinalities the number of bilateral classes and therefore also the number of double cosets. Recall from 2.6 that bilateral classes are the orbits of the following action of a subgroup U of the direct product $G \times G$:

$$U \times G \to G : ((a,b),g) \mapsto agb^{-1}.$$

In order to apply the Lemma of Cauchy-Frobenius to that situation, we have to evaluate the number $|G_{(a,b)}|$ of fixed points of $(a,b) \in U$ on G which is

$$|\{g \mid a = gbg^{-1}\}| = \begin{cases} |C_G(a)| = |C_G(b)|, & \text{if } a, b \text{ are conjugates,} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, by the Cauchy-Frobenius Lemma, we obtain the number of bilateral classes in the following form:

$$|U \backslash\backslash G| = \frac{|G|}{|U|} \sum_{(a,b)\in U} \frac{|C^G(a) \cap C^G(b)|}{|C^G(a)|^2}.$$

It can be simplified since characters are constant on conjugacy classes of elements, and so *the number of bilateral classes* turns out to be ([4])

$$|U \backslash\backslash G| = \frac{|G|}{|U|} \sum_{g\in C} \frac{|(C^G(g) \times C^G(g)) \cap U|}{|C^G(g)|},$$

if C denotes a transversal of the conjugacy classes of elements in G. In particular, *the set*

$$A\backslash G/B := \{AgB \mid g \in G\} = (A \times B)\backslash\backslash G$$

of (A, B)*-double cosets has the order*

3.6 $$|A\backslash G/B| = \frac{|G|}{|A||B|} \sum_{g\in C} \frac{|C^G(g) \cap A||C^G(g) \cap B|}{|C^G(g)|}.$$

◇

In chemical applications, G is usually a symmetry group of a molecular skeleton, and it depends of the particular situation if we allow reflections or not, in which case we consider the subgroup of proper rotations. The proper rotations form (if there are reflections at all) a subgroup of index 2, i.e. a subgroup of half the order of G. In this case, we may use a homomorphism ϵ from G into the group of order two which consists of the integers ± 1, and where we map each proper rotation onto 1 while the other elements of G are mapped onto -1. A similar situation occurs in mathematics, when we consider permutations and separate them into even and odd ones using the *sign* which is defined to be $+1$ on even and -1 on odd permutations. This can be generalized to finite actions ${}_GX$ by putting

$$\epsilon(\bar{g}) = (-1)^{n-c(\bar{g})},$$

where $c(g)$ means the number of cyclic factors of $\bar{g}$, or, in other terms, the number $|\langle g\rangle\backslash\!\backslash X|$ of orbits of the group $\langle g\rangle$ generated by g. Its kernel

$$\bar{G}^+ := \{\bar{g} \in \bar{G} \mid \epsilon(\bar{g}) = 1\}$$

is either $\bar{G}$ itself or a subgroup of index 2, this is easy to see. Denoting its inverse image by

$$G^+ := \{g \in G \mid \epsilon(\bar{g}) = 1\},$$

we obtain a useful interpretation of the *alternating sum* of fixed point numbers:

3.7 Lemma *For any finite action ${}_GX$ such that $G \neq G^+$, the number of orbits of G on X which split over G^+ (i.e. which decompose into more than one and hence into two G^+ orbits) is equal to*

$$\frac{1}{|G|}\sum_{g\in G}\epsilon(\bar{g})|X_g| = \frac{1}{|\bar{G}|}\sum_{\bar{g}\in\bar{G}}\epsilon(\bar{g})|X_{\bar{g}}|.$$

3.8 Corollary *In the case when $G \neq G^+$, the number of G orbits on X which do not split over G^+ is equal to*

$$\frac{1}{|G|}\sum_{g\in G}(1-\epsilon(\bar{g}))|X_g|.$$

$$\begin{array}{ccccc} & \otimes & & & \\ & \vdots & \text{the selfenantiomeric orbits} & & \\ & \otimes & & & \\ \oplus & & \ominus & & \\ \vdots & & \vdots & \text{the enantiomeric pairs} & \\ \oplus & & \ominus & & \end{array}$$

Figure 1: Enantiomeric pairs and selfenantiomeric orbits

Note what this means. If G acts on a finite set X in such a way that $G \neq G^+$, then we can group the orbits of G on X into a set of orbits which are also G^+–orbits. In figure 1 we denote these orbits by the symbol $\otimes$. The other G–orbits split into two G^+ orbits, we indicate one of them by $\oplus$, the other one by $\ominus$, and call the pair $\{\oplus, \ominus\}$ an *enantiomeric pair* of G^+–orbits.

Hence 3.7 gives us the number of enantiomeric pairs of orbits, while 3.8 yields the number of *selfenantiomeric* orbits of G on X. The elements $x \in X$ belonging to selfenantiomeric orbits are called *achiral* objects, while the others are called *chiral*. These notions of *enantiomerism* and *chirality* are taken from chemistry, as it was mentioned already, where G is usually the symmetry group of the molecule while G^+ is its subgroup consisting of the proper rotations. We call ${}_GX$ a *chiral* action if and only if $G \neq G^+$. Using this notation we can now rephrase 3.7 and 3.8 in the following way:

3.9 Corollary *If ${}_GX$ is a finite chiral action, then the number of selfenantiomeric orbits of G on X is equal to*

$$\frac{1}{|G|}\sum_{g\in G}(1-\epsilon(\bar{g}))|X_g| = 2|G\backslash\!\backslash X| - |G^+\backslash\!\backslash X|,$$

while the number of enantiomeric pairs of orbits is

$$\frac{1}{|G|}\sum_{g\in G}\epsilon(\bar{g})|X_g| = |G^+\backslash\!\backslash X| - |G\backslash\!\backslash X|.$$

Now we are going to enumerate the symmetry classes of mappings introduced above. Their total number can be obtained by an application of the Cauchy Frobenius Lemma as soon as we know the number of fixed points $f \in Y^X$, for each element

$g \in G$. Assume that

$$\bar{g} = \prod_{\nu=1}^{c(\bar{g})} (x_\nu \, gx_\nu \ldots g^{l_\nu - 1} x_\nu)$$

is the disjoint cycle decomposition of $\bar{g}$, the permutation of X which corresponds to g. Then f is a fixed point of g if and only if, for each x_ν, the other values of f arise from the values $f(x_\nu)$ according to the following equations:

$$f(x_\nu) = f(g^{-1}x_\nu) = f(g^{-2}x_\nu) = \ldots \,.$$

This means that f is fixed under g if and only if f is *constant on each cycle (or orbit) of* $\bar{g}$. This, together with the Cauchy–Frobenius Lemma, yields the number of symmetry classes of mappings, since, for each cyclic factor of $\bar{g}$ we have a choice of $|Y|$ values for f:

3.10 Theorem

$$|\, G \backslash\backslash Y^X \,| = \frac{1}{|G|} \sum_{g \in G} |\, Y \,|^{c(\bar{g})},$$

where $c(\bar{g}) = |\langle g \rangle \backslash\backslash X|$ *denotes the number of cyclic factors of* $\bar{g}$.

A nice application is the enumeration of *necklaces*[2], which means equivalence classes of colourations of the vertices of a regular n-gon in m colours, say. In this case X is the set of n vertices of the regular n-gon, while Y is the set of m colours, and the group G is the cyclic group C_n if we do not allow reflections.

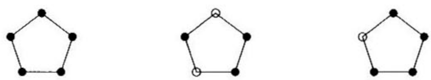
In order to apply the preceding theorem to this case we use that the cyclic group contains, for each divisor t of n, exactly $\varphi(t)$ elements that consist of (exactly n/t) cycles of length t, where $\varphi(-)$ means the Euler function, i.e. $\varphi(t)$ is the number of nonnegative integers smaller than t that are relatively prime to t. (An easy way of calculating these values is to use the fact that $\varphi(1) = 1$ and $t = \sum_{d|t} \varphi(d)$, the sum of the values of φ on the divisors of t.)

[2]It is applied and refined in the ***Educational Note*** by H. Dolhaine and E.K. Lloyd in this issue of MATCH

3.11 Corollary *The number of necklaces consisting of n pearls in m colours, with respect to the cyclic group C_n, is equal to*

$$\frac{1}{n}\sum_{t|n}\varphi(t)m^{n/t}.$$

Here is an example: Three colourings of the regular 5–gon.



