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Yet Another Generalization of Pólya's Theorem: Enumerating Equivalence Classes of Objects with a Prescribed Monoid of Endomorphisms

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Abstract

Let a monoid A of endomorphisms with the identity e act on a nonempty finite set X ($|X| = n$). A submonoid $S \subseteq A$ induces *the set* $S\backslash\backslash X = \{X_1(S), X_2(S), \ldots, X_l(S)\}$ ($1 \le l \le n$) *of orbits on* X; $\cup_{i=1}^{l} X_i = X$ and $X_i \cap X_j = \emptyset$ if $i \neq j$. Let further the *coloration function* $\phi(x)$ ($x \in X$; $\phi(x) \in K = \{1, 2, \ldots, k\}$) take on a constant (*color*) on each *orbit* X_i ($1 \le i \le l$): $\phi(x) \equiv \phi(y)$ $\forall x, y$ ($x, y \in X_i$). Two coloration functions, or *colorations*, ϕ_1 and ϕ_2 are *A-equivalent* iff there exist elements $a, b \in A$, such that $\phi_1(ax) = \phi_2(bx)$ $\forall x$ ($x \in X$); otherwise, they are *A-nonequivalent*.

The following problem is resolved: Given a submonoid $S \subseteq A$ of endomorphisms of X and the set K of colors, *enumerate A-equivalence classes of S-invariant colorations* ϕ *of* X *by content.*

The results obtained are new extensions of *Pólya's counting theorem* and presented in terms of the *generalized cycle indicators*.

Some illustrative examples and open problems are also proposed.

1 Preliminaries

Pólya's theorem [1, 2] and diverse refinements thereof [2–12] play an outstanding role in combinatorics and its applications. In particular, counting objects with a prescribed automorphism group is of paramount significance in chemistry and physics, where there exists the necessity to enumerate the so-called substitutional isomers of molecules (see [7–13]) and large unit cells of crystallographic lattices with color symmetry [7].

Herein, we shall expound for the first time some original results that allow us to count equivalence classes of objects, by weight, with a prescribed monoid of endomorphisms. This novel line of investigation is based on our normalizer approach [8] (see also [9, 10]) that was initially applied to enumeration of objects with a prescribed automorphism group.

Let a monoid A of endomorphisms with the identity e act on a nonempty finite set X; $|X| = n$. A submonoid S $(\{e\} \subseteq S \subseteq A)$ induces the *set* $S\backslash\backslash X = \{X_1(S), X_2(S), \ldots, X_l(S)\}$ $(1 \leq l \leq n)$ *of its orbits on* X (or S-*orbits*, for short). An *orbit* $X_i = X_i(S)$ $(1 \leq i \leq l)$ is a minimal S-closed subset, of X, such that $\forall h \in S$, $\forall x \in X_i$ and $\forall y \in X \setminus X_i$ we have $hx \in X_i$ and $hy \in X \setminus X_i$. Consider the set Φ_k of all mappings K^X from X into the *set* $K = \{1, 2, \ldots, k\}$ *of colors*; $|\Phi_k| = k^n$. Two *coloration functions* (or *colorations*, for short) ϕ_1 and ϕ_2 $(\phi_1, \phi_2 \in \Phi_k)$ are called A-*equivalent* iff there exist elements a and b $(a, b \in A)$ such that $\phi_1(ax) \equiv \phi_2(bx)$ $\forall x$ $(x \in X)$; otherwise, they are A-*nonequivalent*. The colorations ϕ are called *(nonstrictly-)*S-*invariant* $(S \subseteq A)$ if these also include (all) T-invariant $(S \subset T)$ colorations and *strictly-*S-*invariant* if they exclude any T-invariant colorations.

Our major objective is to construct the generalized cycle indicators (see [7–11]) enumerating both types of colorations by weight. Under this, the simplest problem of the two is counting the nonstrictly-S-invariant mappings from Φ_k; enumeration of the strictly-S-invariant colorations is possible only through the exclusion/inclusion argument (see [8–11]). However, the last kind of manipulation is quite a routine procedure [8–11] dealing with the so-called *Möbius function* $\mu(L)$ of the (sub-)lattice L of special (closed) subalgebras S of A (see below); therefore, we shall chiefly focus our attention on solving the first problem.

Our approach is based on the previously obtained results for A being an automorphism group $G = \mathrm{Aut}X$ of X [7–9] and the key feature of this is employing remarkable combinatorial properties of the normalizer $N_G(\hat{H})$ of an automorphic subgroup $\hat{H}$ $(\{e\} \subseteq \hat{H} \subseteq G)$ in the group G of automorphisms of X. A subgroup $\hat{H}$ above is synonymously called by Rota and

Smith *closed* and *periodic* [5], or *automorphic* [8, 9], iff for any subgroup $H' \subseteq G$ satisfying the equality $H'\backslash\backslash X = \hat{H}\backslash\backslash X$ it follows that $H' \subseteq \hat{H}$; thus, $|H''\backslash\backslash X| < |\hat{H}\backslash\backslash X|$ if $\hat{H} \subset H'' \subseteq G$. In other words, $\hat{H}$ is the maximum among all subgroups inducing one and the same set of orbits $(\hat{H}\backslash\backslash X)$ and contains all these equiorbital subgroups, if there exist more than one. By this reason, $\hat{H}$ is also termed the *closure* of all its equiorbital subgroups H with respect to their common orbits $\hat{H}\backslash\backslash X$ on X.

We shall rigorously prove (group- and semigroup-theoretically) that the normalizer $N_G(\hat{H})$ of a closed subgroup $\hat{H}$ $(\{e\} = E \subseteq \hat{H} \subseteq G)$ is the maximum subgroup, in G, that (either fixes or) permutes intact H-orbits in $H\backslash\backslash X = \hat{H}\backslash\backslash X$. This same result had been known long before, as folkloristic or even empirical, in theoretical papers by crystallographers, where it was used without any demonstration. Amazingly, the mentioned properties of the normalizer $N_G(\hat{H})$ of a closed subgroup $\hat{H}$ had been overlooked by the purely mathematical community. However, namely, the great German mathematician Hermann Weyl was the first who began studying the so-called "hidden symmetry" (see [11], p. 50) which can be attributed to the action of the factor group $N_G(\hat{H})/\hat{H}$ on the set $\hat{H}\backslash\backslash X$ of intact $\hat{H}$-orbits exhibiting the "obvious symmetry". Physicists even give the name *Weyl's Recipe* for his recommendation to consider both *obvious and hidden symmetries* of objects. The present author independently came to studying the latter type of symmetry in [7], wherein that was profitably used in enumerating the large unit cells of crystals.

