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Group Theory Applied to Combinatory Analysis

J. H. Redfield[1]

The Theory of Groups of Finite Order, and the subject of Combinatory Analysis, are both of them largely concerned with permutations, and so we should expect to find a good deal of use made of Groups in work on permutations and combinations. Actually, however, we find very little of such application of the theory of groups, and my purpose today is to show how group theory can be effectively brought to bear on a considerable range of combinatorial problems which are not readily handled by the conventional methods.

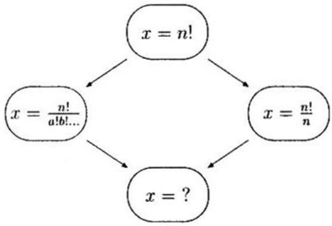
Now, in order that we may get as quickly as possible a clear notion of what this is all about, I will begin by showing how the subject ties up with some of the simple problems of the elementary theory of permutations and combinations. Take first the most fundamental problem of all: to find the number of ways of arranging n different given objects in n different given positions. For this the elementary theory gives the formula $x = n!$

Next we introduce a complication by providing that the objects are not all distinct, but that a are alike and of one kind, b are alike and of another kind, and so on; and for this we have the formula $x = n!/(a!b!\ldots) \quad (a + b + \ldots = n)$.

Going back to the original problem, we complicate it in another direction by providing that the given positions are equally spaced on the circumference of a circle, and that no account is taken of any first or last position, but only of the cyclic order of the positions. For this we have the formula $x = n!/n$.

[1]A manuscript note at the top of the typescript reads "U. of P. 12/10/37 before Grad. Math. Club" indicating that this is the typescript of a lecture given at the University of Pennsylvania on 10 December 1937, before the Graduate Mathematics Club.

Observe that the one derived problem has a qualification respecting the *objects*, the other a qualification respecting the *positions*. These qualifications are independent of one another. Suppose we combine the two in one problem, and inquire: how many distinct cyclic arrangements can be made of n objects which are not all distinct but fall into classes of like objects? (INDICATE $x = ?$)



I will not go so far as to say that this last problem could not with a little ingenuity, be solved by the standard elementary methods, but it may fairly be said to be beyond the normal range of such methods; what I wish to lead up to is a method of considerable generality capable of solving not only this problem but an extensive class of problems of which this one is only a comparatively simple instance.

To see how the theory of groups can be applied to such problems, let us examine what these four cases have in common, and wherein they are diverse. In all the cases we have n objects and n positions, and each of the objects may be placed in any of the positions; but there is diversity as regards what we agree to count as distinct arrangements.

In this problem (INDICATE $n!/a!b!\ldots$) any given distinct arrangement is supposed not to change when we make any permutation of the a like objects among themselves, or of the b other like objects among themselves, and so on, or any combination of such permutations. These admissible permutations, which do not change any arrangement into another distinct arrangement, of course form a group, and we may say that every distinct arrangement is *invariant* under the operations of that group, acting on the *objects*.

Similarly in this problem (INDICATE $n!/n$) every distinct arrangement is invariant under a cyclic group of degree n acting on the *positions*.

And in the combined problem (INDICATE $x = ?$) every distinct arrangement is invariant under both groups, one acting on the objects and one acting on the positions.

Of all the cases we may say that every distinct arrangement is invariant under two groups, one acting on the objects and one acting on the positions; but in two of the cases (INDICATE) one of the groups is the group of identity, while in this case (INDICATE $n!$) both groups are the group of identity.

The only essential thing which distinguishes the four cases from each other is the groups involved; when we know what these groups are, the problem is fully determinate, and we have a right to expect to find that the required number of distinct arrangements is a function of those groups, or, if you prefer, a function of certain numbers associated with the respective groups and known when the groups are known.

Such a function turns out to exist, but before writing an expression for it, it will be necessary to take up some details of notation.

The groups which we have to deal with are groups of permutations; their operations are permutations. One of the usual notations for a permutation consists in writing all the symbols involved in one or more cycles, thus: ach.be.df.g, which stands for a permutation which changes each letter into the next letter in the same cycle, except the last letter of a cycle which goes into the first letter. The numbers of letters in the various cycles form a partition of the total number of letters; in the present example we have the partition 32^21 of 8. This partition we will call the *cycle-partition* of the permutation.

A permutation group will in general be made up of operations of various cycle-partitions, which are all partitions of the number giving the degree of the group. For any group G of degree n we shall use the symbol $N(G; p_1^{\pi_1} p_2^{\pi_2} \ldots)$ to denote the number of operations in G which are of cycle-partition $p_1^{\pi_1} p_2^{\pi_2} \ldots$, that is, which when written in the ordinary cycle notation show π_1 cycles of p_1 letters each, π_2 cycles of p_2 letters each, and so on.