According to 3.10 there are altogether

$$\frac{1}{5}\left(\varphi(1)2^5+\varphi(5)2^1\right)=\frac{1}{5}\left(1\cdot 2^5+4\cdot 2^1\right)=8$$

such symmetry classes of mappings.

4 Enumeration by Weight

In the preceding chapter the Cauchy Frobenius Lemma was mentioned, and we studied certain actions of groups on sets of the form Y^X in some detail. We saw that various structures like necklaces can be defined as orbits on such sets of mappings, and so we already have a method at hand to evaluate the *total number* of such structures. The question arises how these methods can be refined in such a way that we can also derive the *number of orbits with certain prescribed properties* like, for example, the number of necklaces with n pearls in m colours, where we, in addition, prescribe *how many pearls should be there for each of the given colours.* The answer to many such questions can be given by introducing a *weight* which means a mapping, defined on the set X on which the group is acting and which is supposed to be *constant on each orbit.* The range of the weights which we will consider is mostly a polynomial ring over $\mathbb{Q}$. The final result will be a *generating function* for the enumeration problem in question, i.e. we shall obtain a polynomial which has the desired numbers of orbits as coefficients of its different monomial summands. The basic tool is

4.1 The Cauchy–Frobenius Lemma, weighted form *Let ${}_GX$ denote a finite action and $w: X \to R$ a map from X into a commutative ring R containing $\mathbb{Q}$ as a subring. If w is constant on the orbits of G on X, then we have, for any transversal T of the orbits:*

$$\sum_{t\in T} w(t) = \frac{1}{|G|}\sum_{g\in G}\sum_{x\in X_g} w(x) = \frac{1}{|\bar{G}|}\sum_{\bar{g}\in\bar{G}}\sum_{x\in X_{\bar{g}}} w(x).$$

This result will be used in the following way for the enumeration of orbits with prescribed properties. We consider weights w with values in a *vector space*, for example in a polynomial ring. These weight functions will be chosen in such a way that the values are the same on orbits with the considered property, while their values on orbits which we should like to distinguish are *linearly independent*. Then we obviously can read off from $\sum w(t)$ the number of orbits with the property in question. It is just the coefficient of the vector $w(t_0)$, if t_0 has that property, in the vector $\sum w(t)$.

The lemma 4.1 implies 3.1 (put $w: x \mapsto 1$), which we call the *constant form* of the Cauchy–Frobenius Lemma. In order to apply 4.1 to the enumeration of symmetry classes of mappings f in Y^X we introduce, for a given $W: Y \to R$, R a commutative ring with $\mathbb{Q}$ as a subring, the *multiplicative weight* w, defined by

$$w: Y^X \to R: f \mapsto \prod_{x\in X} W(f(x)),$$

and notice that for any finite actions ${}_GX$ and every W, the corresponding multiplicative weight w is constant on the orbits of G on Y^X. Thus 4.1 can be applied as soon as we have evaluated the sum of the weights of those f which are fixed under $g \in G$. An application of the weighted form of the Cauchy Frobenius Lemma yields the desired generating function for the enumeration of symmetry classes by the multiplicative weight w:

4.2 Theorem *Let ${}_GX$ be a finite action, $W: Y \to R$ a mapping into a commutative ring containing $\mathbb{Q}$ as a subring, and denote by w the corresponding multiplicative weight function on Y^X. Then for each transversal T of these orbits, we have that* *the sum of its values on a transversal of the orbits is equal to*

$$\frac{1}{|G|}\sum_{g\in G}\prod_{i=1}^{|X|}\Big(\sum_{y\in Y}W(y)^i\Big)^{a_i(\bar{g})},$$

where $a_i(\bar{g})$ means the number of cyclic factors of length i in the cycle decomposition of $\bar{g}$, i.e. the number of orbits of order i of $\langle g\rangle$ on X.

The most general weight function is obtained when we take for W a mapping which sends each $y \in Y$ to an indeterminate of a polynomial ring. For the sake of notational simplicity we can do this by taking the elements $y \in Y$ themselves as indeterminates and putting

$$W\colon Y \to \mathbb{Q}[Y]\colon y \mapsto y,$$

where $\mathbb{Q}[Y]$ denotes the polynomial ring over $\mathbb{Q}$ in the set Y of commuting indeterminates. This yields the multiplicative weight $w(f) = \prod_x f(x)$, a monomial in $\mathbb{Q}[Y]$. If we define the *content* of $f \in Y^X$ to be the mapping

4.3 $$c(f,-)\colon Y \to \mathbb{N}\colon y \mapsto |f^{-1}(y)|,$$

i.e. $c(f,y)$ is the multiplicity with which f takes the value y, then we get

4.4 Corollary *The number of $G-$classes on Y^X, the elements of which have the same content as $f \in Y^X$, is equal to the coefficient of the monomial $\prod_y y^{c(f,y)}$ in the polynomial*

$$\frac{1}{|G|}\sum_{g\in G}\prod_{i=1}^{|X|}\Big(\sum_{y\in Y}y^i\Big)^{a_i(\bar{g})}.$$

A nice example is the enumeration of necklaces by weight:

4.5 Application (necklaces by weight) We ask for the number of different necklaces with n pearls in up to m colours and given content. In order to bring this problem within reach of unromantic mathematics, we consider such a necklace again as a colouring of the set X of vertices of a regular $n-$gon, the colours taken from a set Y of colours, i.e. a necklace is considered as a mapping $f \in Y^X$. Two such necklaces or colourings are different if and only if none of them can be obtained from the other one by a rotation. Hence we are faced with an action of the form ${}_G(Y^X)$, namely the natural action of a cyclic group $G := C_n$ of order $n = |X|$ on the set Y^X. (In the case when we want to allow reflections, we have to consider $G := D_n$, the dihedral group.) We are now in a position to count these orbits by content. In order to do this we use the cycle structure of the elements of C_n which was mentioned already, obtaining by an application of 4.4 the desired solution of the necklace problem:

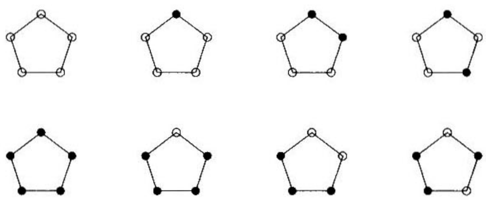
4.6 Corollary *The number of different necklaces containing b_i pearls of the i-th colour $y_i \in Y$, is (if $Y = \{y_1, \ldots, y_m\}$) the coefficient of $y_1^{b_1} \ldots y_m^{b_m}$ in the polynomial*

$$\frac{1}{n}\sum_{d|n} \varphi(d)(y_1^d + \ldots + y_m^d)^{n/d}.$$

For an example we take $m := 2$ and $n := 5$, obtaining the generating function

$$\frac{1}{5}\left((y_1 + y_2)^5 + 4(y_1^5 + y_2^5)\right) = y_1^5 + y_1^4 y_2 + 2y_1^3 y_2^2 + 2y_1^2 y_2^3 + y_1 y_2^4 + y_2^5.$$

The presence of the monomial summand $2y_1^3y_2^2$ means that there are exactly two different necklaces consisting of 5 pearls three of which are of the colour y_1 and two of which are of the colour y_2. The drawing shows representatives of the symmetry classes:



◇

5 Counting by Symmetry Group

We just discussed the enumeration of orbits by weight, a problem that can be solved by an easy refinement of the Cauchy-Frobenius Lemma. In the present section we shall introduce another variation of this lemma in order to count orbits by stabilizer class, or, in terms of applications in science, *by symmetry group.*

Recall that the elements of an orbit have as their stabilizers a full conjugacy class of subgroups of G,

$$\forall\, x \in X\colon\ \{G_{x'} \mid x' \in G(x)\} = \widetilde{G_x} = \{gG_xg^{-1} \mid g \in G\}.$$

We say that $\widetilde{G_x}$ is the *type* of this orbit. The *set* of orbits of type $\widetilde{U}$, for a given subgroup U of G, is called the $\widetilde{U}$*-stratum* and indicated by

$$G\backslash\!\!\backslash_{\widetilde{U}} X.$$

Consider the lattice $L(G)$ of subgroups U^i of G. The group G is supposed to be finite. $L(G)$ is a partially ordered set, partially ordered by the inclusion order $\subseteq$. This order can be expressed in terms of the *zeta function* ζ on $L(G)$, defined by

$$\zeta(U^i, U^k) = \begin{cases} 1, & \text{if } U^i \subseteq U^k, \\ 0, & \text{otherwise,} \end{cases}$$

for any two subgroups U^i and U^k of G. We can assume that the subgroups U^i of G are numbered in such a way that

5.1 $$U^i \subseteq U^k \implies i \leq k.$$

The *zeta-matrix* $\zeta(G)$ is defined by

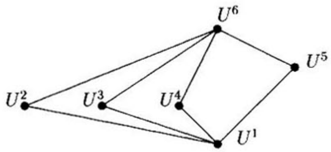
$$\zeta(G) := \left(\zeta(U^i, U^k)\right),$$

it is an upper triangular matrix with ones along its main diagonal, because of 5.1, and so it can be inverted, and the inverse is a matrix with integer elements as well:

$$\mu(G) := \left(\mu(U^i, U^k)\right) = \zeta(G)^{-1}.$$

This matrix is called the *Möbius-matrix* of G, and the *Möbius function* $\mu(-,-)$ on $L(G)$ is defined to be the function with the values given by the Möbius matrix.