As the next step, our exposition will be extended to the normalizer $N_M(\hat{S})$ of a closed submonoid $\hat{S}$ in the monoid M of endomorphisms of X (for more details, see §3). By analogy, a submonoid $\hat{S}$ $(E \subseteq \hat{S} \subseteq M)$, of M, is called *closed* iff for any submonoid S' satisfying the equality $S'\backslash\backslash X = \hat{S}\backslash\backslash X$ it follows that $S' \subseteq \hat{S}$. Another necessary result immediately follows from the theory of semigroups [14, 15]: the factor monoid N/S by submonoid S, where $N = N_M(S)$ and S is (not necessarily) closed, is homomorphic to the maximum group P that can be a homomorphic image of N.

In what follows, this group P plays a very important role because it may be displayed as a group of permutations acting on a set $\hat{S}\backslash\backslash X$ of intact orbits of a closed submonoid $\hat{S} \subseteq M$. It resembles the above when we dealt with $N_G(\hat{H})$ (the action of the factor group $N_G(\hat{H})/\hat{H}$ on $\hat{H}\backslash\backslash X$ will also be considered later on); however, what is now absolutely novel is that P may thus also permute $\hat{S}$-orbits that are not equipotent, as blocks within P-orbits on $\hat{S}\backslash\backslash X$.

Here, the reader may well take the set $I = \{1, 2, \ldots, l\}$ $(1 \leq l \leq n)$ of S-orbit indices instead of $S\backslash\backslash X$ itself; this will not violate our approach. To enumerate *nonstrictly-S-invariant colorations*, one can construct the *generalized cycle indicator* $Q(S; S\backslash\backslash X; z_1, z_2, \ldots, z_n)$ (in notation) for a closed submonoid $\hat{S}$ operating on the set $\hat{S}\backslash\backslash X = S\backslash\backslash X$ of intact S-orbits on X, which is, in fact, defined as the usual Pólya's cycle indicator $Z(P; I; z_1, z_2, \ldots, z_n)$ (see [1–12]) for a group of permutations P acting on the set I of S-orbit indices.

We have drawn on the key features of our exposition below. The latter will also include some illustrative examples and open problems that may stimulate the further development of the subject.

2 When A is the group $G = \mathrm{Aut} X$ of automorphisms

2.1 Combinatorial properties of certain normalizers

Let X be a finite set on which a finite group G is acting from the left:

$$G \times X \mapsto X : (g, x) \mapsto gx,$$

such that

$$g(g'x) = (gg')x \text{ and } 1x = x \qquad (\forall x \in X \text{ and } \forall g, g' \in G)\,.$$

If H is a subgroup of G, then it also induces an action on this set X, we simply need to restrict attention to the subgroup. It forms *orbits*

$$H(x) := \{hx \mid h \in H\},$$

and we denote the set of all these orbits by

$$H\backslash\backslash X := \{H(x) \mid x \in X\}.$$

There may be other subgroups with this same set of orbits, and it is not difficult to check that there is the maximum element in this set of subgroups; let us denote it by

$$\hat{H} := \{g \in G \mid g(\omega) = \omega,\ \forall\, \omega \in H\backslash\backslash X\}.$$

Now we remark that for every $g \in G$ we obtain, for any orbit of H,

$$gH(x) = H^g(gx), \text{ if } H^g := gHg^{-1}.$$

We shall apply this to the maximal subgroup $\hat{H}$ introduced above, which we shall call (following Rota and Smith [5]) *closed*, or *periodic* (it was also called by us *automorphic* in [8, 9]).

Theorem 1. *Let H be a subgroup of G. Then the normalizer $N_G(\hat{H})$ is the maximum subgroup, in G, the elements of which permute the intact orbits of H on X.*

Proof. Clearly, $\forall g \in G$ and $\forall x \in X$ $g\hat{H}(x) = \hat{H}^g(gx)$, where the L.H.S. corresponds to the action of g on an orbit $\hat{H}(x)$ while the R.H.S. does to a derivative orbit $\hat{H}^g(x')$ generated by a subgroup $\hat{H}^g \subseteq G$ from an element $x' = gx \in X$. Note also that an element g^{-1} uniquely transfers the image orbit $\hat{H}^g(x')$ back into its preimage $\hat{H}(x)$. Thus, the above action of an element g simultaneously performed on all $\hat{H}$-orbits, on X, induces a one-valued invertible mapping from the set $\hat{H}\backslash\backslash X$ of all $\hat{H}$-orbits into the set $\hat{H}^g\backslash\backslash X$ of $\hat{H}^g$-orbits. Obviously, under this mapping, $\hat{H}\backslash\backslash X \equiv \hat{H}^g\backslash\backslash X$ and we thus obtain a permutation, or automorphism, of $\hat{H}\backslash\backslash X$ (when the action of g either fixes or permutes intact $\hat{H}$-orbits). Since $\hat{H}$ is closed and $|\hat{H}^g| = |\hat{H}|$ (i. e., $\hat{H}^g$ could not be any equiorbital subgroup of $\hat{H}$), $\hat{H}^g\backslash\backslash X = \hat{H}\backslash\backslash X$ iff $\hat{H}^g = \hat{H}$, by definition. Clearly, the subset of all elements g, in G, for which $\hat{H}^g = g\hat{H}g^{-1} = \hat{H}$, is $N_G(\hat{H})$ and we at once arrive at the proof. □

Theorem 1 can also be derived within the framework of the theory of semigroups [14, 15], as a corollary for groups (see below). To the best of our knowledge, in the English-language literature, this statement was employed for the first time in our paper [8] (which extended a text in Russian [7]).

Fueled with Theorem 1, one can consider the combinatorial action of $N_G(\hat{H})$ on $H\backslash\backslash X = \hat{H}\backslash\backslash X$ in the same methodological manner as the action of G on X.

2.2 The cycle indicators for a closed subgroup $\hat{H}$

This subsection will chiefly cite the results of [8] or their improved versions proposed by Kerber [10].

