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Redfield's 1937 Lecture

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Abstract

Redfield was a pioneer in applying group theory to enumerative combinatorics but only one of his papers on the subject was published (in 1927) during his lifetime; a second was published posthumously in 1984, many years after it was originally submitted (in 1940). The typescript of a lecture which he delivered in 1937 is now published in this issue of *Match*. The present paper examines the material in the lecture and its relation to the 1927 and 1940 papers.

Introduction

During his lifetime, J. Howard Redfield (1879-1944) published a single paper [18]. Although it appeared in the *American Journal of Mathematics*, it was generally overlooked for about 30 years. It was cited briefly by Littlewood in both editions of his book [9], but there seems to be no other citation of the paper until Harary [5] pointed out that Redfield had anticipated various major developments which had taken place in combinatorial enumeration in the intervening years.

Not surprisingly, combinatorialists began to ask the question "Who was Redfield?". A little information was obtained by Harary and the text of a letter (dated 19 December 1963) from C. Oakley to him is included in a footnote in Harary and Palmer's book [6]. Oakley was writing some twenty years after he had known Redfield, so one or two details in his letter are wrong, but the overall picture of him as a man with broad interests, earning his living for much of his life as an engineer, is correct. The letter ends with the information that Redfield "has a very distinguished brother (Alfred Redfield) at Woods Hole, Massachusetts". For some reason no-one seems to have followed up that information - Alfred was still alive when [6] was published and a letter addressed to him at the Woods Hole Oceanographic Institution would certainly have reached him.

The *Dictionary of Scientific Biography* contains an entry [2] by Harold L. Burstyn on William C. Redfield (1789-1857). The present author, suspecting (correctly) that William might be an ancestor of Howard, wrote to Burstyn in 1976 enquiring if he knew anything of J. Howard Redfield. Burstyn passed the enquiry to Alfred C. Redfield who responded to the author with some information. Later Howard's daughter, Mrs Priscilla Redfield Roe, contacted the author by telephone when she visited the United Kingdom. She recalled that her father had submitted a second paper which the *American Journal of Mathematics* had rejected, but she was certain that a copy of it still existed. Thus the Redfield family learnt, rather belatedly, that Howard had done significant mathematical research and, in addition, the mathematical world was to learn that Redfield had continued his researches in enumerative combinatorics, but it was not until 1981 that the rejected typescript was tracked down in the family papers. Other material was also found at that time and later.

It was in 1940 that Redfield had submitted the rejected paper; after its rediscovery, it was published in 1984 [21] in a special issue of the *Journal of Graph Theory* dedicated to Redfield. Other articles in that issue include a paper [7] about the rediscovery of Redfield's work, another [4] comparing the material in the newly discovered work with that published by other authors in the meantime and a biography of Redfield [10]. The content of the 1940 paper is also discussed in [22].

Over a number of generations, several members of the Redfield family have been interested in scientific matters. William Redfield's interests, for example, included steam navigation and meteorology, and he was the first president of the American Association for the Advancement of Science. William also produced some genealogical tables of the Redfield family in the United States; these were revised and extended [17] by his eldest son, John Howard Redfield (1815-1895), the grandfather of the "mathematician" John Howard Redfield. Perhaps "polymath" would be a better description of the younger John Howard, since he obtained degrees both in engineering and Romance languages, but although he had a few short-term university appointments (both in languages and in mathematics) he never obtained a permanent post. For much of his life, he earned his living as a civil engineer, but during part of the 1930s he was unemployed and did what work he could find such as translating patents (see [10] for further details).

Enumeration problems

Enumerative combinatorics is concerned with problems such as how many different (or inequivalent) ways are there to place a collection of n objects into a set of n positions (one object in each position). The answer, of course, depends on what one means by "different". As an example, in his 1927 paper, Redfield asks for the number of structures which can be formed by placing four identical black balls and four identical white balls at the eight vertices of a cube (one ball in each corner). The set of balls has a symmetry group and so does the cube. Two distributions of balls are regarded as equivalent if one can be obtained from the other under the actions of one or both of these two groups. Redfield introduced two methods for solving such problems, both of which make use of a polynomial (which he called the *group reduction function* or GRF) associated with a permutation group. The same polynomial was introduced independently by Pólya in the 1930s (see in particular [15] of which an English translation appears in [16]). Since Pólya's paper was noticed by both combinatorialists and chemists, his name for the polynomial has been adopted as the standard term; in English it is called the *cycle index.*