Let us consider an easy example, $G := S_3$. Here is the lattice of subgroups:



where $U^1 = \langle 1\rangle$, $U^2 = \langle(12)\rangle$, $U^3 = \langle(13)\rangle$, $U^4 = \langle(23)\rangle$, $U^5 = \langle(123)\rangle$, $U^6 = S_3$. This gives the zeta-matrix

$$\zeta(S_3) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 0 & 1 \\ & & 1 & 0 & 0 & 1 \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix},$$

and so we obtain, by inversion, the Möbius-matrix

$$\mu(S_3) = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 3 \\ & 1 & 0 & 0 & 0 & -1 \\ & & 1 & 0 & 0 & -1 \\ & & & 1 & 0 & -1 \\ & & & & 1 & -1 \\ & & & & & 1 \end{pmatrix}.$$

In order to evaluate $|G\backslash\!\backslash_{\widetilde{U}} X|$, we consider, for a finite action ${}_GX$ and for the subgroups U of G, the sets

$$X_U := \{x \in X \mid \forall\, g \in U\colon\ gx = x\}.$$

Burnside called the order $|X_U|$ of this set the *mark* of U on X. We express it in terms of lengths of strata:

$$|X_U| = \sum_V \zeta(U,V)\frac{|G/V|}{|\widetilde{V}|}|\, G\backslash\!\backslash_{\widetilde{V}} X\,|.$$

By Möbius Inversion (which simply means by an application of the fact that the Möbius-matrix is the inverse of the zeta-matrix) this equation is equivalent to:

5.2 $$|\, G\backslash\!\backslash_{\widetilde{U}} X\,| = \frac{|\widetilde{U}|}{|G/U|}\sum_V \mu(U,V)|X_V|.$$

In order to simplify this expression we consider the set $\widetilde{L}(G)$ consisting of the conjugacy classes of subgroups of G:

$$\widetilde{L}(G) := \{\widetilde{U}_0, \ldots, \widetilde{U}_{d-1}\}, \text{ with representatives } U_i \in \widetilde{U}_i.$$

U^i must not be mixed up with U_i, and d will denote the number of conjugacy classes of subgroups of G. Putting

5.3
$$b_{ik} := \frac{|\widetilde{U}_i|}{|G/U_i|} \sum_{V \in \widetilde{U}_k} \mu(U_i, V) = \frac{|\widetilde{U}_i|}{|G/U_i|} \mu(U_i, \widetilde{U}_k),$$

where $\mu(U_i, \widetilde{U}_k) := \sum_{V \in \widetilde{U}_k} \mu(U_i, V)$, we can introduce the matrix

5.4
$$B(G) := (b_{ik}) = \begin{pmatrix} \ddots & & 0 \\ & |N_G(U_i)/U_i|^{-1} & \\ 0 & & \ddots \end{pmatrix} \left(\mu(U_i, \widetilde{U}_k)\right).$$

I suggest we call this matrix the *Burnside matrix* of G, although Burnside considered in fact the inverse of $B(G)$,

$$M(G) := B(G)^{-1}.$$

For example,

$$M(S_3) = \begin{pmatrix} 6 & 3 & 2 & 1 \\ & 1 & 0 & 1 \\ & & 2 & 1 \\ & & & 1 \end{pmatrix}.$$

Burnside called the inverse of $B(G)$ *the table of marks;* we shall return to this matrix later.

Using the Burnside matrix of G we can now reformulate 5.2, since the number $|X_V|$ of invariants of the subgroup V is clearly constant on the conjugacy class $\widetilde{V}$,

5.5
$$X_U \to X_{gUg^{-1}} : x \mapsto gx \text{ is a bijection.}$$

The crucial result is

5.6 Burnside's Lemma *If ${}_GX$ is a finite action, then the vector of the lengths $|G\backslash\!\backslash_{\widetilde{U}_i} X|$ of the strata of G on X satisfies the equation*

$$\begin{pmatrix} \vdots \\ |G\backslash\!\backslash_{\widetilde{U}_i} X| \\ \vdots \end{pmatrix} = B(G) \cdot \begin{pmatrix} \vdots \\ |X_{U_i}| \\ \vdots \end{pmatrix}.$$

We can apply this now in order to enumerate the G–classes on Y^X by type. Since $f \in Y^X$ is fixed under each $g \in U_i$ if and only if f is constant on each orbit of U_i on X, we obtain:

5.7 Corollary *The number of symmetry classes of G on Y^X of type $\widetilde{U}_i$ is the i-th entry of the one column matrix*

$$B(G) \cdot \begin{pmatrix} \vdots \\ |Y|^{|U_i \backslash\backslash X|} \\ \vdots \end{pmatrix}.$$

If X is again the pentagon, $Y = \{y_1, y_2\}$ and $G = C_5$, we have (since C_5 contains only one proper subgroup, the trivial subgroup $\{1\}$) that

$$\zeta(C_5) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \mu(C_5) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

so that

$$M(C_5) = \begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B(C_5) = \begin{pmatrix} 1/5 & -1/5 \\ 0 & 1 \end{pmatrix}.$$

Hence, by Burnside's Lemma, we obtain the following result on the numbers of orbits by stabilizer class:

$$B(C_5) \cdot \begin{pmatrix} 2^5 \\ 2^1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix},$$

in accordance with the above drawing of a transversal of the symmetry classes.

6 Tables of Marks and Burnside Matrices

The inverse $M(G) = (m_{ik})$ of the Burnside matrix $B(G) = (b_{ik})$ was introduced by Burnside (1911,[1]) and called the *table of marks,* as it was mentioned already. Sometimes it is also called the *supercharacter table* for a reason which will become clear later. The definition of b_{ik} together with the identity

6.1 $$|\widetilde{U}_i|\zeta(U_i, \widetilde{U}_k) = |\widetilde{U}_k|\zeta(\widetilde{U}_i, U_k),$$

where

$$\zeta(U_i, \widetilde{U}_k) = \sum_{V \in \widetilde{U}_k} \zeta(U_i, V), \; \zeta(\widetilde{U}_i, U_k) = \sum_{V \in \widetilde{U}_i} \zeta(V, U_k).$$

This shows that the elements of the table of marks are of the following form:

6.2 $$m_{ik} = \frac{|G/U_k|}{|\widetilde{U}_k|}\zeta(U_i, \widetilde{U}_k) = \frac{|G/U_k|}{|\widetilde{U}_i|}\zeta(\widetilde{U}_i, U_k) \in \mathbb{N}.$$

The claim that $m_{ik} \in \mathbb{N}$ follows from the first identity since the order of a conjugacy class of a subgroup is equal to the index of the normalizer, and so

6.3 $$\frac{|G/U_k|}{|\widetilde{U}_k|} = \frac{|N_G(U_k)|}{|U_k|} \in \mathbb{N}.$$

Equation 6.2 shows that

$$M(G) = \left(\zeta(U_i, \widetilde{U}_k)\right) \begin{pmatrix} \ddots & & 0 \\ & |N_G(U_i)/U_i| & \\ 0 & & \ddots \end{pmatrix},$$

and hence the entries of $M(G)$ describe the poset

$$(\widetilde{L}(G), \preceq).$$

It consists of the conjugacy classes $\widetilde{U}_i$ of subgroups and the partial order

6.4 $$\widetilde{U}_i \preceq \widetilde{U}_k \ :\Longleftrightarrow\ \exists\, U \in \widetilde{U}_i, V \in \widetilde{U}_k :\ U \leq V.$$