Let $G = \mathrm{Aut}X$ be the automorphism (or permutation) group acting on a nonempty finite set X of objects, as above. As was shown by Pólya, *G-equivalence classes of objects*, or *G-orbits*, can be enumerated by weight, by means of the special polynomial $Z(G; X; z_1, z_2, \ldots, z_n)$ called the *cycle indicator* (or *index*) [1, 2] (see also [3–12]). It may be written down as follows:

$$Z(G;X) = Z(G;X;z_1,z_2,\ldots,z_n) = \frac{1}{|G|}\sum_{g\in G}\prod_{i||G|} z_i^{\alpha_i(<g>)}\,, \tag{1}$$

where $|G|$ is the cardinality of a group G; z_i's are weight-indeterminates used for convenience of notation; $\alpha_i(< g >)$ is the number of orbits of length i induced by the cyclic group $< g >$ generated by an element $g \in G$; the sum runs over all elements of G and the product is taken over all divisors i of $|G|$.

We should mention that here we changed the standard notation $\alpha_i(g)$ to a new, equivalent, notation $\alpha_i(< g >)$ not in vain but because we shall violate the old tradition to use for enumeration purposes just cyclic (1-generator) subgroups, using subgroups $< g\hat{H} >$ generated by the complete coset $g\hat{H}$ of $N_G(\hat{H})$.

Let a weight-indeterminate w_t $(1 \leq t \leq k)$ stand for the t-th color above. The following statement is a version of Pólya's counting theorem [1–12], viz.:

Theorem 2. *The number of G-equivalence classes of K-colorations of X with a given assortment of K-colors equals the corresponding coefficient of the polynomial*

$$Z(G;X;w_1,w_2,\ldots,w_n) = Z(G;X;z_1,z_2,\ldots,z_n)\Big|_{z_i=\sum_{t=1}^{k} w_t^i \quad (i||G|)} \,. \tag{2}$$

Here, we can turn from coloring individual elements of X to coloring intact H-orbits (i.e., elements of $H\backslash\backslash X$). The following statement is an elementary common corollary of Theorem 1 and Theorem 2:

Lemma 3. *The number of G-equivalence classes of K-colorations of $H\backslash\backslash X$ with a given assortment of K-colors equals the corresponding coefficient of the polynomial*

$$Z(N_G(\hat{H}); H\backslash\backslash X; w_1,w_2,\ldots,w_n) = \left[\frac{1}{|N_G(\hat{H})|}\sum_{g\in N_G(\hat{H})}\prod_{i||N_G(\hat{H})|} z_i^{\alpha_i(<g>)}\right]_{z_i=\sum_{t=1}^{k} w_t^i \quad (i||G|)} . \tag{3}$$

***Proof*.** The result follows by applying the cycle index definition to the set $H\backslash\backslash X$ and normalizer $N_G(\hat{H})$ considered as its automorphism group $\mathrm{Aut}(H\backslash\backslash X)$ in Theorem 1. Under this, substituting $N_G(\hat{H})$ and $H\backslash\backslash X$, respectively, for G and X in Theorem 2 immediately gives the proof. □

By virtue of Theorem 2, the maximum orbit length (cardinality) that may be realized in the set $N_G(\hat{H})\backslash\backslash(H\backslash\backslash X)$ is $s = |N_G(\hat{H})|/|\hat{H}|$. Also, we mention that $\hat{H}$ lies in the kernel of the action of $N_G(\hat{H})$ on $H\backslash\backslash X$; by this reason, we can replace $N_G(\hat{H})$ with $N_G(\hat{H})/\hat{H}$ and, therefore, restrict the summation over the complete normalizer $N_G(\hat{H})$ to that over the left cosets $g\hat{H}$ of $\hat{H}$ in $N_G(\hat{H})$. As a result, we can obtain the following

Lemma 4. *Let $Z(N_G(\hat{H}); H\backslash\backslash X)$ be the above cycle indicator. Then*

$$Z(N_G(\hat{H}); H\backslash\backslash X; z_1, z_2, \ldots, z_n) = \frac{1}{s} \sum_{g\hat{H} \in N_G(\hat{H})/\hat{H}} \prod_{i=1}^{|H\backslash\backslash X|} z_i^{\alpha_i(<g\hat{H}>)} = \frac{1}{s} \sum_{g\in J} \prod_{i|s} z_i^{\alpha_i(<g>)}, \tag{4}$$

where J $(|J| = s)$ is an arbitrary fixed transversal of the left cosets $g_j\hat{H}$ $(1 \le j \le s)$ of $\hat{H}$ in $N_G(\hat{H})$ $(\{g_j\hat{H}\}_{j=1}^{s} = N_G(\hat{H})/\hat{H};\ N_G(\hat{H}) = J\hat{H})$; $< g\hat{H} >$ is the subgroup generated by all elements of a left coset $g\hat{H}$ $(g \in J)$; the first summation goes over all the left cosets of $\hat{H}$ in $N_G(\hat{H})$; the first product is taken over all divisors of $|H\backslash\backslash X|$; the second summation goes over all the transversal J; and the last product is taken over all divisors of s.

Proof. By virtue of Theorem 1, $< g_j\hat{H} >$ induces the same dissection of $H\backslash\backslash X$ into orbits as does any cyclic subgroup $< g >$ $(g \in g_j\hat{H}; g_j \in J; 1 \le j \le s)$. However, (4) uses every element of $g_j\hat{H}$ $(1 \le j \le s)$ which is tantamount to $|\hat{H}|$-fold using of each element g $(g \in J)$, hence we have

$$\frac{1}{|N_G(\hat{H})|} \sum_{g\in N_G(\hat{H})} (\cdots) = \frac{|\hat{H}|}{|N_G(\hat{H})|} \sum_{g\hat{H}\in N_G(\hat{H})/\hat{H}} (\cdots) = \frac{1}{s} \sum_{g\in J} (\cdots)$$

which completes the proof. □

It is worth noting that the transversal J above may be just a subset (or *complex*, in group-theoretical terms) of $N_G(\hat{H})$, with or without the unity e, not necessarily a subgroup; furthermore, in general, there may exist no subgroup $H' \cong N_G(\hat{H})/\hat{H}$ in $N_G(\hat{H})$, at all.