Now if, in our problem, G_1 is the group acting on the objects, and G_2 the group acting on the positions, we have the formula:

$$x = \frac{\sum [p_1^{\pi_1} p_2^{\pi_2} \ldots \pi_1^! ! \pi_2^! ! \ldots N(G_1; p_1^{\pi_1} p_2^{\pi_2} \ldots).N(G_2; p_1^{\pi_1} p_2^{\pi_2} \ldots)]}{\mu(G_1)\mu(G_2)}$$

in which $\mu(G_1)$ and $\mu(G_2)$ are the orders of G_1 and G_2 respectively, and the summation $\sum$ covers all the partitions of n the degree of G_1 and G_2, which in practice means however only those partitions which occur as cycle-partitions of operations both of G_1 and of G_2, since if a group happens not to contain any operations of a given cycle-partition, we shall have $N(\quad) = 0$, and the summand containing it as a factor will drop out.

The derivation of this formula will be found in my paper [2]; we shall see that it is only a special case of a more general result which I shall come to presently.

Let us apply this to a numerical example, and determine the number of distinguishable ways of seating three pairs of indistinguishable twins about a round table.

The groups are of degree 6 ($n = 6$). The group of permutations among the six persons under which any distinguishable arrangement is invariant contains: 1 operation, the identity, which leaves matters as they were originally, of cycle-partition 1^6; 3 operations interchanging the members of one pair of twins only, which pair can be chosen in 3 ways, of cycle-partition 21^4; 3 operations interchanging the members of 2 pairs (3 choices), of cycle-partition $2^2 1^2$; 1 operation interchanging the members of all three pairs (1 choice only), of cycle-partition 2^3. Total, 8 operations, so that $\mu(G_1) = 8$.

Next, as to the group G_2 acting on the positions at the table. There is first the identical operation of cycle-partition 1^6. Then there is 1 operation which turns the table half-way round, thus interchanging the pairs of opposite seats, cycle-partition 2^3. Then there are 2 operations turning the table through 120^o and 240^o respectively, causing the seats to permute in 3 cycles of 3 seats each, cycle-partition 3^2. Lastly there are 2 permutations turning the table through 60^o and 300^o respectively, giving one cycle of 6 seats, cycle-partition 6. Total, 6 operations, so that $\mu(G_2) = 6$.

In filling out our table[2] (INDICATE) we note that zeros appear for G_1 for cy.p's 3^2 and 6, and for G_2 for cy.p's 21^4 and 2^21^2. We next put down the factors $p_1^{\pi_1}p_2^{\pi_2}\ldots$; observe that in this numerical factor the π's are true exponents, whereas in the $N(\quad)$ symbols they indicate the number of times a part is repeated in a partition. We have: $1^6.6! = 720$; $21^4.1!4! = 48$; $2^21^2.2!2! = 16$; (I put all of these down for better illustration, through some of them will not be used); $2^3.3! = 48$; $3^2.2! = 18$; $6.1! = 6$. Multiplying out and adding, we get 768, which is the numerator in our formula. The denominator is the product of the orders: $\mu(G_1)\mu(G_2) = 8 \times 6 = 48$. Finally, the number of distinguishable arrangements is 768/48=16.

Order	8	6	(12)					
	Objs.	Pos'ns						
Cyc Part.	G_1	G_2	G_2					
1^6	1	1	1	$1^6.6!$	=	720	720	720
21^4	3	0	0	$21^4.4!$	=	48	0	
2^21^2	3	0	3	$2^21^2.2!.2!$	=	16	0	144
2^3	1	1	4	$2^3.3!$	=	48	48	192
3^2	0	2	2	$3^2.2!$	=	18	0	
6	0	2	2	$6.1!$	=	6	0	
							768	1056

As a further illustration we may vary this problem by supposing that we no longer distinguish between right-hand and left-hand cyclic order around the table; or, to put it otherwise, suppose we wish to determine the number of distinguishable ways of placing on a ring a pair of indistinguishable brass keys, a pair of indistinguishable copper keys, and a pair of indistinguishable iron keys. The group for the objects is the same as before, but the group for the positions is now of order 12, there being added to the original 6 operations 6 new operations = 3 operations in which the ring is turned over about an axis passing through two opposite keys; the keys on this axis remain where they were, but the other four keys interchange in pairs, so that we have the cy.p. 2^21^2, and there are 3 such operations because there are just 3 pairs of keys which can be taken as axis; also there are added 3 operations in which the axis passes

[2]The figures in the first of the two columns headed G_2 refer to the twins problem, where the position group is a cyclic group; the second such column refers to the keys problem (discussed in the next paragraph) where the object group is a dihedral group. The column totalling 768 also refers to the twins and that totalling 1056 to the keys.

between keys, leaving on each side of the axis a set of 3 keys which interchange in pairs with the opposite set of 3, giving the cy.p 2^3; again there are 3 operations because there are 3 distinct positions possible for the axis. We now have 3 operations of cy.p. $2^2 1^2$ where we had none before, and 4 of cy.p. 2^3 where we had only one before. We modify our calculation accordingly and get $(720+144+192)/(8\times 12)=1056/96=11$ distinguishable arrangements.