6.2 implies the following equivalence:

6.5 $$m_{ik} \neq 0 \iff \widetilde{U}_i \preceq \widetilde{U}_k.$$

Burnside called $M(G)$ the table of marks for the following reason:

6.6 Lemma *The entry m_{ik} is the number of left cosets of U_k in G which remain fixed under left multiplication by the elements of U_i.*

Hence m_{ik} is, so to speak, the *mark* which U_i leaves when it is acting on the left cosets of U_k. We now derive further properties of these elements. As G is assumed to be finite, we can choose a numbering of the conjugacy classes $\widetilde{U}_i$ such that the following implication holds:

6.7 $$|U_i| < |U_k| \implies i < k.$$

This guarantees in particular that the partial order is respected:

6.8 $$\widetilde{U}_i \preceq \widetilde{U}_k \implies i \leq k.$$

Under the assumptions of 6.7, $M(G)$ is upper triangular, $U_0 = \{1\}$, $U_{d-1} = G$, and so the table of marks takes the following form:

6.9 $$M(G) = \begin{pmatrix} |G| & \dots & |G/U_k| & \dots & 1 \\ & \ddots & & * & \vdots \\ & & |N_G(U_i)/U_i| & & \vdots \\ & 0 & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}.$$

Other consequences of 6.2 are divisibility properties:

6.10 Lemma *If $\widetilde{U}_i \preceq \widetilde{U}_k$, then $m_{ik} = m_{kk}\zeta(U_i, \widetilde{U}_k)$, and so m_{kk} divides all the m_{ik} in the same column.*

Moreover, certain nonzero elements in a column form a monotonic sequence, for $m_{ik} \neq 0 \neq m_{jk}$ means that $\widetilde{U}_i \preceq \widetilde{U}_k \succeq \widetilde{U}_j$. If in addition $\widetilde{U}_i \preceq \widetilde{U}_j$ holds, which implies that $i \leq j$, then we have

$$\widetilde{U}_i \preceq \widetilde{U}_j \preceq \widetilde{U}_k.$$

As we may assume $U_i \leq U_j$ without restriction, we obtain:

$$gU_jg^{-1} \leq U_k \implies gU_ig^{-1} \leq U_k,$$

so that an application of

6.11 $$m_{ik} = \frac{1}{|U_k|} |\{g \in G \mid gU_ig^{-1} \leq U_k\}|$$

finally yields

6.12 Corollary *If $\widetilde{U}_i \preceq \widetilde{U}_j \preceq \widetilde{U}_k$, then the corresponding elements in the k-th column of the table of marks of G satisfy*

$$m_{ik} \geq m_{jk} \geq m_{kk}.$$

The evaluation of the table of marks is usually quite difficult since one has to know $L(G)$, its Hasse diagram, and the orders of the U_i and their normalizers. Fortunately there exists several subgroup lattice programs, for example GAP and MAGMA, which can be used in order to evaluate tables of marks and Burnside matrices.

Burnside's original motivation for introducing the table of marks was the problem of decomposing a given action into its orbits or, in other words, to decompose a permutation representation into its *transitive constituents.* The question was whether it suffices to consider only the *character* of the action ${}_GX$, i.e. the function $\chi: g \mapsto |X_g|$. In order to explain character theoretically what is meant by the decomposition of the action ${}_GX$ into its transitive constituents we first recall that there exists a natural equivalence relation on the set of actions of G on finite sets. Two actions, ${}_GX$ and ${}_GY$, say, were called *similar* if and only if there exists a bijection $\theta: X \to Y$ which is G-*invariant,* i.e. for which the following holds: $\forall\, x \in X, g \in G: \ \theta(gx) = g\theta(x)$. We know from 2.5 that there are exactly as many similarity classes of transitive actions as there are conjugacy classes of subgroups. An immediate implication is

6.13 Lemma *The set* $\{\,{}_G(G/U_i) \mid i = 1, \ldots, d\}$ *is a transversal of the similarity classes of transitive actions of* G.

6.14 Corollary *The characters*

$$\chi_i: G \to \mathbb{C}: g \mapsto |(G/U_i)_g|,$$

of the actions ${}_G(G/U_i), U_i \in \widetilde{U}_i, i \in d$, *are the transitive characters of* G. *These characters have the following values (recall that* $C^G(g)$ *denotes the conjugacy class and* $C_G(g)$ *the centralizer of* g*):*

$$\chi_i(g) \;=\; \frac{|G|}{|U_i|}\frac{|C^G(g) \cap U_i|}{|C^G(g)|}.$$

Burnside saw that a knowledge of the character χ of ${}_GX$ together with a table of the χ_i does *not* suffice to decompose χ into its transitive constituents. Such a decomposition is equivalent to the evaluation of the coefficients $n_i \in \mathbb{N}$ in the equation

$$\chi = \sum_{i \in d} n_i \chi_i,$$

where n_i is the number of orbits of G on X similar to ${}_G(G/U_i)$, as G–sets. It turns out that replacing the χ_i by the rows of the table of marks we get a unique linear combination. We first mention that the χ_i form part of the table of marks.

6.15 Lemma *If $U_i = \langle g \rangle$ is a cyclic subgroup of G and χ_k the character of ${}_G(G/U_k)$, then $m_{ik} = \chi_k(g)$.*

This is the reason why $M(G)$ is sometimes called the table of *supercharacters* of G, and it also shows that it is helpful to indicate the columns of $M(G)$ which belong to cyclic subgroups so that we can easily identify the transitive characters from $M(G)$. For example the table of marks of S_4 is

6.16
$$M(S_4) = \begin{pmatrix} 24 & 12 & 12 & 8 & 6 & 6 & 6 & 4 & 3 & 2 & 1 \\ & 2 & . & . & 2 & . & . & 2 & 1 & . & 1 \\ & & 4 & . & 2 & 2 & 6 & . & 3 & 2 & 1 \\ & & & 2 & . & . & . & 1 & . & 2 & 1 \\ & & & & 2 & . & . & . & 1 & . & 1 \\ & & & & & 2 & . & . & 1 & . & 1 \\ & & & & & & 6 & . & 3 & 2 & 1 \\ & & & & & & & 1 & . & . & 1 \\ & & & & & & & & 1 & . & 1 \\ & & & & & & & & & 2 & 1 \\ & & & & & & & & & & 1 \end{pmatrix} \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \\ \\ \leftarrow \\ \\ \\ \\ \\ \\ \end{array}$$

a table which corresponds to the following numbering of subgroups U_i:

$$U_1 = \langle 1 \rangle,\ U_2 = \langle (24) \rangle,\ U_3 = \langle (13)(24) \rangle,\ U_4 = \langle (132) \rangle,$$

$$U_5 = \langle (13), (24) \rangle,\ U_6 = \langle (1234) \rangle,\ U_7 = \langle (12)(34), (14)(23) \rangle,$$

$$U_8 = \langle (132), (13) \rangle,\ U_9 = \langle (1234), (24) \rangle,\ U_{10} = \langle (132), (142) \rangle,$$

$$U_{11} = \langle (1324), (1342) \rangle.$$

The rows which correspond to cyclic groups are marked by an arrow $\leftarrow$, and hence the table of transitive permutation characters of S_4 is as follows:

	(1^4)	$(1^2 2)$	(2^2)	(13)	(4)
χ_1	24	.	.	.	.
χ_2	12	2	.	.	.
χ_3	12	.	4	.	.
χ_4	8	.	.	2	.
χ_5	6	2	2	.	.
χ_6	6	.	2	.	2
χ_7	6	.	6	.	.
χ_8	4	2	.	1	.
χ_9	3	1	3	.	1
χ_{10}	2	.	2	2	.
χ_{11}	1	1	1	1	1

A few words on the entries b_{ik} of the Burnside matrix $B(G) = M(G)^{-1}$ are in order since in fact it is this matrix which we usually apply for enumerative purposes (see section 3.1). By the definition of b_{ik} we have

6.17 $$b_{ik} \in \mathbb{Q}.$$

Moreover, the following can easily be derived:

6.18 Corollary *If the numbering of the conjugacy classes of subgroups of G satisfies 5.1, then we have for the Burnside matrix of G:*

$$B(G) = \begin{pmatrix} \frac{1}{|G|} & \cdots & \frac{\mu(1,U_k)}{|N_G(U_k)|} & \cdots & \frac{\mu(1,G)}{|G|} \\ & \ddots & & * & \vdots \\ & & \frac{|U_i|}{|N_G(U_i)|} & & \frac{\mu(U_i,G)}{|N_G(U_i)/U_i|} \\ & 0 & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}.$$