Let us return to the base set X now. Reconsidered as the corresponding subset of X, each orbit of the orbit set $< g\hat{H} > \backslash\backslash(H\backslash\backslash X)$ is the union of an integer number of complete orbits from the orbit set $H\backslash\backslash X$. Also, in the case of the groups we consider, all orbits of the latter set comprising one orbit of the former set have one and the same cardinality; however, the last circumstance, which may be helpful here, nevertheless, is not essential to what follows below for other algebras that may disregard such a condition. Somehow or other, one can derive from the above statements the following

Lemma 5. *Let H $(E \subseteq H \subseteq G)$ be a subgroup of G. The number of G-equivalence classes of nonstrictly-H-invariant K-colorations $\phi \in \Phi_k$, of X, with a given assortment of colors equals the corresponding coefficient of the generalized cycle indicator*

$$Q(H;X;w_1,w_2,\ldots,w_n) = Q(H;X;z_1,z_2,\ldots,z_n)\Big|_{z_i=\sum_{t=1}^{k} w_t^i \quad (1\le i\le n)}, \tag{5}$$

where

$$Q(H;X;z_1,z_2,\ldots,z_n) := \frac{1}{s}\sum_{g\in J}\prod_{p|s;q||\hat{H}|} z_{p\cdot q}^{\alpha_{p\cdot q}(<g\hat{H}>)}. \tag{6}$$

Proof. By virtue of Theorem 1, each orbit of cardinality r induced by $N_G(\hat{H})$ on X is composed of q complete H-orbits of cardinality p, on X, $(p\cdot q = r)$; therefore the second side of (6), dealing with the base set X, is nothing but specification of the third side of (4), dealing with $H\backslash\backslash X$. Indeed, substituting z_p for $z_{p\cdot q}$ $(p \mid s; q \mid |\hat{H}|)$ on the R.H.S. of (5), conversely, yields an expression equivalent to the third side of (4). □

The following statement may be proposed as a generalization of Pólya's theorem:

Theorem 6. *The number of G-equivalence classes of H-invariant K-colorations of X with a given assortment of K-colors equals the corresponding coefficient of the polynomial*

$$Q(H;X;w_1,w_2,\ldots,w_n) = \left[\frac{1}{s}\sum_{g\in J}\prod_{i||N_G(\hat{H})|} z_i^{\alpha_i(<g\hat{H}>)}\right]_{z_i=\sum_{t=1}^{k} w_t^i \quad (1\le i\le n)} \tag{7}$$

where

$$\left[\frac{1}{s}\sum_{g\in J}\prod_{i||N_G(\hat{H})|} z_i^{\alpha_i(<g\hat{H}>)}\right] = Q(H;X;z_1,z_2,\ldots,z_n) \tag{8}$$

Proof. Substituting i for every $p\cdot q = i$ in (6) and doing a novel, universal condition $p\cdot q \mid |N_G(\hat{H})|$ for previous $p \mid s$ and $q \mid |\hat{H}|$, in (6), we at once arrive at the proof. □

Evidently, setting $H = E$ $(|E| = 1)$ in (7) gives the famous Pólya's counting theorem as an immediate corollary of Theorem 6.

Obviously, for treating all possible K-colorations of X we need to have all closed subgroups $\hat{H} \subseteq G$ only. Besides, since all conjugated subgroups have the same cycle index Q, as well as Z, we may, undoubtedly, confine ourselves, in any specific case, by considering just a certain, fixed transversal $\mathcal{T}$ of conjugacy classes of closed subgroups.

The cycle indicators $R(H;X)$'s that correspond to the strictly-H-invariant K-colorations can be calculated through all needed indicators $Q(H;X)$'s, using the exclusion/inclusion argument and combinatorial incidency functions (ζ- and μ-, or *Möbius*, functions). Directly solving the corresponding system of simultaneous linear equations for all $R(H;X)$'s, as its roots, is possible as well (see a case in point below).

2.3 An illustrative example

Let us consider colorations of the necklace B with six beads (or the so-called substitutional isomers of a benzene molecule) with a given assortment of bead colors (resp. chemical substituents) [8, 9]. According to the crystallographic notation $G = \mathrm{Aut}B = D_6$, where D_6 is a dihedral group ($|D_6| = 12$). A transversal of closed (automorphic) subgroups of $G = D_6$ is $\mathcal{T} = \{C_1 \equiv E, C_2, C_2', C_2'', D_2, D_3, D_6\}$ (see below). Neither C_3 nor C_6 is closed since $C_3\backslash\backslash B = D_3\backslash\backslash B$ and $C_6\backslash\backslash B = D_6\backslash\backslash B$ where the necklace B is displayed as the base set X above and may simply be represented as the set $\{1,2,3,4,5,6\}$ of naturals, due to its six beads. We may describe all necessary subgroups of G by means of the following permutations:

$$g_1 = \begin{pmatrix} 123456 \\ 123456 \end{pmatrix}; \quad g_2 = \begin{pmatrix} 123456 \\ 456123 \end{pmatrix}; \quad g_3 = \begin{pmatrix} 123456 \\ 165432 \end{pmatrix}; \quad g_4 = \begin{pmatrix} 123456 \\ 321654 \end{pmatrix};$$
$$g_5 = \begin{pmatrix} 123456 \\ 543216 \end{pmatrix}; \quad g_6 = \begin{pmatrix} 123456 \\ 216543 \end{pmatrix}; \quad g_7 = \begin{pmatrix} 123456 \\ 432165 \end{pmatrix}; \quad g_8 = \begin{pmatrix} 123456 \\ 654321 \end{pmatrix};$$
$$g_9 = \begin{pmatrix} 123456 \\ 345612 \end{pmatrix}; \quad g_{10} = \begin{pmatrix} 123456 \\ 561234 \end{pmatrix}; \quad g_{11} = \begin{pmatrix} 123456 \\ 234561 \end{pmatrix}; \quad g_{12} = \begin{pmatrix} 123456 \\ 612345 \end{pmatrix}.$$

Owing to such notation, for instance, the two nonautomorphic subgroups above can be displayed as $C_3 = \{g_1, g_9, g_{10}\}$ and $C_6 = \{g_1, g_2, g_9, g_{10}, g_{11}, g_{12}\}$.

Table 1 below contains all necessary information concerning closed subgroups $\hat{H} \subseteq G$, their normalizers $N_G(\hat{H})$, a fixed transversal $\mathcal{T}$ of these subgroups and the corresponding families $\{< g_1\hat{H} >, < g_2\hat{H} >, \ldots, < g_s\hat{H} >\}$ of subgroups as well.