Consider again the problem of the twins at table. Every distinguishable arrangement either is or is not symmetrical about some diameter of the table. If an arrangement has this kind of symmetry, it is unaltered when the table is turned over as in the problem of the keys, so that each symmetrical arrangement of the table problem corresponds to one arrangement of the key problem. But if the table arrangement is unsymmetrical, it is interchanged with the arrangement opposite-handed to it, which is also present among the possible table arrangements; but these two count as only one arrangement in the key problem. Thus, since we have 5 fewer arrangements in the key problem than in the table problem, there must be five pairs of such unsymmetrical arrangements in the table problem, leaving 6 arrangements which must be symmetrical.

All these enumerations you can easily verify at your leisure by forming all the distinguishable arrangements with the letters *a a b b c c* written on the circumference of a circle.

Before passing on I wish to notice an application to organic chemistry which led me to choose these particular examples. I refer to the benzine molecule and its derivatives. The benzine molecule consists of 6 C atoms combined with 6 H atoms. Any or all of the H atoms can, in the derivatives, be replaced by other univalent atoms or radicals, such as Cl, Br, NH_4, HO, CH_3. Derivatives having equal numbers of the same respective constituent atoms or radicals, but differing in their properties, are called isomers. The difference in properties is supposed to be caused by differences in the arrangement of the constituents in the molecule. The isomers thus correspond to the distinguishable arrangements of our theory. Now the chemists do not seem to have come to an agreement about the structure of the benzine molecule. The most generally held hypothesis is that the carbon atoms form a ring, as in the key problem, to which are attached the other constituents, which in the various derivatives may consist of various combinations of single constituents, twins, triplets, etc. But there are difficulties about disposing of the fourth valence bonds of the C atoms, and other hypotheses have been proposed, one being that the C's are placed at the vertices of a triangular prism. What can our theory say about this question? A large number of benzine derivatives have been produced and studied in the laboratory, and the number of members in the various sets of isomers determined. There may be a few errors in these determinations, but the evidence as a whole seems to show beyond a doubt that the only position-group which can be assumed consistently with the observed facts is the dihedral group of degree 6, order 12, which we used in the key-ring problem. The cyclic group of the table problem, of degree and order 6, will not work; its order is too small and therefore it would give more isomers in many cases than are found to exist. The same is true of the position-group of the triangular prism, which is also of order 6 but different from the cyclic group. That is, provided we suppose the prism to be rigid. If we imagine a triangular prism made, as it were, of rubber rods, so that it could be turned inside out, then we should again have the dihedral group, and this would be an admissible structure so far as the group theorist could judge; the chemists might of course rule it out for other reasons. The number of distinct types of permutation groups of degree 6 is strictly limited; I suppose there are not more than 40 such[3]. If we examine them all, we shall find none other than the dihedral group which answers the requirements.

Until now we have spoken of certain objects placed in certain positions, but from a more general point of view we should speak rather of two classes of abstract entities, of which one class is put in one-one correspondence with the other class. Further, we need not limit ourselves to two classes; we may think of three classes put in one-one-one correspondence, or any larger number q of classes in 1-1-1-...-1 correspondence; in other words we may suppose all the entities made up into parcels, each parcel to contain one representative of each class. With each of the q classes, which we call *ranges*, we associate a group G_r $(r = 1, 2, \ldots, q)$, which we call its

[3] A manuscript note in the margin reads "There are nearly 60".

range-group. It is found that our previous formula can be extended to cover this more general case: in the denominator we adjoin a new factor $\mu(G_r)$ for each added group G_r, and in the numerator an additional factor $p_1^{\pi_1} p_2^{\pi_2} \ldots \pi_1! \pi_2! \ldots$, so that our formula now reads:

$$x = \frac{\sum [(p_1^{\pi_1} p_2^{\pi_2} \ldots \pi_1! \pi_2! \ldots)^{q-1} N(G_1; p_1^{\pi_1} p_2^{\pi_2} \ldots).N(G_2; p_1^{\pi_1} p_2^{\pi_2} \ldots) \ldots N(G_q; p_1^{\pi_1} p_2^{\pi_2} \ldots)]}{\mu(G_1)\mu(G_2) \ldots \mu(G_q)}$$

As an example of a problem of more than two dimensions, consider again the key-ring problem, and suppose we have two red tags, two white tags, and two blue tags, and that we attach a tag to each key on the ring; how many distinguishable arrangements do we get? Here we have 1 - 1 - 1 correspondence connecting a class of tags, a class of keys, and a class of ring-positions. Performing the calculation (INDICATE) we find 696 distinguishable arrangements.

Key ring with tags[4]: (2 red, 2 white, 2 blue)}

	G_1	G_2	G_3		
1^6	1	1	1	720^2	518400
$2^2 1^2$	3	3	3	16^2	6912
2^3	1	4	1	48^2	9216
	8	12	8		8)534528
					6)66816
					8)11136
					2)1392
					696

In all examples just given we have had to do with groups of rather low order, for which the numbers $N(\quad)$ could be determined by direct counting. When the groups are of high orders this is not practicable, and the matter is often one of some difficulty and must be dealt with by various artifices, there being no universal method available. This is a question which belongs rather to pure group theory than to the present subject, but I should like to give you some notion of the sort of trick which serves us here.