Each of the structures formed in this type of construction has its own symmetry group and in his 1927 paper [18], Redfield asks if it is possible to break down the counting so as to keep track of the number of structures with each possible symmetry group. He defines a binary operation on GRFs and shows that this composition of two (or more generally q) GRFs is equal to a linear combination of GRFs. The coefficients in this expansion are, in some cases, the desired numbers. The method fails to work in general, however, since different groups may have the same GRF and furthermore the GRFs involved are not always linearly independent. In an unpublished typescript dated 1935 [19]. Redfield indicates that he now believes that he has the solution to the problem, but he has not yet been able to prove that the method works in general. A later typescript [20] (published in this issue of *Match*) is the text of a lecture which he delivered at the University of Pennsylvania on 10 December 1937 to the Graduate Mathematics Club. In it he states that he has proved that the procedure works "but none of these proofs has yet been brought to a degree of elegance which would permit me to give any intelligible account of them in the time available". A proof is included in his 1940 paper [21].

The text of Redfield's lecture is very readable, and anyone wishing to study his work would be well advised to read the lecture before passing on to his 1927 and 1940 papers. The text is also of historical interest, since it establishes that by 1937 Redfield had already worked out the theory in his 1940 paper.

Content of Redfield's lecture

In his lecture, Redfield starts with some elementary problems: (1) permutations of distinct objects, (2) permutations where some objects are indistinguishable from others (here a non-trivial symmetry group acts on the objects) and (3) arrangements of distinct objects around a circular table (here a symmetry group acts on the positions). These three problems can be solved by elementary arguments but, in his fourth problem, there is a group acting on the objects as well as one acting on the positions, so the problem is more difficult.

Some of Redfield's notation and terminology is no longer current. For example, he speaks of group operations rather than group elements, he writes $\mu(G)$ rather than $|G|$ for the order of G and he uses dots rather than parentheses in permutations written in cyclic form, so, for example, he writes *ach.be.df.g* rather than $(ach)(be)(df)(g)$. This permutation has cycle-partition type 321^2 (one cycle of length 3, one cycle of length 2 and 2 cycles of length 1).

Redfield now quotes the following formula from his 1927 paper:

$$x = \frac{\sum [p_1^{\pi_1} p_2^{\pi_2} \ldots \pi_1! 1^{\pi_1} \pi_2! 2^{\pi_2} \ldots N(G_1; p_1^{\pi_1} p_2^{\pi_2} \ldots).N(G_2; p_1^{\pi_1} p_2^{\pi_2} \ldots)]}{\mu(G_1)\mu(G_2)}.$$

This expresses the number x of arrangements as a sum over all partitions of the degree of the group G_1 acting on the objects (which is, of course, the same as the degree of the group G_2 acting on the positions). Each summand involves the product of the number $N(G_1; p_1^{\pi_1} p_2^{\pi_2} \ldots)$ of elements in G_1 with cycle-partition $p_1^{\pi_1} p_2^{\pi_2} \ldots$ and the number $N(G_2; p_1^{\pi_1} p_2^{\pi_2} \ldots)$ of elements in G_2 with the same cycle-partition (so the summand is non-zero only when both G_1 and G_2 contain elements with that cycle-partition). Redfield then applies the formula to various problems. First he asks for the number of distinguishable ways of seating three pairs of identical twins around a circular table (there are 16). He then varies the problem by not distinguishing between right-hand and left-hand cyclic order around the table or, more realistically, by considering an equivalent problem with three pairs of identical keys (brass, copper and iron) on a ring (the number reduces to 11). In his first calculation, Redfield includes in his table a row for each partition of the degree of the groups, but subsequently he omits those rows which contribute zero to the sum. He also returns to the twins problem and considers which of the arrangements are symmetrical about some diameter of the table (there are 6).

In his work, Pólya [15], following up work of Cayley and others, includes the enumeration of several chemical compounds. Redfield's inspiration, however, had come from a different direction, in particular from MacMahon's *Combinatory Analysis* [14], and there is no mention of chemistry in either his 1927 paper or his 1940 paper. It is interesting, therefore, to find that in his lecture he does have a short section on the *benzine* (*sic*) molecule and its derivatives. The spelling which he adopts is that more usually used in the USA as a generic term for petrols and petroleum distillates; the particular compound is usually spelt *benzene*.