We shall represent incidency ζ- and μ-functions [8–11] as two corresponding matrices $\underline{\underline{\zeta}} =$

Table 1:

A closed subgroup $\hat{H}$	A normalizer $N_G(\hat{H})$ of $\hat{H}$	A transversal $J[N_G(\hat{H})/\hat{H}]$	A family $\{< g_j\hat{H} >\}_{j=1}^{s}$
$C_1 = \{g_1\}$	D_6	$\{g_1, g_2, \ldots, g_{12}\} = D_6$	$\{C_1, C_2, 3C_2', 3C_2'', 2C_3, 2C_6\}$
$C_2 = \{g_1, g_2\}$	D_6	$\{g_1, g_3, g_4, g_5, g_9, g_{10}\} = D_3$	$\{C_2, 3D_2, 2D_3\}$
$C_2' = \{g_1, g_3\}$	D_2	$\{g_1, g_2\} = C_2$	$\{C_2', D_2\}$
$C_2'' = \{g_1, g_7\}$	D_2	$\{g_1, g_3\} = C_2'$	$\{C_2'', D_2\}$
$D_2 = \{g_1, g_2, g_3, g_7\}$	D_2	$\{g_1\} = C_1$	$\{D_2\}$
$D_3 = \{g_1, g_3, g_4, g_5, g_9, g_{10}\}$	D_6	$\{g_1, g_2\} = C_2$	$\{D_3, D_6\}$
$D_6 = \{g_1, g_2, \ldots, g_{12}\}$	D_6	$\{g_1\} = C_1$	$\{D_6\}$

$||\zeta_{ij}||_{i,j=1}^{7}$ and $\underline{\underline{\mu}} = ||\mu_{ij}||_{i,j=1}^{7}$, where $\underline{\underline{\mu}} = \underline{\underline{\zeta}}^{-1}$, viz.:

$$\underline{\underline{\zeta}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 1 & 0 & 1 \\ & & 1 & 0 & 1 & 1 & 1 \\ & & & 1 & 1 & 0 & 1 \\ & & & & 1 & 0 & 1 \\ & & & & & 1 & 1 \\ & & & & & & 1 \end{bmatrix}; \qquad \underline{\underline{\mu}} = \begin{bmatrix} 1 & -1 & -1 & -1 & 2 & 0 & 0 \\ & 1 & 0 & 0 & -1 & 0 & 0 \\ & & 1 & 0 & -1 & -1 & 1 \\ & & & 1 & -1 & 0 & 0 \\ & & & & 1 & 0 & -1 \\ & & & & & 1 & -1 \\ & & & & & & 1 \end{bmatrix}.$$

Here, we recall that an entry ζ_{ij} $(1 \le i, j \le 7)$ of the matrix $\underline{\underline{\zeta}}$ is equal to 1 iff (if and only if) a subgroup $\hat{H}_i$, corresponding to the i-th row of this matrix, is a not necessarily proper subgroup of a subgroup $\hat{H}_j$, corresponding to the j-th matrix column, $(E \subseteq \hat{H}_i \subseteq \hat{H}_j \subseteq G = D_6)$. To all other automorphism groups $G = \mathrm{Aut}X$, one may adopt the same approach as we demonstrate, in our example, for $D_6 = \mathrm{Aut}B$ (see [10, 11]).

In our case, the exclusion/inclusion argument, or inclusion and exclusion principle, affords the following matrix equality involving just all closed subgroups $\hat{H}$ of $D_6 = \mathrm{Aut}B$:

$$\begin{pmatrix} R(C_1) \\ R(C_2) \\ R(C_2') \\ R(C_2'') \\ R(D_2) \\ R(D_3) \\ R(D_6) \end{pmatrix} = \underline{\underline{\mu}} \begin{pmatrix} Q(C_1) \\ Q(C_2) \\ Q(C_2') \\ Q(C_2'') \\ Q(D_2) \\ Q(D_3) \\ Q(D_6) \end{pmatrix} .$$

Hence, we can write down all the cycle indicators $R(H;B)$'s $(E \subsetneq H \subseteq G)$ for strictly-H-invariant K-colorations through the cycle indicators $Q(H;B)$'s for nonstrictly-H-invariant colorations as follows:

$$R(C_1) = Q(C_1) - Q(C_2) - Q(C_2') - Q(C_2'') + 2Q(D_2);$$
$$R(C_2) = Q(C_2) - Q(D_2);$$
$$R(C_2') = Q(C_2') - Q(D_2) - Q(D_3) + Q(D_6);$$
$$R(C_2'') = Q(C_2'') - Q(D_2);$$
$$R(D_2) = Q(D_2) - Q(D_6);$$
$$R(D_3) = Q(D_3) - Q(D_6);$$
$$R(D_6) = Q(D_6) .$$

Using Theorem 6 and the last (fourth) column of Table 1, we can calculate all the $Q(H;B;z_1,z_2,\ldots,z_6)$'s. Thus, we get

$$Q(C_1) = \frac{1}{12}\left(z_1^6 + 3z_1^2z_2^2 + 4z_2^3 + 2z_3^2 + 2z_6\right) \equiv Z(D_6;B);$$
$$Q(C_2) = \frac{1}{6}\left(z_2^3 + 3z_2z_4 + 2z_6\right);$$
$$Q(C_2') = \frac{1}{2}\left(z_1^2z_2^2 + z_2z_4\right);$$
$$Q(C_2'') = \frac{1}{2}\left(z_2^3 + z_2z_4\right);$$
$$Q(D_2) = z_2z_4;$$
$$Q(D_3) = \frac{1}{2}\left(z_3^2 + z_6\right);$$
$$Q(D_6) = z_6 .$$

Hence, we can find all the $R(H;B;z_1,z_2,\ldots,z_6)$'s, which completes the task, viz.:

$$R(C_1) = \frac{1}{12}\left(z_1^6 - 3z_1^2z_2^2 - 4z_2^3 + 6z_2z_4 + 2z_3^2 - 2z_6\right);$$

$$R(C_2) = \frac{1}{6}\left(z_2^3 - 3z_2z_4 + 2z_6\right);$$
$$R(C_2') = \frac{1}{2}\left(z_1^2z_2^2 - z_2z_4 - z_3^2 + z_6\right);$$
$$R(C_2'') = \frac{1}{2}\left(z_2^3 - z_2z_4\right);$$
$$R(D_2) = z_2z_4 - z_6;$$
$$R\left(D_3^2\right) = \frac{1}{2}\left(z_3^2 - z_6\right);$$
$$R(D_6) = z_6 \ .$$