[4]Comparing this table with the previous one, it will be seen that Redfield now omits any row with a zero entry; this is why the numbers at the foot of columns G_1, G_2, G_3 exceed the sum of the numbers above them. Redfield originally wrote 6 at the foot of column G_2 and then changed it to 12: this is why he has divided 66816 by 6 and then divided by 2 at the end.

I have found it convenient to adopt a symbolic function, which I have called the *group reduction function* (GRF), and in which the numbers $N(\quad)$ appear as coefficients of terms whose literal parts indicate the cycle-partitions. This function has the general expression $(1/\mu(G))\sum\left[(N(G;p_1^{\pi_1}p_2^{\pi_2}\ldots)s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\ldots\right]$. The quantities s_k can be interpreted as the sums of the k-th powers of the roots of a polynomial, and since these are symmetric functions of the roots, every GRF is also a symmetric function of the roots. However, I do not wish to dwell today on this interpretation; for our purpose it is sufficient merely to regard the GRF as a convenient symbolic shorthand to show the properties of the groups which we need.

The simple algebraic properties of GRF's prove to be useful; in particular, the product of any two GRF's is also a GRF, of higher degree of course. For example, the group of degree 2 which gives all the permutations of two objects has the GRF $(1/2)(s_1^2+s_2)$. The cube of this is $(1/8)(s_1^6+3s_2s_1^4+3s_2^2s_1^2+s_2^3)$, which is the GRF of the group which we used for the independent permutations of three pairs of twins. The cyclic group used for the table-positions has the GRF $(1/6)(s_1^6+s_2^3+2s_3^2+2s_6)$; for any cyclic group, each term exhibits a power of a single letter s_k, whose suffix k is a divisor of n, and whose coefficient is the well-known totient function giving the number of integers less than and prime to k. For the dihedral group we have to adjoin additional terms in s_2^3 and $s_2^2s_1^2$, giving $(1/12)(s_1^6+3s_2^2s_1^2+4s_2^3+2s_3^2+2s_6)$.

As an example of the derivation of GRF's of more complex groups, let us consider a modification of the table problem in which we now have, instead of three pairs of twins, one set of sextuplets, and inquire in how many distinguishable ways they can join hands two and two across the table. We now have as object-group one which includes all possible interchanges of the members of pairs, for each of which we have the GRF $(1/2)(s_1^2+s_2)$, and also includes all the permutations of the three pairs treated as units, with the GRF $(1/6)(s_1^3+3s_2s_1+2s_3)$. We combine these in the following way:

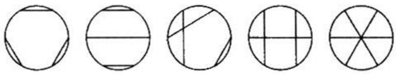
$$\frac{1}{6}\left[\left(\frac{s_1^2+s_2}{2}\right)^3+3\left(\frac{s_2^2+s_4}{2}\right)\left(\frac{s_1^2+s_2}{2}\right)+2\left(\frac{s_3^2+s_6}{2}\right)\right]$$

$$=\frac{1}{48}\left(s_1^6+3s_2s_1^4+9s_2^2s_1^2+7s_2^3+8s_3^2+6s_4s_1^2+6s_1s_2+8s_6\right)$$

where you will observe that we have written $(1/2)(s_k^2 + s_{2k})$, a modification of the first GRF, in place of every s_k of the second GRF. Now if we take this group of order 48 as object-group, and the cyclic group of degree 6 as position group as before, we may calculate as follows (INDICATE):

1^6	1	1	720	720
2^3	7	1	48	336
3^2	8	2	18	288
6	8	2	6	96
				6)1440
				48)240
				5

– giving 5 distinguishable arrangements, which may be pictured thus: (INDICATE).



Note that all these diagrams have diametral symmetry; therefore we should expect to get also 5 distinguishable arrangements in the corresponding key-ring problem, where we might suppose the keys on the ring tied together with strings two and two. I will leave this to you to verify, using the dihedral group instead of the cyclic group as position-group.

This process of combining group reduction functions is of general application; thus if we were to suppose the table-sitters to join hands to form two triangles, we should combine the GRF's the other way round, getting:

$$\frac{1}{2}\left[\left(\frac{s_1^3 + 3s_2s_1 + 2s_3}{6}\right)^2 + \left(\frac{s_2^3 + 3s_4s_2 + 2s_6}{6}\right)\right]$$

$$= \frac{1}{72}\left(s_1^6 + 6s_2s_1^4 + 9s_2^2s_1^2 + 6s_2^3 + 4s_3s_1^3 + 12s_3s_2s_1 + 4s_3^2 + 18s_4s_2 + 12s_6\right).$$

We should then find 3 distinguishable arrangements, both in the table problem and in the key-ring problem; again I leave the verification to you.

Up to this point we have taken account of *all* the possible ways of placing the objects in the positions, of *all* the possible correspondences among the given classes

of entities; we have not counted them all as distinguishable, but we have included them all. We may however wish to exclude certain arrangements from the count. The arrangements which we retain, considered as permutations of some assumed initial arrangement, may or may not form a group; thus if we wished to determine the number of distinguishable anagrams of the word *success*, excluding all those in which two like letters are adjacent, the arrangements retained would not form a group. Many such cases can be treated effectively by our methods, but the devices at present available are of no great generality, and so I shall speak only of problems in which the retained arrangements do form a group.