¿From the definitions of m_{ik} and b_{ik} we obtain interesting relations between various elements or products of elements of these two matrices, e.g. that the following products are rational integral:

6.19 $$m_{ik}b_{kj} = \zeta(U_i, \widetilde{U}_k)\mu(U_k, \widetilde{U}_j) \in \mathbb{Z}.$$

7 Weighted Enumeration by Stabilizer Class

The proof of the weighted form 4.1 of the Cauchy-Frobenius Lemma is as easy as the proof of its constant form 3.1. The same holds for the corresponding weighted form of Burnside's Lemma, which we are going to introduce next. Recall that, by Burnside's result 5.2, we obtain for a transversal T of $G\backslash\backslash X$:

$$|G\backslash\backslash_{\widetilde{U}}X| = \sum_{t\in T:\ G_t\in\widetilde{U}} 1 = \frac{|\widetilde{U}|}{|G/U|}\sum_{V\leq G}\mu(U,V)\sum_{x\in X_V} 1.$$

If we now replace the 1's on both sides by the weight of the element to which they correspond, then we obtain the identity

$$\sum_{t\in T:\ G_t\in\widetilde{U}} w(t) = \frac{|\widetilde{U}|}{|G/U|}\sum_{V\leq G}\mu(U,V)\sum_{x\in X_V} w(x).$$

Since the bijection described in 5.5 is weight preserving, we get

7.1 $$\sum_{t\in T:\ G_t\in\widetilde{U}_i} w(t) = \frac{|\widetilde{U}_i|}{|G/U_i|}\sum_{k}\mu(U_i,\widetilde{U}_k)\sum_{x\in X_{U_k}} w(x).$$

A direct consequence is the desired result on the enumeration by weight and stabilizer class:

7.2 Burnside's Lemma, weighted form *Let ${}_GX$ denote a finite action and $w\colon X\to R$ a weight function from X into a commutative ring R which contains $\mathbb{Q}$ as a subring. If w is constant on the orbits of X, then we have, for the elements t of a transversal T of the orbits and the vector of the sums of weights of transversals of strata $G\backslash\backslash_{\widetilde{U}_i}X$ of G on X, the equation*

$$\begin{pmatrix}\vdots\\ \sum_{t:G_t\in\widetilde{U}_i} w(t)\\ \vdots\end{pmatrix} = B(G)\cdot\begin{pmatrix}\vdots\\ \sum_{x:U_i\leq G_x} w(x)\\ \vdots\end{pmatrix}.$$

This weighted form of Burnside's Lemma was, as far as I know, first stated and proved in P. Stockmeyer's thesis ([13]). He provided applications of the following immediate consequence to the enumeration of graphs (resp. symmetry classes of mappings) by weight and type:

7.3 Corollary *The generating function for the enumeration of G-classes on Y^X of type $\widetilde{U}_j$ by weight w: $f \mapsto \prod f(x) \in \mathbb{Q}[Y]$ is the j-th row of the following one column matrix:*

$$B(G) \cdot \begin{pmatrix} \vdots \\ \prod_{\nu \in |U_i \backslash\backslash X|} \sum_y y^{l_\nu(U_i)} \\ \vdots \end{pmatrix},$$

where $l_\nu(U_i)$ denotes the length of the ν-th orbit of U_i on X.

For example, we can easily enumerate necklaces by weight and stabilizer class. A particularly simple case is when the number of pearls of the necklaces is a prime number p, say, and the symmetry group G is the cyclic group C_p. This group has just two subgroups, the trivial ones, $\{1\}$ and C_p. Hence the table of marks is

$$M(C_p) = \begin{pmatrix} p & 1 \\ 0 & 1 \end{pmatrix},$$

while its inverse, the Burnside matrix, is

$$B(C_p) = \begin{pmatrix} 1/p & -1/p \\ 0 & 1 \end{pmatrix}.$$

Hence, according to the weighted form of Burnside's Lemma, the corresponding vector of generating functions is

$$B(C_p) \cdot \begin{pmatrix} (\sum_{y \in Y} y)^p \\ \sum_{y \in Y} y^p \end{pmatrix}.$$

For the prime number $p = 5$, for example, we obtain this way the vector

$$\begin{pmatrix} y_1^4 y_2^1 + 2 \cdot y_1^3 y_2^2 + 2 \cdot y_1^2 y_2^3 + y_1^1 y_2^4 \\ y_1^5 y_2^0 + y_1^0 y_2^5 \end{pmatrix}.$$

The presence of the summand $2 \cdot y_1^3 y_2^2$ in the polynomial in the first row says, for example, that there are exactly two necklaces with symmetry group $\{1\}$ and three pearls in the first and two pearls in the second colour, while the second row shows that the necklaces with symmetry group (=stabilizer) C_5 are the two necklaces where all the pearls have the same colour.

7.4 Application (asymmetric symmetry classes by weight) A case of particular interest is the enumeration by weight of *asymmetric* symmetry classes, i.e. of orbits with trivial stabilizer class $\widetilde{1}$. The corresponding generating function is direct from 7.1:

7.5
$$\sum_{t \in T:\ G_t = 1} w(t) = \sum_k \frac{\mu(1, U_k)}{|N_G(U_k)|} \sum_{x:\ U_k \leq G_x} w(x).$$

This series is called the *asymmetry series*. For example, the asymmetry series of the action ${}_G(Y^X)$ is

7.6
$$A(G, X \mid \sum y) := \sum_k \frac{\mu(1, U_k)}{|N_G(U_k)|} \prod_i \left(\sum_{y \in Y} y^i \right)^{|U_k \backslash\!\backslash_i X|},$$

where $U_k \backslash\!\backslash_i X$ denotes the set of orbits of length i of U_k on X.

7.7 Corollary *The generating function for the enumeration of asymmetric G-classes on Y^X by multiplicative weight is*

$$A(G, X \mid \sum y).$$

In the case of the cyclic group the asymmetry indicator is

7.8
$$A(C_n, n) = \frac{1}{n} \sum_{d \mid n} \mu(d) \left(\sum_{y \in Y} y^d \right)^{n/d}.$$

Here μ denotes the number theoretic Möbius function, the Möbius function on the set of natural numbers ordered by divisibility. ◇

8 Examples of Tables of Marks and Burnside Matrices

This section contains tables of marks and Burnside matrices of several cyclic, dihedral, symmetric and alternating groups.

8.1 Cyclic Groups

The cyclic groups C_p, where p is a prime number, were already mentioned. They contain trivial subgroups only, and so we have

$$M(C_p) = \begin{pmatrix} p & 1 \\ & 1 \end{pmatrix}, \quad B(C_p) = \begin{pmatrix} 1/p & -1/p \\ & 1 \end{pmatrix}.$$

There is also an explicit expression for the table of marks in the general case since the subgroup lattice of $C_n, n > 0$, is the lattice of divisors of n, but in order to make life easier for the interested reader we show these matrices and their inverses, which are less trivial.

The group C_4: The subgroups are

$$U_1 = \langle 1 \rangle, U_2 = \langle (13)(24) \rangle, U_3 = \langle (1234) \rangle = C_4.$$

The table of marks and the Burnside matrix:

$$\begin{pmatrix} 4 & 2 & 1 \\ & 2 & 1 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1/4 & -1/4 & . \\ & 1/2 & -1/2 \\ & & 1 \end{pmatrix}$$

The group C_6: The subgroups are

$$U_1 = \langle 1 \rangle, U_2 = \langle (14)(25)(36) \rangle, U_3 = \langle (135)(246) \rangle, U_4 = C_6.$$

The table of marks and the Burnside matrix:

$$\begin{pmatrix} 6 & 3 & 2 & 1 \\ & 3 & . & 1 \\ & & 2 & 1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1/6 & -1/6 & -1/6 & 1/6 \\ & 1/3 & . & -1/3 \\ & & 1/2 & -1/2 \\ & & & 1 \end{pmatrix}.$$

The group C_8: The subgroups are

$$U_1 = \langle 1 \rangle, U_2 = \langle (15)(26)(37)(48) \rangle, U_3 = \langle (1357)(2468) \rangle, U_4 = C_8.$$