2.4 Further generalizations

One may ask the following question—If T_1 and T_2 are two not necessarily closed subgroups of an automorphism group $G = \mathrm{Aut}X$ of a nonempty finite set X, then whether or not $T_1\backslash\backslash X = T_2\backslash\backslash X$ implies that any two T_1-invariant colorations, of X, are always T_2-invariant colorations or vice versa? Regrettably, the answer is negative; nevertheless, sometimes, it can be profitably used in practice. To illustrate this, we shall examine below an example dealing with enumerating enantiomorphic stereoisomers in chemistry [9–12]. However, at first, we have to approach that from the general standpoint.

Let $\varepsilon \subseteq G = \mathrm{Aut}X$, call $N_\varepsilon(H) = \varepsilon \cap N_G(H)$ the "generalized normalizer of a subgroup H with respect to a subgroup ε" since $N_\varepsilon(H)$ can well be redefined as the maximum subset of ε such that $\forall g \ (g \in \varepsilon) \ gH = Hg$.

Lemma 7 ([16, 17]). *Let G' be a group and let $T \subseteq G'$, $H \trianglelefteq G'$. Then*

$$TH/H \cong T/T \cap H \ . \tag{9}$$

Corollary 7.1. *Let $T = N_\varepsilon(\hat{H})$ and $G' = N_G(\hat{H})$ (see Lemma 7). Then*

$$\hat{H}N_\varepsilon(\hat{H})/\hat{H} \cong N_\varepsilon(\hat{H})/\varepsilon \cap \hat{H} \ . \tag{10}$$

Proof. By virtue of Lemma 7, $\hat{H}N_\varepsilon(\hat{H})/\hat{H} \cong N_\varepsilon(\hat{H})/(\varepsilon \cap N_G(\hat{H}) \cap \hat{H})$. Since we have $N_G(\hat{H}) \cap \hat{H} = \hat{H}$, the statement is immediate. □

Evidently, Lemma 7 enables one to calculate all G-nonequivalent colorations. In order to generalize it for enumerating all ε-noneqivalent objects ($\varepsilon \subseteq G$), we additionally introduce two generalized cycle indicators $Q_\varepsilon(H;X)$ and $R_\varepsilon(H;X)$, accordingly. Then we can state the following

Lemma 8. *The cycle indicator Q_ε is*

$$Q_\varepsilon(H;X;z_1,z_2,\dots,z_n)=\frac{|\hat{H}|}{|\hat{H}N_\varepsilon(\hat{H})|}\sum_{g\in J'}\prod_{i\,||\hat{H}N_\varepsilon(\hat{H})|} z_i^{\alpha_i(<g\hat{H}>)}\ , \tag{11}$$

where J' $(|J'|=s')$ is an arbitrary fixed transversal of the left cosets $g_j\hat{H}$ $(1\le j\le s')$ of $\hat{H}$ in $\hat{H}N_\varepsilon(\hat{H})=[N_\varepsilon(\hat{H})]\hat{H}$ $(\{g_j\hat{H}\}_{j=1}^{s'}=N_\varepsilon(\hat{H})/\hat{H};J'\hat{H}=[N_\varepsilon(\hat{H})]\hat{H})$.

Proof. The statement is simply a generalization of Lemma 5 and (8), with $N_G(\hat{H})$ replaced by $\hat{H}N_\varepsilon(\hat{H})$. This corresponds to substituting ε-nonequivalent objects for G-nonequivalent ones. QED. □

Corollary 8.1. *The cycle indicator Q_ε is*

$$Q_\varepsilon(H;X;z_1,z_2,\dots,z_n)=\frac{1}{s'}\sum_{g\in J''}\prod_{i\,||\hat{H}N_\varepsilon(\hat{H})|} z_i^{\alpha_i(<g\hat{H}>)}\ , \tag{12}$$

where $s'=|N_\varepsilon(\hat{H})|/|\varepsilon\cap\hat{H}|\equiv|\hat{H}N_\varepsilon(\hat{H})|/|\hat{H}|$; J'' $(|J''|=s')$ is an arbitrary fixed transversal of the left cosets $g_j(\hat{H}\cap\varepsilon)$ $(1\le j\le s')$ of $\hat{H}\cap\varepsilon$ in $N_\varepsilon(\hat{H})$ $(\{g_j(\hat{H}\cap\varepsilon)\}_{j=1}^{s'}=N_\varepsilon(\hat{H})/\hat{H}\cap\varepsilon;N_\varepsilon(\hat{H})=J''(\hat{H}\cap\varepsilon))$.

Proof. Applying Corollary 7.1 to Lemma 8 at once gives the proof. □

We can specially extend Pólya's counting theorem to the case of ε-nonequivalent colorations, which is also relevant to enumerating stereoisomers below, and state the following generalization of Theorem 6:

Theorem 9. *The number of ε-equivalence $(\varepsilon\subseteq G)$ classes of K-colorations of X with a given assortment of K-colors equals the corresponding coefficient of the polynomial*

$$Q_\varepsilon(H;X;w_1,w_2,\dots,w_n)=\left[\frac{1}{s'}\sum_{g\in J''}\prod_{i\,||\hat{H}N_\varepsilon(\hat{H})|} z_i^{\alpha_i(<g\hat{H}>)}\right]_{z_i=\sum_{t=1}^{k}w_t^i\quad(1\le i\le n)}\ . \tag{13}$$

2.5 Another example: enumerating enantiomeric molecules

A pair of stereoisomeric objects (molecules etc.) corresponding to each other as the left hand to the right hand is called a pair of enantiomers (see [12, 13]); under this, each of the two, 'left' or 'right', is characterized by the adjective "chiral". An object that cannot correspond to any other one as a 'left' or 'right' enantiomer is alternatively called "achiral". Note, in passing, that both of these definitions, 'left' and 'right', are also applicable to molecules for their ability to turn the polarization plane of light either left or right, which enables one to use them in a direct rather than conditional sense.