Our results from now on are to hold for abstract groups. But as we know, every abstract group can be represented as a group of permutations, so that it is legitimate to use permutation groups for illustration and as aids to reasoning whenever it suits us.

First of all we must specify the group which determines the arrangements or correspondences, or the abstract equivalent thereof, which we are going to retain as admissible. We call this group the *frame group*, F. In the problems previously taken up, we had a frame group, namely the symmetric group on n symbols, only we did not find occasion to mention it explicitly; in fact, I had worked out that part of the theory long before I suspected that there was such a thing as a frame group to be considered.

We may now think of our abstract system as represented by a set of q classes of n entities each, and that these classes, or *ranges*, can be placed in one-one-... correspondence in certain admissible ways. We begin by placing the classes in an initial correspondence, which is arbitrary except that it must be one of the correspondences which we agree to regard as admissible. We then apply to each range some arbitrary operation of the frame group F. This gives a correspondence which may or may not differ from the initial one, but in any case it is by definition also a correspondence of the admissible set. We now introduce the *range groups* $G_1, G_2, \ldots, G_q$. These must be subgroups of the frame group F, (including it may be F itself), but subject to this limitation they may be any groups which may be assigned by the data of the problem. Now taking the derived correspondence, let us apply to range 1 an operation g_1 of the range group G_1, to range 2 an operation g_2 of the range group G_2, and so on. This gives a second derived correspondence. If now it is possible to choose $g_1, g_2, \ldots, g_q$ (that is, if the requisite suitable operations are contained in the respective range groups) so that the second derived correspondence is identical with the initial correspondence, we say that the initial correspondence and the first derived correspondence are both of them representatives or aspects of one and the same distinguishable arrangement, as we previously called it; but if this is impossible because the range groups do not contain the needed operations, then the initial correspondence and the first derived correspondence are aspects of different distinguishable arrangements. On this basis it is evident that we can divide the whole of the admissible correspondences into classes, each class standing for one distinguishable arrangement and including all its representatives or aspects. When we have counted these classes, we have counted the distinguishable arrangements.

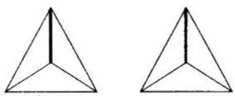
When the frame group was a symmetric group, we classified our operations according to their cycle-partitions; now, with a general abstract frame group, we divide the operations into classes, each of which classes coincides with a conjugate set of operations under the frame group F. Suppose now that one of the conjugate sets of the operations of F contains L members, and that of these L operations, l_1 are found in G_1, l_2 in $G_2, \ldots, l_q$ in G_q. Then one summand in our new formula will be $(\mu(F))^{q-1} l_1 l_2 \ldots l_q / (\mu(G_1)\mu(G_2)\ldots\mu(G_q)L^{q-1})$, and the complete formula will be

$$x = \frac{(\mu(F))^{q-1}}{\mu(G_1)\mu(G_2)\ldots\mu(G_q)} \sum \left[\frac{l_1 l_2 \ldots\ldots l_q}{L^{q-1}}\right],$$

with a summand for each of the conjugate sets into which all the operations of F are divided: but (as before) a summand exists effectively only when it corresponds to a conjugate set, under F, which has representatives in all the range groups G_1, $G_2, \ldots, G_q$.

Where F is a symmetric group, this formula reduces to the previous one, since all the operations of a given cycle-partition form a conjugate set under the symmetric group; the analogy of the two formulae is otherwise evident. There is not time for me to give the proof, but I think anyone who is reasonably familiar with the theory of groups would be able to adapt it from the very similar proof of the first formula as given in my paper [2].

We ought to have some examples of this, so I will suppose that we have two regular tetrahedra, made of thin sheet material, or of wire if you like, so that one can be superposed on the other in any orientation. The two tetrahedra can then be fitted together in 12 different ways, the group of rotations of this figure being a group of the 12th order isomorphic with the alternating group on four symbols (which may be taken to represent the vertices or the faces); this is our frame group F, but we regard it here as an abstract group. Now we mark one edge of one tetrahedron in red; of the 12 ways in which this tetrahedron can be made to coincide with itself, there are just 2 which keep the red edge where it was; so one range group G_1 is a group of order 2 and a subgroup of F. Let us mark one edge of the other tetrahedron in blue, and we have a precisely similar second range group G_2. Now we can state the problem: To find the number of distinguishable ways of superposing the two marked tetrahedra.



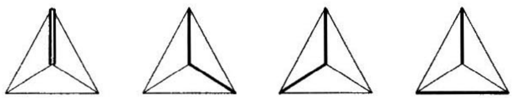
Now the frame group F, which is the group of rotations of the regular tetrahedron, contains 12 operations, which divide into four conjugate sets. One set contains only the identical operation, so that $L = 1$; this operation occurs of course in both the range groups, so that $l_1 = l_2 = 1$. A second set contains 3 operations of order 2, which correspond to the rotations about the three axes which pass through the mid-point of opposite edges; we have here $L = 3$; one of these operations occurs also in G_1 and one in G_2, so we have $l_1 = l_2 = 1$. A third set contains 4 operations of order 3, which correspond to the rotations through 120^o about the 4 axes through a vertex and the center of the opposite face; none of these occur in the range groups, so we have $l_1 = l_2 = 0$. The fourth set contains 4 operations similar to those of the third set, corresponding to rotations through 240^o, but these though similar are not conjugate with those of the third set; again we have $l_1 = l_2 = 0$.