Redfield discusses the problem of determining the structure of the benzene molecule, something which had taxed the minds of chemists over a long period. But, in writing that "the chemists do not seem to have come to an agreement about the structure of the benzine molecule", he is a little out of date, since by 1937 very few chemists would have doubted that the carbon atoms in benzene form a ring. Nonetheless, Redfield's argument is sound. The number of theoretically possible derivatives can be determined for various position-groups, and these numbers can be compared with the known number of derivatives. The cyclic group gives the best match between the theoretical number and the known number. Similar reasoning had been used by chemists but Redfield had a systematic method for calculating the theoretically possible numbers.

After this diversion, Redfield returns to the main theme of the lecture. He points out that placing objects in positions can be described abstractly as putting two classes (objects and positions) in one-one correspondence; this formulation can then be generalised to putting q classes into 1-1-1-... -1 correspondence. The earlier formula generalises to the following:

$$x = \frac{\sum [(p_1^{\pi_1} p_2^{\pi_2} \dots \pi_1! \pi_2! \dots)^{q-1} N(G_1; p_1^{\pi_1} p_2^{\pi_2} \dots).N(G_2; p_1^{\pi_1} p_2^{\pi_2} \dots) \dots N(G_q; p_1^{\pi_1} p_2^{\pi_2} \dots)]}{\mu(G_1)\mu(G_2)\dots\mu(G_q)}.$$

As an example of a 1-1-1 correspondence, Redfield returns to the key-ring problem and adds a tag to each key where there are two red, two white and two blue tags. This simple change to the problem increases the number of solutions to 696.

At this point in the lecture, Redfield introduced his *group reduction function* (GRF) and he calculated the GRFs for the twins problem. Then he considers a set of identical sextuplets joining hands two and two across a table. The object group here is a *wreath product* (but he does not give it a name). Forming its GRF involves first calculating two GRFs, $(1/2)(s_1^2 + s_2)$ and $(1/6)(s_1^3 + 3s_2s_1 + 2s_3)$, and then writing $(1/2)(s_k^2 + s_{2k})$ in place of every s_k in the second GRF. If the sextuplets join hands to form two triangles, then the group is a wreath product with the factors interchanged, so the GRFs are composed the other way round.

Up to this point (apart from the chemistry) the main ideas in the lecture are in Redfield's 1927 paper, though the particular illustrative examples differ. Now he moves on to new ideas and instead of allowing any object to go in any position, restrictions are introduced. Arbitrary restrictions are not allowed since the permitted permutations must form a group which Redfield calls the *frame group*. The object group and position group, or, in the general case, the q groups acting on the classes (called *ranges*), must all be subgroups of the frame group. The subgroups are denoted by $G_1, G_2, \dots, G_q$ and are called *range groups*.

In the unrestricted case, a frame group did not feature explicitly, but to fit in with the general theory, the full symmetric group plays the role of the frame group.

In that case, the summands in his formula correspond to splitting the set of group elements into cycle-partition types or, equivalently, into conjugacy classes with respect to the symmetric group. In the general case, the set of group elements is divided into conjugacy classes with respect to the frame group. The new formula for the number of arrangements (or structures) is:

$$x = \frac{(\mu(F))^{q-1}}{\mu(G_1)\mu(G_2)\ldots\mu(G_q)} \sum \left[\frac{l_1 l_2 \ldots\ldots l_q}{L^{q-1}}\right].$$

Here each summand involves a product $l_1 l_2 \ldots l_q$ where the corresponding conjugacy class contains l_1 elements of G_1, l_2 elements of $G_2, \ldots, l_q$ elements of G_q; the total number of elements in the conjugacy class is denoted by L. Redfield did not have time to prove this formula in his lecture but he indicates that the proof is similar to that of the earlier formula where the frame group is a symmetric group.

As an example of the new formula, Redfield studies two regular tetrahedra made of wire so that one can be superposed on the other in any orientation; thus the frame group is the alternating group A_4. Next, one edge of the first tetrahedron is marked in red, so the range group for it is a cyclic group of order 2 (in modern parlance the group is the stabilizer of the marked tetrahedron). The second tetrahedron has one edge marked in blue, so the second range group is also a cyclic group of order 2. Applying his new formula, Redfield calculates that there are 4 ways to superpose the two marked tetrahedra; he also gives the calculation for superposing three marked tetrahedra and obtains 30 as the number.