Let here $G = \mathrm{Aut}X$ be a symmetry group of a skeleton (core) of a substitutional isomer X [8–13] (i.e., changing the assortment of ligands of the latter, as well as rearranging them, may give other substitutional isomers possessing the same skeleton). Let further $\varepsilon \subseteq G$ be the maximum subgroup, of G, all nontrivial elements of which correspond to proper rotations of the core of X. Then it is known [8–13] that all chiral substitutional isomers are ε-nonequivalent, whether they belong to the same pair of left-right ones or not. Alternatively, if G contains at least one element not being a strict proper rotation of a skeleton X (e.g., a mirror axis or an inversion; if $\varepsilon \subset G$, then $|G : \varepsilon| = 2$ and, therefore, G contains strictly $|\varepsilon|$ such elements), then any two substitutional isomers comprising a pair of left-right enantiotwins are G-equivalent.

Thus, in the above terms, $Q(H; X)$ enumerates every achiral object once and every pair of enantiomeric stereoisomers, taken as a whole, once too (i.e., an automorphism group G does not distinguish left and right stereotwins, at all, certainly, if $G = \mathrm{Aut}X$ is not a group of proper rotations only). On the other hand, $Q_\varepsilon(H; X)$ just distinguishes left and right chiral isomers, if they only exist (i.e., it takes into account every pair of stereoisomers already twice) and counts as the former index, $Q(H; X)$, every achiral object once. Hence, the difference $U(H; X) := Q_\varepsilon(H; X) - Q(H; X)$ enumerates just enantiomeric pairs, which is of a considerable importance to chemistry (to tell the truth, the calculation scheme itself is well-known from the literature (see [8–13]) expounding it in other terms). Likewise, $V(H; X) := R_\varepsilon(H; X) - R(H; X)$.

To sum up all that has been said about possible applications, one may conclude that the six cycle indicators $Q(H; X)$, $R(H; X)$, $Q_\varepsilon(H; X)$, $R_\varepsilon(H; X)$, $U(H; X)$ and $V(H; X)$, introduced above, cover by themselves most problems in chemistry that arise in connection with enumerating various isomeric objects, taking into account their symmetry and assortment of ligands.

2.6 Differential properties of the cycle indicators

All the cycle indicators above, both classical and generalized, have some common differential properties. The following selected statements will be given herein without any demonstration; we propose to the reader to derive his/her own proofs, as respective exercises to the text.

Proposition 10. *Let $G_x \subseteq G = \mathrm{Aut}X$ be the stabilizer of an element x in X. Let further $Z(G; X)$ be Pólya's cycle indicator. Then*

$$z_1 \frac{\partial}{\partial z_1} Z(H; X; z_1, z_2, \ldots, z_n) = \frac{1}{|G|} \sum_{x \in X} |G_x| Z(G_x; X; z_1, z_2, \ldots, z_n) \;, \tag{14}$$

where the summation runs over all the set X.

The second statement symbolizes a converse transition from Pólya's theorem to the Cauchy-Frobenius lemma (see [10, 11]), which was just used for deriving this theorem. Viz.:

Proposition 11. *Let $Z(G; X)$ be Pólya's cycle indicator. Then*

$$|G\backslash\backslash X| = \frac{\partial}{\partial z_1} Z(G; X; z_1, z_2, \ldots, z_n) \big|_{z_i=1 \quad (1 \le i \le n)} \;. \tag{15}$$

In a similar vein, one can state

Proposition 12. *Let $Z(H; X)$ be any of the cycle indicators above; $H \subseteq G$. Then*

$$\frac{1}{n} \sum_{s=1}^{n} z_s \frac{\partial}{\partial z_s} Z(H; X; z_1, z_2, \ldots, z_n) \equiv Z(H; X; z_1, z_2, \ldots, z_n) \;. \tag{16}$$

Also, we mention that differentiating classical cycle indicators was earlier used by de Bruijn [3] for his famous generalization of Pólya's theorem.

2.7 Some open problems

We start with some introductory background. Define the s-th normalizer $N_G^{(s)}(H)$ of a subgroup H ($E \subseteq H \subseteq G$) in G as follows: $N_G^{(0)}(H) := H$, $N_G^{(1)}(H) := N_G(H), \ldots, N_G^{(s)}(H) := N_G(N_G^{(s-1)}(H))$ ($s \ge 0$). Evidently, if G is a finite group, there exists the minimum value h of the index s, such that $N_G^{(h+j)}(H) \equiv N_G^{(h)}(H)$ $(j \geq 0)$. Moreover, $N_G^{(s)}(H) \trianglelefteq N_G^{(s+1)}(H)$ $(s \geq 0)$. With some restrictions, one can define the s-th centralizer $C_G^{(s)}(H)$, namely: $C_G^{(0)}(H) = H$ iff H is commutative (otherwise, it is not defined, at all) and all the other members are defined as in the case of the respective generalized normalizers above. Here, we also recall that $C_G(H) \trianglelefteq N_G(H)$. Seemingly, it is worth additionally consulting [16–18], where classical definitions of solvable and nilpotent groups are given; in some cases, the latter are relevant to the notions considered herein.

Theorem 1 above describes remarkable combinatorial properties of the usual normalizer $N_G(\hat{H})$. In order to stimulate further development of the subject along the same lines, we want to propose the following two questions:

***Problem* 1.** Obtain possible generalizations of Theorem 1 to the case of the s-th normalizer $N_G^{(s)}(\hat{H})$ of a closed subgroup $\hat{H}$ $(E \subseteq \hat{H} \subseteq G)$ in G; $s \geq 1$. In other words, investigate combinatorial properties of the generalized normalizer in question.

***Problem* 2.** Investigate combinatorial properties of the s-th centralizer $C_G^{(s)}(\hat{H})$ of a subgroup $\hat{H}$ $(E \subseteq \hat{H} \subseteq G)$ in G; $s \geq 1$.

Above all, posing these problems is targeted at the further generalization of Weyl's ideas concerning the "hidden symmetry" (see p. 50 in [11] or our short remark above). But a complete gamut of mathematical tools for treating all possible symmetries should also involve more general sorts of universal algebras than groups [19]; and, among all such algebras, semigroups hold a central position.