The table of marks and the Burnside matrix:

$$\begin{pmatrix} 8 & 4 & 2 & 1 \\ & 4 & 2 & 1 \\ & & 2 & 1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1/8 & -1/8 & . & . \\ & 1/4 & -1/4 & . \\ & & 1/2 & -1/2 \\ & & & 1 \end{pmatrix}.$$

The group C_9: The subgroups are

$$U_1 = \langle 1 \rangle, U_2 = \langle (147)(258)(369) \rangle, U_3 = C_9.$$

The table of marks and the Burnside matrix:

$$\begin{pmatrix} 9 & 3 & 1 \\ & 3 & 1 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1/9 & -1/9 & . \\ & 1/3 & -1/3 \\ & & 1 \end{pmatrix}.$$

The group C_{10}: The subgroups are

$$U_1 = \langle 1 \rangle, U_2 = \langle (16)(27)(38)(49)(5,10) \rangle, U_3 = \langle (13579)(2,4,6,10) \rangle, U_4 = C_{10}.$$

The table of marks and the Burnside matrix:

$$\begin{pmatrix} 10 & 5 & 2 & 1 \\ & 5 & . & 1 \\ & & 2 & 1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1/10 & -1/10 & -1/10 & 1/10 \\ & 1/5 & . & -1/5 \\ & & 1/2 & -1/2 \\ & & & 1 \end{pmatrix}.$$

8.2 Dihedral Groups

This section contains a system of representatives of the conjugacy classes of subgroups, the table of marks and the Burnside matrix of the dihedral groups D_3 up to D_6 (of order 12).

The group D_3: A transversal of the conjugacy classes of subgroups is

$$U_1 = \langle 1 \rangle, U_2 = \langle (12) \rangle, U_3 = \langle (123) \rangle, U_4 = \langle (123), (12) \rangle = D_3.$$

The table of marks is

$$\begin{pmatrix} 6 & 3 & 2 & 1 \\ & 1 & . & 1 \\ & & 2 & 1 \\ & & & 1 \end{pmatrix}.$$

The Burnside matrix looks as follows

$$\begin{pmatrix} 1/6 & -1/2 & -1/6 & 1/2 \\ & 1 & . & -1 \\ & & 1/2 & -1/2 \\ & & & 1 \end{pmatrix}.$$

The group D_4: A transversal of the conjugacy classes of subgroups is

$$U_1 = \langle 1 \rangle, U_2 = \langle (13)(24) \rangle, U_3 = \langle (14)(23) \rangle, U_4 = \langle (13) \rangle,$$

$$U_5 = \langle (14)(23), (12)(34) \rangle, U_6 = \langle (13), (24) \rangle, U_7 = \langle (1234) \rangle, U_8 = D_4.$$

The table of marks is

$$\begin{pmatrix} 8 & 4 & 4 & 4 & 2 & 2 & 2 & 1 \\ & 4 & . & . & 2 & 2 & 2 & 1 \\ & & 2 & . & 2 & . & . & 1 \\ & & & 2 & . & 2 & . & 1 \\ & & & & 2 & . & . & 1 \\ & & & & & 2 & . & 1 \\ & & & & & & 2 & 1 \\ & & & & & & & 1 \end{pmatrix}.$$

The Burnside matrix looks as follows

$$\begin{pmatrix} 1/8 & -1/8 & -1/4 & -1/4 & 1/4 & 1/4 & . & . \\ & 1/4 & . & . & -1/4 & -1/4 & -1/4 & 1/2 \\ & & 1/2 & . & -1/2 & . & . & . \\ & & & 1/2 & . & -1/2 & . & . \\ & & & & 1/2 & . & . & -1/2 \\ & & & & & 1/2 & . & -1/2 \\ & & & & & & 1/2 & -1/2 \\ & & & & & & & 1 \end{pmatrix}.$$

The group D_5: A transversal of the conjugacy classes of subgroups is

$$U_1 = \langle 1 \rangle, U_2 = \langle (12)(35) \rangle, U_3 = \langle (12345) \rangle, U_4 = D_5.$$

The table of marks is

$$\begin{pmatrix} 10 & 5 & 2 & 1 \\ & 1 & . & 1 \\ & & 2 & 1 \\ & & & 1 \end{pmatrix}.$$

The Burnside matrix looks as follows

$$\begin{pmatrix} 1/10 & -1/2 & -1/10 & 1/2 \\ & 1 & . & -1 \\ & & 1/2 & -1/2 \\ & & & 1 \end{pmatrix}.$$

The group D_6: A transversal of the conjugacy classes of subgroups is

$$U_1 = \langle 1 \rangle, U_2 = \langle (14)(25)(36) \rangle, U_3 = \langle (16)(25)(34) \rangle,$$

$$U_4 = \langle (15)(24) \rangle, U_5 = \langle (135)(246) \rangle, U_6 = \langle (16)(25)(34), (13)(46) \rangle,$$

$$U_7 = \langle (135)(246), (14)(25)(36) \rangle, U_8 = \langle (135)(246), (16)(25)(34) \rangle,$$

$$U_9 = \langle (135)(246), (15)(24) \rangle, U_{10} = D_6.$$

The table of marks is

$$\begin{pmatrix}
12 & 6 & 6 & 6 & 4 & 3 & 2 & 2 & 2 & 1 \\
 & 6 & . & . & . & 3 & 2 & . & . & 1 \\
 & & 2 & . & . & 1 & . & 2 & . & 1 \\
 & & & 2 & . & 1 & . & . & 2 & 1 \\
 & & & & 4 & . & 2 & 2 & 2 & 1 \\
 & & & & & 1 & . & . & . & 1 \\
 & & & & & & 2 & . & . & 1 \\
 & & & & & & & 2 & . & 1 \\
 & & & & & & & & 2 & 1 \\
 & & & & & & & & & 1
\end{pmatrix}.$$

The Burnside matrix is

$$\begin{pmatrix}
1/12 & -1/12 & -1/4 & -1/4 & -1/12 & 1/2 & 1/12 & 1/4 & 1/4 & -1/2 \\
 & 1/6 & . & . & . & -1/2 & -1/6 & . & . & 1/2 \\
 & & 1/2 & . & . & -1/2 & . & -1/2 & . & 1/2 \\
 & & & 1/2 & . & -1/2 & . & . & -1/2 & 1/2 \\
 & & & & 1/4 & . & -1/4 & -1/4 & -1/4 & 1/2 \\
 & & & & & 1 & . & . & . & -1 \\
 & & & & & & 1/2 & . & . & -1/2 \\
 & & & & & & & 1/2 & . & -1/2 \\
 & & & & & & & & 1/2 & -1/2 \\
 & & & & & & & & & 1
\end{pmatrix}$$

8.3 Alternating Groups

This section contains a system of representatives of the conjugacy classes of subgroups, the table of marks and the Burnside matrix of A_3 and A_4.

The group A_3: A transversal of the conjugacy classes of subgroups is

$$U_1 = \langle 1 \rangle, U_2 = \langle (123) \rangle.$$

The table of marks is

$$\begin{pmatrix} 3 & 1 \\ & 1 \end{pmatrix}.$$

The Burnside matrix looks as follows

$$\begin{pmatrix} 1/3 & -1/3 \\ & 1 \end{pmatrix}.$$

The group A_4: A transversal of the conjugacy classes of subgroups is

$$U_1 = \langle 1 \rangle, U_2 = \langle (13)(24) \rangle, U_3 = \langle (123) \rangle,$$

$$U_4 = \langle (13)(24), (14)(23) \rangle, U_5 = \langle (123), (142) \rangle = A_4.$$

The table of marks is

$$\begin{pmatrix} 12 & 6 & 4 & 3 & 1 \\ & 2 & . & 3 & 1 \\ & & 1 & . & 1 \\ & & & 3 & 1 \\ & & & & 1 \end{pmatrix}.$$

The Burnside matrix looks as follows

$$\begin{pmatrix} 1/12 & -1/4 & -1/3 & 1/6 & 1/3 \\ & 1/2 & . & -1/2 & . \\ & & 1 & . & -1 \\ & & & 1/3 & -1/3 \\ & & & & 1 \end{pmatrix}.$$

8.4 Symmetric Groups

This section contains a system of representatives of the conjugacy classes of subgroups, the table of marks and the Burnside matrix of S_3 and S_4, as well as a system of representatives of the conjugacy classes of their subgroups.