L	l_1	l_2
1	1	1
3	1	1
4	0	0
4	0	0

Now substituting in our formula we have

$$x = \frac{12}{2 \times 2}\left(\frac{1 \times 1}{1} + \frac{1 \times 1}{3}\right) = 4.$$

(INDICATE[5]).



4 6 6 4

In 3 dimensions, with 3 marked tetrahedra, we should have:

$$x = (12.12/2.2.2)(1.1.1/1.1 + 1.1.1/3.3) = 20;$$

and so on.

It should be noted that the enumeration depends on the choice of the frame group as well as the range groups. If the tetrahedra are made of wire and can be turned inside out as well as turned around, the frame group is of order 24 and the distinguishable arrangements are in general fewer; on the other hand if the tetrahedra are not regular, so that fewer rotations are possible, the frame group is of lower order and the distinguishable arrangements become more numerous.

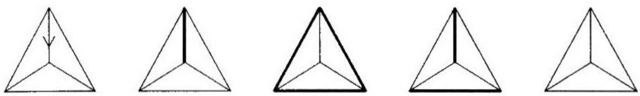
[5]It should be remembered that one tetrahedron has an edge marked red and the other tetrahedron an edge marked blue and Redfield probably drew the edges on the blackboard with coloured chalk. In the first figure the marked edges are superposed but are drawn side by side for emphasis; in the others the "vertical" edge may be assumed to be red and the other marked edge blue. With this colouring, the second figure cannot be rotated into the third one. The number under each tetrahedron is the number of rotationally different ways of superposing a third marked tetrahedron (with a green edge, for example), but there is no reference to these numbers in the text. The sum $4 + 6 + 6 + 4 = 20$ agrees, of course, with the calculation in the next equation.

Let us now re-write our formula with the summands factored in a special way for a special purpose which will appear shortly. Then

$$x = \sum \left[\left(\frac{\mu(F)l_1}{\mu(G_1)L} \right) \left(\frac{\mu(F)l_2}{\mu(G_2)L} \right) \cdots\cdots \left(\frac{\mu(F)l_q}{\mu(G_q)L} \right) \cdot \left(\frac{L}{\mu(F)} \right) \right].$$

Notice that the last factor involves only the frame group, and that each of the other factors involves only the frame group and one range group. It is sometimes convenient to make a table showing, for a given frame group F, the values of $L/\mu(F)$, and the values of $\mu(F)l_r/(\mu(G_r)L)$ for all the different types of subgroups of F which might be required to be used as range groups; then we have the figures at hand for various calculations which we may wish to make. Suppose we do this for the frame group of order 12 which we have been using in our examples. We get this table[6]:

m_0		$L/\mu(F)$	:	1/12	3/12	4/12	4/12
m_1	Identity	(order 1)	:	12	0	0	0
m_2	Cyclic group	(order 2)	:	6	2	0	0
m_3	Cyclic group	(order 3)	:	4	0	1	1
m_4	Noncyc. grp.	(order 4)	:	3	3	0	0
m_5	Frame group	(order 12)	:	1	1	1	1



This table could be used for the calculations of the tetrahedron problems which we just worked; a similar table for the table and key-ring problems could be made, but that table would have 11 columns and 58? rows and so would not be worth while making unless for a more extensive series of calculations.

In this table however (INDICATE) we notice that the numbers in the body of the table are all integers, and perhaps some of you have noticed something else: these numbers are the so-called *characters* of the subgroups of the particular frame group considered. There is a fairly extensive literature on group characters, but most of

[6]This table is repeated at the end of Redfield's paper but with a marked tetrahedron added on the right-hand side of each row; each tetrahedron has the corresponding group as its symmetry group. The five tetrahedra are reproduced below the table, but there is no mention of them in the original text.

it does not happen to have been conveniently accessible to me, so that I am unable to say whether anyone has used group characters in anything like this way for the enumeration of what we have called distinguishable arrangements.

The characters which appear here are so-called *compound* characters appertaining to a group considered as a subgroup of a larger group. They are expressible as linear functions with non-negative integral coefficients of a smaller set of so-called *simple* characters. The simple characters are not associated with any particular subgroups of the frame group, and they may involve negative, fractional, and even irrational and complex numbers. I have not discovered as yet any interesting way of relating the simple characters to the present subject, but the question is well worth further investigation.