3 When A is the monoid $M = \mathrm{End}X$ of endomorphisms

A semigroup $\mathcal{S}$ is an associative groupoid $\mathcal{G}$, with a one-valued binary operation $(*)$ defined thereon [14–15]. If $(*)$ is interpreted as multiplying of respective elements, this is equivalent to necessarily obeying the associativity equality $(ab)c = a(bc)$ $(a, b, c \in \mathcal{S})$. A monoid M is a semigroup $\mathcal{S}$ possessing a unity element e or 1 (see above); $\mathcal{S}^1$ is also used to indicate the monoid case (or the case when a unity element is additionally introduced into some semigroup without it) and, sometimes, $\mathcal{S}$ to denote $M \setminus e$.

A subset N of a semigroup $\mathcal{S}$ is called a normal complex [14–15] if $\forall a, b \in N$ and $\forall x, y \in$

S^1 $xay \in N$ implies $xby \in N$. A normal complex N is some congruency class in $\mathcal{S}$ (resp. $\mathcal{S}^1$); besides, if N contains any subsemigroup it is a subsemigroup of $\mathcal{S}$ as well (in particular, if N contains any idempotent). A subsemigroup or normal complex N of a semigroup $\mathcal{S}$ (resp. $\mathcal{S}^1$) is called a normal subsemigroup (submonoid) if $\forall a \in N$ and $\forall x, y \in \mathcal{S}^1$ $(xy \in \mathcal{S})$ $xy \in N$ implies $xay \in N$ and vice versa. A normal subsemigroup N is the preimage of e under the homomorphism of a monoid M or $\mathcal{S}^1$ onto some monoid M'.

Define the normalizer $N_M(S)$ of a submonoid S $(E \subseteq S \subseteq M)$ in M as the maximum submonoid, in M, for which S is the normal submonoid $(S \trianglelefteq N_M(S))$. The semigroup normalizer is a direct generalization of the widely used group normalizer. The following known result lies in the very fundamentals of the theory of semigroups [14–15], viz.:

Lemma 13. *Let S be a subsemigroup (submonoid) of a semigroup (monoid) M. The factor semigroup (factor monoid) M/S is homomorphic to a group L iff S is a normal subsemigroup (normal monoid) of M.*

Now, we can derive from the last statement a result that has to be crucial for our inferences below, viz.:

Theorem 14. *Let $\hat{S}$ be a closed submonoid of a monoid M $(E \subseteq \hat{S} \subseteq M)$. Then the normalizer $N_M(\hat{S})$ is the maximum submonoid, in M, whose elements (either fix or) permute intact S-orbits in $S\backslash\backslash X$, or, equivalently, $N_M(\hat{S})/\hat{S}$ homomorphically acts as the maximum permutation group, induced by M, on the set I of indices of the S-orbits in $S\backslash\backslash X$; $|I| \equiv |S\backslash\backslash X|$.*

Proof. First, we mention that the submonoid $\hat{S}$ (resp. S) simultaneously defines two congruences that respectively generate two factor-sets $N_M(\hat{S})/\hat{S}$ and $\hat{S}\backslash\backslash X \equiv S\backslash\backslash X \equiv X/S$. This circumstance allows us to homomorphically represent the combinatorial action of $N_M(\hat{S})$ on X by an induced action of $N_M(\hat{S})/\hat{S}$ (or even $N_M(\hat{S})$ itself) on $S\backslash\backslash X$. Since a submonoid $\hat{S}$ is the kernel of the homomorphism from M onto a factor monoid $N_M(\hat{S})/\hat{S}$, elements g of $N_M(\hat{S})$ can either fix (some) orbits in $S\backslash\backslash X$ or map (some) intact S-orbits into other S-orbits therein. But the mentioned factor monoid $N_M(\hat{S})/\hat{S}$ is homomorphic to a group L and, thereby, any such mapping from $S\backslash\backslash X$ into itself is reversible with respect to intact S-orbits. Thus, $N_M(\hat{S})/\hat{S}$ indeed acts as a group on I (and similarly acts on $N_M(\hat{S})$ as well). The maximality of such action follows from the definition of the normalizer $N_M(\hat{S})$ which immediately completes the proof. □

Here, we remind the reader that the cardinalities of the S-orbits corresponding to mutually-permutable indices, in I, are not necessary equal when M is not a group. The equality of their cardinalities is also insufficient for mapping one intact orbit into another.

Corollary 14.1. *Let $\hat{S}$ be a closed subgroup of a group M acting on a base set X. Then $N_M(\hat{S})$ acts as the maximum permutation group, on $S\backslash\backslash X$, that (either fixes or) permutes intact S-orbits in $S\backslash\backslash X$.*

Proof. Evidently, the normal subsemigroups of a group are exactly its normal subgroups. By virtue of Theorem 14, $N_M(\hat{S})/\hat{S}$ (either fixes or) permutes the respective indices in I. However, in the case of a group M, to mutually-permutable S-orbits there necessarily correspond equipotent S-orbits; therefore, $N_G(\hat{S})/\hat{S}$ must also similarly act on the set $S\backslash\backslash X$. But, because $\hat{S}$ is the kernel of this action, the normalizer $N_G(\hat{S})$ too (either fixes or) permutes intact S-orbits in $S\backslash\backslash X$ which completes the proof. □

Apparently, Corollary 14.1 is nothing but Theorem 1, proven above by other methods; here we reproved it only in order to show that the proof may also be obtained from the more general semigroup-theoretical standpoint, as it was promised in the very beginning. But what is rather more essential is that the finite combinatorial actions of groups are in fact the core of combinatorial actions of semigroups and hence all finite algebras as well.

In our opinion, the above results may find the first applications to enumerating various graphs with a given monoid M of endomorphisms (and/or other functional properties) and the functions ϕ of the k-valued logic, which are, in particular, applied in the theory of finite automata. Chemical applications are possible as well (see [20]).

The author's experience over a period of years enables him to declare that the normalizer method [8] (locally, call it so) demands rather less theoretical knowledge, time, and typed space for dealing with the respective task than any of the well-accustomed methods like those employing double cosets and table of marks. In particular, for the sake of chemical or physical applications, to all subgroups of point crystallographic groups O_h and D_{6h} as well as fullerenes' groups possessing elements of noncrystallographic orders, the corresponding generalized cycle indicators (discriminating enantiomers or not) [8] may readily be tabulated in a reference book. Finally, we mention that the fundamental encyclopedic book [10] and, especially, its extended version [11] may play an invaluable role in developing any combinatorial method in context.

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