The group S_3: A transversal of the conjugacy classes of subgroups is

$$U_1 = \langle 1 \rangle, U_2 = \langle (12) \rangle, U_3 = \langle (123) \rangle, U_4 = \langle (123), (12) \rangle = S_3.$$

The table of marks is

$$\begin{pmatrix} 6 & 3 & 2 & 1 \\ & 1 & . & 1 \\ & & 2 & 1 \\ & & & 1 \end{pmatrix}.$$

The Burnside matrix looks as follows

$$\begin{pmatrix} 1/6 & -1/2 & -1/6 & 1/2 \\ & 1 & . & -1 \\ & & 1/2 & -1/2 \\ & & & 1 \end{pmatrix}.$$

The group S_4: A transversal of the conjugacy classes of subgroups is

$$U_1 = \langle 1 \rangle, U_2 = \langle (13) \rangle, U_3 = \langle (13)(24) \rangle, U_4 = \langle (132) \rangle,$$

$$U_5 = \langle (13), (24) \rangle, U_6 = \langle (1234) \rangle,$$

$$U_7 = \langle (12)(34), (14)(23) \rangle, U_8 = \langle (132), (13) \rangle,$$

$$U_9 = \langle (1234), (24) \rangle, U_{10} = \langle (132), (142) \rangle, U_{11} = S_4.$$

The table of marks is

$$\begin{pmatrix} 24 & 12 & 12 & 8 & 6 & 6 & 6 & 4 & 3 & 2 & 1 \\ & 2 & . & . & 2 & . & . & 2 & 1 & . & 1 \\ & & 4 & . & 2 & 2 & 6 & . & 3 & 2 & 1 \\ & & & 2 & . & . & . & 1 & . & 2 & 1 \\ & & & & 2 & . & . & . & 1 & . & 1 \\ & & & & & 2 & . & . & 1 & . & 1 \\ & & & & & & 6 & . & 3 & 2 & 1 \\ & & & & & & & 1 & . & . & 1 \\ & & & & & & & & 1 & . & 1 \\ & & & & & & & & & 2 & 1 \\ & & & & & & & & & & 1 \end{pmatrix}.$$

The Burnside matrix looks as follows

$$\begin{pmatrix} 1/24 & -1/4 & -1/8 & -1/6 & 1/4 & . & 1/12 & 1/2 & . & 1/6 & -1/2 \\ & 1/2 & . & . & -1/2 & . & . & -1 & . & . & 1 \\ & & 1/4 & . & -1/4 & -1/4 & -1/4 & . & 1/2 & . & . \\ & & & 1/2 & . & . & . & -1/2 & . & -1/2 & 1/2 \\ & & & & 1/2 & . & . & . & -1/2 & . & . \\ & & & & & 1/2 & . & . & -1/2 & . & . \\ & & & & & & 1/6 & . & -1/2 & -1/6 & 1/2 \\ & & & & & & & 1 & . & . & -1 \\ & & & & & & & & 1 & . & -1 \\ & & & & & & & & & 1/2 & -1/2 \\ & & & & & & & & & & 1 \end{pmatrix}$$

Further tables can be found in [3], [2] and [5].

9 Transversals of Symmetry Classes

The most ambitious enterprise is the redundancy free construction of a transversal of the G classes on Y^X. For technical reasons we put

$$X = \{x_1, \ldots, x_n\} \text{ and } Y = \{y_1, \ldots, y_m\}.$$

Moreover, in order to decrease the complexity we restrict attention to G–classes of fixed content

$$\lambda = (\lambda_1, \ldots, \lambda_m),$$

where $\lambda_i = |f^{-1}(y_i)|$, the multiplicity with which f takes the value y_i. Here is the *canonical mapping* f with this content:

$$f_\lambda := (f_\lambda(x_1), \ldots, f_\lambda(x_n)) := (\underbrace{y_1, \ldots, y_1}_{\lambda_1}, \underbrace{y_2, \ldots, y_2}_{\lambda_2}, \ldots, \underbrace{y_m, \ldots, y_m}_{\lambda_m}).$$

The set of all the mappings of this content will be indicated as follows:

$$Y_\lambda^X = \{\pi f_\lambda = f_\lambda \circ \pi^{-1} \mid \pi \in S_n\}.$$

Since a permutation of the arguments does not change the content, this set Y_λ^X is a union of orbits of G on Y^X:

$$G \backslash\!\backslash Y_\lambda^X \subseteq G \backslash\!\backslash Y^X.$$

We should like to construct a transversal of $G \backslash\!\backslash Y_\lambda^X$. This reduces the complexity since the desired transversal of $G \backslash\!\backslash Y^X$ is the union of the transversals of the sets $G \backslash\!\backslash Y_\lambda^X$, taken over all the contents λ:

$$G \backslash\!\backslash Y^X = \dot{\bigcup}_\lambda G \backslash\!\backslash Y_\lambda^X.$$

To begin with, we note *that Y_λ^X is an orbit of the symmetric group S_X on X* :

$$Y_\lambda^X = S_X(f_\lambda).$$

Hence we can use the Fundamental Lemma 2.1 in order to derive the following decisive result of Ruch, Hässelbarth and Richter ([11]):

9.1 Lemma *The following mapping is a bijection between the set of symmetry classes of mappings of content λ and the set of (G, S_λ)-double cosets in the symmetric group S_X :*

$$G\backslash\!\!\backslash Y_\lambda^X \to G\backslash S_X/S_\lambda,$$

where

$$S_\lambda = \bigoplus_i S_{f_\lambda^{-1}(y_i)},$$

the stabilizer of f_λ.

Since the set of double cosets is a set of orbits,

$$G\backslash S_X/S_\lambda = G\backslash\!\!\backslash S_X/S_\lambda,$$

it is helpful to note that there are *lexicographically smallest* elements in the cosets, from which we can get lexicographically smallest elements of the double cosets and hence a *canonical transversal* of $G\backslash\!\!\backslash Y_\lambda^X$. In order to describe these elements we use that a mapping $f \in Y_\lambda^X$ can be displayed by its λ–*tabloid*

$$\begin{array}{c} \overline{\underline{i_1 \ldots i_{\lambda_1}}} \\ \underline{j_1 \ldots j_{\lambda_2}} \\ \ldots \end{array},$$

the rows of which are formed by the *indices* of the inverse images in increasing order,

$$f(x_{i_1}) = \ldots = f(x_{i_{\lambda_1}}) = y_1, \text{ and } i_1 < \ldots < i_{\lambda_1},$$

$$f(x_{j_1}) = \ldots = f(x_{j_{\lambda_2}}) = y_2, \text{ and } j_1 < \ldots < j_{\lambda_2},$$

$$\ldots$$

Let us consider again, as an example, the necklaces with 5 pearls in two colours, $n = 5$, $m = 2$, and, as a particular content, $\lambda = (3, 2)$. Before we write down all the 10 $(3, 2)$ tabloids in full detail, it is practical to note that in each tabloid the first (or any other) row is uniquely determined by the remaining rows which form the *truncated* tabloid. Hence in the present case we need only to display the second rows, here they are:

$$\overline{\underline{12}},\ \overline{\underline{13}},\ \overline{\underline{14}},\ \overline{\underline{15}},\ \overline{\underline{23}},\ \overline{\underline{24}},\ \overline{\underline{25}},\ \overline{\underline{34}},\ \overline{\underline{35}},\ \overline{\underline{45}}.$$

According to the above arguments we have to find the orbits of the cyclic group

$$G = C_5 = \langle (12345) \rangle$$

on this set. Applying the generating permutation (12345) of the cyclic group and all its different powers to such a truncated tabloid, we obtain the (two different) orbits of the tabloids,

$$\omega_1 = \{\overline{\underline{12}}, \overline{\underline{23}}, \overline{\underline{34}}, \overline{\underline{45}}, \overline{\underline{15}}\},$$

$$\omega_2 = \{\overline{\underline{13}}, \overline{\underline{24}}, \overline{\underline{35}}, \overline{\underline{14}}, \overline{\underline{25}}\}.$$

The lexicographically smallest elements of these orbits ω_1, ω_2 are the tabloids corresponding to $\overline{\underline{12}} \in \omega_1$ and $\overline{\underline{13}} \in \omega_2$. Adding the truncated first lines we obtain the full tabloids

$$\begin{array}{l}\overline{\underline{345}}\\ \underline{12}\end{array} \text{ and } \begin{array}{l}\overline{\underline{245}}\\ \underline{13}\end{array}.$$

Hence there are exactly two different necklaces with 5 pearls in two colours, containing exactly two pearls in the second colour. Here they are:

These methods, using double cosets, can be applied in many other cases, too. Permutation isomers can be constructed, diamutamers, combinatorial libraries. They are discussed in further papers of the present issue.