To those familiar with matrices, it will be helpful to consider the rows of the table as the main diagonals of a set of matrices whose other elements are all zeros. Then a symbol such as (4,0,1,1) may be regarded as an abbreviation for

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M_3 \quad \text{(say)}$$

and the *trace* $tr(M_0M_3M_3)$ of the product

$$M_0M_3M_3 = (1/12, 1/4, 1/3, 1/3)(4,0,1,1)(4,0,1,1)$$

will be

$$(1/12 \times 4 \times 4) + (1/4 \times 0 \times 0) + (1/3 \times 1 \times 1) + (1/3 \times 1 \times 1) = 4/3 + 1/3 + 1/3 = 2,$$

which is the number of distinguishable arrangements of two superposed tetrahedra each having one marked face. (The sums and products of diagonal matrices are of course also diagonal matrices, and their products are commutative).

This brings us to a new consideration. Let us omit the matrix M_0 and take the product $M_3^2 = (4,0,1,1)^2$, which is the matrix $(16,0,1,1)$. This we find can be decomposed into $(12,0,0,0) + (4,0,1,1)$, so that $M_3^2 = M_1 + M_3$. Or, to take the problem which we solved a few minutes ago, of the tetrahedra with one marked edge, we have $M_2^2 = (36,4,0,0) = 2M_1 + 2M_2$, while we already had found that $tr(M_0M_2^2) = 4$ The same thing happens with the product of three or more matrices:

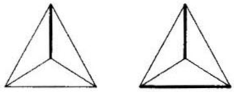
we have $tr(M_0M_2^3) = 20$, and[7] $M_2^3 = (216, 8, 0, 0)$, decomposing into $16M_1 + 4M_2$. This is found to be a general property: the product of any number of matrices of the set is equal to a sum of matrices of the set, the number of such summand matrices being equal to the trace of M_0 times the given product. That is $M_{r_1}M_{r_2}\ldots M_{r_q} = M' + M'' + M''' + \ldots + M^{(p)}$, where the number of terms p (sum of coefficients) $= tr(M_0M_{r_1}M_{r_2}\ldots M_{r_q})$.

The existence of this decomposition is proved for symmetric frame groups in my AJM paper [2], and the proof can easily be extended to the general case.

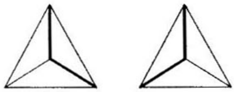
The number of summand matrices in the linear decomposition of a product of matrices is also seen to be equal to the number of distinguishable arrangements enumerated by the trace of M_0 times the given product, and in fact it turns out that to each distinguishable arrangement there corresponds a particular matrix, which in turn corresponds to a certain group which specifies a certain type of invariance possessed by the distinguishable arrangement in question. To define this invariance, let us return to our one-one-... correspondences of q ranges of n entities each. If we take any such correspondence, and apply to each r-th range some operation g_r of the associated range group G_r, we obtain in general another correspondence which is an aspect of the same distinguishable arrangement as the first correspondence. But it may happen that we get exactly the same correspondence, in which case all the operations $g_1, g_2, \ldots, g_q$ must be similar operations, belonging in fact all to the same conjugate set under the frame group F, and further, for each range R_r the operation g_r must be in a certain subgroup Y_r of G_r; the groups $Y_1, Y_2, \ldots, Y_q$ are all conjugate to one another under the frame group F, and the character matrix M_Y of this conjugate set is the summand matrix corresponding to the distinguishable arrangement represented by the one-one-... correspondence with which we started.

Now let us see what this type of invariance amounts to in the examples which we have worked. Take the case of two tetrahedra with one marked edge each. Of the 4 distinguishable arrangements which we found, the following two:

[7] The original reads $M_0M_2^3 = (216, 8, 0, 0,)$, but this is not what was intended.



can be rotated so as to turn the marked edge or edges end for end, while keeping every marked edge coincident with itself; these arrangements correspond to the two summands giving $2M_2$, and they are invariant under a group of rotations of order 2. The other two arrangements:



do not admit any rotation which does not shift one or both marked edges; they correspond to the two summands giving $2M_1$, and the invariance group is merely the group of identity.

Now I would remind you that these methods do not enable us to *discover* the actual distinguishable arrangements in any problem, but only to *count* them; that is of course equally true of the elementary theory of permutations and combinations. The practical utility of our theory, if it may be said to have any, is that, when we have occasion to make a complete list of any fairly numerous set of distinguishable arrangements, it is of great advantage to know exactly how many we ought to look for, so that we may avoid omissions and duplications. This being the case, it is of still greater advantage to be able in advance to subclassify the distinguishable arrangements by the invariance groups which they admit.

To do this, we must find the linear decomposition of the product of matrices, and you may have noticed that I have given no general method of doing so. I am not ready to offer any such general method. In the simple cases we have treated, the decomposition was effected by trial, and you might perhaps suppose that in more complicated cases it would, failing a better way, be after all only a question of patience and industry, since the decomposition can be proved to exist in every case. Unfortunately, there is sometimes more than one way to decompose the product, and when we have found a decomposition we have no way of telling whether it is the only one possible, or whether it is the one which applies to the problem in hand. There are frame groups for which one identical matrix corresponds to different conjugate set of subgroups, and in such cases there may be different linear decompositions corresponding to the various ways in which these conjugate sets enter the product of matrices. It is possible also that a decomposition which looks all right will mean nothing at all, will not be interpretable in terms of any possible set of arrangements. I mean to say, we must be prepared for this possibility; I have not met with such a case, and I cannot say whether or not such a case can occur[8].

The root of the difficulty here is that the character matrices do not contain all the needed information, and in an attempt to better the position of affairs I have been led to develop what might be called *extended* character matrices. The matrices we have been using have a constituent for each conjugate set of the *operations* of F. The extended matrices will have a constituent for each conjugate set of the *subgroups* of F. That means that in a constituent $\mu(F)l_r/(\mu(G_r)L)$, L is now to stand for the number of *subgroups* of F which are contained in a certain conjugate set, and l_r is now to stand for the number of subgroups, out of those L, which occur also as subgroups of the group G_r, or of any of the conjugates of G_r under F. The old characters will coincide, as far as they go, with the new extended characters, except in one particular: where we have in the old table two or more columns corresponding to similar operations which, though powers of one another, form distinct conjugate sets, these columns will be combined to represent the single conjugate set of cyclic subgroups which contain the operations; thus in the table for the frame group of order 12 the third and fourth columns will be thus merged. When columns are merged in this way the fractions $L/\mu(F)$ in the upper row (M_0) are to be added together, so that the sum of the whole row continues to be 1. The old columns, as modified by merging, correspond to the cyclic subgroups of F: the new columns added will correspond to the non-cyclic

[8]A manuscript note added later reads: "Such a case occurs with the 9 subgroups of the simple group of order 168 (1.1.38)".

subgroups of F, including the last column for F itself. The new columns will have 0 instead of $L/\mu(F)$ in the upper row.

The new table will be as follows:

M_0 :			:	1/12	1/4	2/3	0	0
M_1 :	Identity	(order 1)	:	12	0	0	0	0
M_2 :	Cyclic group	(order 2)	:	6	2	0	0	0
M_3 :	Cyclic group	(order 3)	:	4	0	1	0	0
M_4 :	Noncyc. grp.	(order 4)	:	3	3	0	3	0
M_5 :	Frame group	(order 12)	:	1	1	1	1	1

The extended character matrices can be multiplied together, and the product so obtained may be multiplied by M_0 and the trace taken, just as in the case of the unextended character matrices; but the decomposition is now unique, because, as you may readily see, the matrices are now all linearly independent, which the unextended matrices were not. When a product of matrices has been obtained, there is now a very simple rule for the linear decomposition: the last non-vanishing constituent points out the summand matrix corresponding to the highest invariance group order; we subtract a suitable multiple of this tabular matrix, so as to annihilate this final constituent, and then we repeat the process until the whole decomposition is obtained. The present table is too simple to offer any examples of much complexity, but we may take the product M_2^2 already used. In extended matrices this is (36,4,0,0,0). The figure 4 points to M_2 as the first summand; we must double this to annihilate the 4, and so we have $M_2^2 - 2M_2 = (24, 0, 0, 0, 0)$. We now see that we must next subtract $2M_1$, and we have $M_2^2 - 2M_2 - 2M_1 = (0, 0, 0, 0, 0) = 0$, so that finally we get $M_2^2 = 2M_2 + 2M_1$.

A proof of the validity of this procedure has been obtained. It depends in great part on the results which I previously outlined; but none of these proofs has yet been brought to a degree of elegance which would permit me to give any intelligible account of them in the time available.

The use of extended character matrices gives a theoretically complete solution of the problem of classifying distinguishable arrangements according to their invariance groups; but the practical difficulty and labor of working out the tables of matrices is very considerable, so that I have not been able to make applications comparable in interest with those of the simpler, purely enumerative theory.

Before leaving the subject I should like to indicate one interesting thing which emerges. Suppose we tabulate, instead of the values of $\mu(F)l_r/(\mu(G_r)L)$, the values of l_r simply. We then get a table like this:

	C_1	C_2	C_3	C_4	C_5
G_1 :	1	0	0	0	0
G_2 :	1	1	0	0	0
G_3 :	1	0	1	0	0
G_4 :	1	3	0	1	0
G_5 :	1	3	4	1	1

From this table we form matrices C_r whose constituents form *columns* of the table, and the products of these can be linearly decomposed just as in the case of the character matrices. Thus we find $C_2^2 = (0, 1, 0, 9, 9) = (0, 1, 0, 3, 3) + (0, 0, 0, 6, 6) = C_2 + 6C_4$. The following interpretation can be made of this result: If we take a group of the conjugate set to which G_2 belongs, and the same group or another from the same conjugate set, choosing them in all the 9 possible ways, and if we then use the two chosen groups as generators, we obtain all the groups (3 in number) of the conjugate set to which G_2 belongs *once*, and the single self-conjugate group G_4 *six times*.

References

[1] MacMahon, P.J., *Combinatory Analysis*, vols. i and ii. Cambridge University Press, Cambridge, 1915-1916.

[2] Redfield, J.H., *The Theory of Group-Reduced Distributions*, Amer. Jour. Math., 1927, pp. 433-455.

[3] Littlewood, D.E., *Group Characters and the Structure of Groups*, Proc. London Math. Soc. 39 (1935) 150-199. (contains bibliography on characters).