In this way double cosets turned out to be a quite general tool for classification in mathematics and in sciences, see [11], and in particular the review article [12].

Tabloids can be used for hand calculations, a computer oriented systematic way of successively evaluating transversals of all the occuring weights, the *ladder game*, is described in [7] and in [5].

10 Constructing by Symmetry Group

The evaluation of a transversal of *all* the orbits of G on Y^X can be refined to an evaluation of a transversal of the orbits of prescribed type $\widetilde{U}$. This was shown in [6] by R. Laue, using the following very general argument that is in fact a set theoretic version of 5.2:

- Since each orbit of type $\widetilde{U}$ contains fixed points of U, we can find a transversal of the orbits of this type among the set

$$X_U := \{x \in X \mid \forall\, g \in U\colon\ gx = x\}$$

of fixed points of U.

- We can restrict attention from X_U to the subset $\hat{X}_U$ of fixed points that have no bigger subgroup as stabilizer. This set is easily obtained by subtracting the fixed points of all the subgroups V of G that contain U as a maximal subgroup:

$$\hat{X}_U = X_U \ - \bigcup_{V:\ U\ max.\ in\ V} X_V.$$

- Moreover, the elements of $\hat{X}_U$ which are in the same orbit of G are in the same orbit with respect to the normalizer $N_G(U)$ or — which is easier to check — in the same orbit with respect to the factor group $N_G(U)/U$.

Summarizing we obtain

10.1 Laue's Lemma *Each transversal T of the set of orbits of $N_G(U)/U$ on $\hat{X}_U$,*

$$T \in \mathcal{T}(N_G(U)/U \backslash\backslash \hat{X}_U)$$

is a transversal of the orbits of type $\widetilde{U}$ of G on X :

$$T \in \mathcal{T}(G \backslash\backslash_{\widetilde{U}} X).$$

Moreover, all these orbits of $N_G(U)/U$ are of the same size $|N_G(U)/U|$, and hence it is easy to generate elements of them uniformly at random.

The application to symmetry classes ([6]) is easy since the mappings $f \in Y^X$ that are fixed points of U are just the mappings which are constant on the orbits of U and so we can easily display them in the following symbolic way:

$$f \in \left(Y^X\right)_U = \prod_{\omega \in U \backslash\backslash X} \sum_{y \in Y} y^{|\omega|}$$

since each monomial summand of this product arises by picking from each one of the factors $\sum_{y \in Y} y^{|\omega|}$ a summand $y^{|\omega|}$ which means that the corresponding mapping f has this value y on the orbit ω.

10.2 Corollary (Laue) *Each transversal T of the set of orbits of $N_G(U)/U$ on*

$$\left(\hat{Y^X}\right)_U = (Y^X)_U - \bigcup_{V:\, U \text{ max. in } V} (Y^X)_V,$$

i.e. each

$$T \in \mathcal{T}\left(N_G(U)/U \backslash\!\backslash \left(\hat{Y^X}\right)_U\right)$$

is a transversal of the orbits of type $\widetilde{U}$ of G on X :

$$T \in \mathcal{T}\left(G \backslash\!\backslash_{\widetilde{U}} Y^X\right).$$

Moreover, all these orbits of $N_G(U)/U$ are of the same size $|N_G(U)/U|$, and hence it is easy to generate elements of them uniformly at random.

For example, if $X = \{1, 2, 3, 4\}$, $G = A_4$ and $Y = \{y_1, y_2\}$, then we can see from the section above, where the table of marks of A_4 is given, that it has the subgroups $U_2 = \langle (13)(24) \rangle$ and $U_4 = \langle (13)(24), (14)(23) \rangle$. Moreover, U_2 is maximal in U_4. Their orbit sets are

$$U_2 \backslash\!\backslash Y^4 = \{\{1,3\}, \{24\}\},\ U_4 \backslash\!\backslash Y^4 = \{\{1, 2, 3, 4\}\}.$$

Hence the set of mappings $f = (f(1), f(2), f(3 =, f(4))$ which are fixed under U_2 is

$$\left(Y^4\right)_{U_2} = \{(y_1, y_1, y_1, y_1), (y_2, y_2, y_2, y_2), (y_1, y_2, y_1, y_2), (y_2, y_1, y_2, y_1).$$

Since U_4 is transitive, we have to subtract from this set the constand mappings, obtaining

$$(\hat{Y^4})_{U_2} = \{(y_1, y_2, y_1, y_2), (y_2, y_1, y_2, y_1).$$

Since U_4 is transitive and contained in the normalizer of U_2, this set $(\hat{Y^4})_{U_2}$ is an orbit of $N_{A_4}(U_2)$ and we find a transversal of the orbits of type U_2 :

$$T = \{(y_1, y_2, y_1, y_2)\}.$$

This situation is discussed in the paper by W. Hässelbarth and the present author in another contribution to this issue of MATCH. An application to C_{60} is given in [8].

11 Generating Orbit Representatives

The evaluation of an orbit transversal is of limited use if the complete catalog is too big (for example there exist chemical formulae that have billions of connectivity isomers). Of course, there exist methods for the generation of selected subsets only. But there are situations where one would prefer to test a hypothesis on the complete catalog without being able to generate it. Hence the question arises how we can generate orbit representatives *uniformly at random,* which means that we should like to generate elements $x \in X$ that belong to an orbit of G with the probability $|G\backslash\backslash X|^{-1}$.

This can be done with the following algorithm:

11.1 The Dixon/Wilf Algorithm *If ${}_GX$ denotes a finite action, then we can generate orbit representatives uniformly at random in the following way:*

- *Choose a conjugacy class C of G with probability*

$$p(C) := \frac{|C||X_g|}{|G||G\backslash\backslash X|}, \text{ where } g \in C.$$

- *Pick any $g \in C$ and generate a fixed point x of g, uniformly at random.*

Then the probability that x is an element of the orbit $\omega \in G\backslash\backslash X$ is $1/|G\backslash\backslash X|$, i.e. x is uniformly distributed over the orbits of G on X.

The application of this method to the generation of representatives of symmetry classes of mappings reads as follows:

11.2 Corollary *For finite ${}_GX$ and Y the following procedure yields elements $f \in Y^X$ that are distributed over the G-classes on Y^X uniformly at random:*

- *Choose a conjugacy class C of G with the probability*

$$p(C) := \frac{|C||Y|^{c(\bar{g}')}}{\sum_g |Y|^{c(\bar{g})}}, g' \in C.$$

- *Pick any $g \in C$, evaluate its cycle decomposition and construct an $f \in Y^X$ that takes values $y \in Y$ on these cycles which are distributed uniformly at random over Y.*

Consider our standard example for this situation: we would like to generate necklaces with 5 pearls in 2 colours uniformly at random. The symmetry group $G = C_5$ is an abelian group, and hence the conjugacy classes consist of single elements, and there are five of them. Assume that the first one is the class of the identity element, it consists of 5 cyclic factors, $c(\bar{1}) = 5$. The elements in the other four conjugacy classes are cyclic permutations, consisting of a single cycle of length 5, and so, for each of them, we have $c(\bar{g}) = 1$. Moreover, we know already that there are altogether 8 orbits, which shows that the probabilities of the conjugacy classes are

$$4/5, 1/20, 1/20, 1/20, 1/20.$$

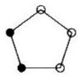
In order to choose a conjugacy class with the prescribed probability we use a generator that generates real numbers in the interval $(0, 1]$ at random.
Assume, for example, that it generated a positive real number $\leq 4/5$. This means that the first one of the conjugacy classes was chosen, the class of the identity element. We have to pick an element of this class, there is not much choice, the identity is the only element in this class. Its cycle decomposition is

$$1 = (x_1)(x_2)(x_3)(x_4)(x_5).$$

Now we have to generate a fixed point of this element, uniformly at random. Fixed points are the mappings $f \in Y^X$, where $X = \{x_1, \ldots, x_5\}$ and $Y = \{\bullet, \circ\}$ that are constant on the cyclic factors. In order to generate such a fixed point uniformly at random, we use a generator which generates either $\bullet$ or $\circ$, uniformly at random. Assume, for example, that five runs of that generator give the sequence

$$\bullet, \circ, \circ, \circ, \bullet.$$

We associate this sequence with the vertices of the regular 5-gon (counterclockwise or clockwise) and obtain the necklace



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