Redfield now factorises his formula in the following way:

$$x = \sum \left[\left(\frac{\mu(F)l_1}{\mu(G_1)L}\right)\left(\frac{\mu(F)l_2}{\mu(G_2)L}\right)\cdots\cdots\left(\frac{\mu(F)l_q}{\mu(G_q)L}\right)\cdot\left(\frac{L}{\mu(F)}\right)\right].$$

Here each summand is a product of $q+1$ terms, where (for $r = 1, 2, \ldots, q$) the r-th factor $\mu(F)l_r/(\mu(G_r)L)$ involves the r-th range group and the frame group, but the last factor $L/\mu(F)$ involves only the frame group. Next he produces a table in which the rows $m_1, m_2, \ldots$ correspond to the various subgroups of the frame group A_4 and the columns to the conjugacy classes of A_4. The table also has an upper row m_0 giving the values $L/\mu(F)$ for the various conjugacy classes. He then observes that the entries in the main part of the table are *compound characters* of the subgroups of the frame group. Each row m_r is then used to form a diagonal matrix M_r which he calls a *character matrix*. This is really a device which enables component-wise multiplication of vectors to be avoided and replaced by matrix multiplication of diagonal matrices, but as Redfield generally writes his diagonal matrices as row vectors the introduction of the matrices is hardly necessary. The earlier calculation can be expressed as the trace $tr(M_0 M_{r_1} M_{r_2} \ldots M_{r_q})$ of the product of diagonal matrices, where the matrix M_0 corresponds to m_0 and the matrices M_{r_i} $(i = 1, 2, \ldots q)$ correspond to the rows indexed by the q range groups. This method is illustrated for two superposed tetrahedra each with one marked *face*, rather than one marked edge.

When the factor M_0 is omitted, the product of the remaining matrices can be expressed as a linear combination of character matrices, and, in some cases, the coefficient of M_r in this decomposition is the number of structures (distinguishable arrangements) which have as their symmetry group the subgroup corresponding to M_r. This is the analogue for general frame groups of the linear decomposition of GRFs in the 1927 paper, but it suffers from similar deficiencies: sometimes different groups correspond to identical diagonal matrices and in some cases the matrices are not linearly independent.

Towards the end of his lecture, Redfield explained how to overcome the difficulties. He introduces what he terms *extended character matrices*. In the corresponding tables, each column corresponds not to a conjugate set of elements but to a conjugate set of *subgroups* of F. The columns in the earlier tables correspond to cyclic subgroups, but, in the new tables some of the old columns are merged and new columns are added for the non-cyclic subgroups. The new table is square and, if the rows are ordered in increasing order of size of the subgroups, then the table is lower triangular with non-zero entries on the diagonal (but Redfield does not explicitly say this). Now the rows of the table[1] are clearly linearly independent, so, when the extended character matrices are used, a unique decomposition is always obtained. Thus the problem of counting structures according to their symmetry groups is solved. The proof of this result was not included in the lecture, but it can be found in Redfield's

[1] Here Redfield uses the same notation M_r for an extended character matrix as he does for the corresponding row of the table.

1940 paper [21].

Although Redfield made use of some of the libraries in the Philadelphia area, he did not always have ready access to material which would have helped him in his researches. For example, the tables containing what he termed *extended characters* appear in the second edition of Burnside's book [1] under the name *tables of marks*. It seems, however, that Redfield had only seen the first edition of the book. Much of the material in Redfield's 1927 paper [18] can be reformulated in terms of group characters and this was done by Foulkes [3]. He also showed that marks can be used to overcome the difficulties which Redfield had encountered in his 1927 paper, but he was, of course, unaware that Redfield had already done this in his 1937 lecture. The table of marks is sometimes called the table of *supercharacters* and some authors (see Kerber [8], for example) write the table as the transpose of that favoured by Burnside and Redfield. For details of how Redfield's work may be used in chemistry see [11] and [13] and references therein.

Finally, at the end of his lecture, Redfield forms a table giving the values of l_r. He then forms diagonal matrices from the *columns* of the table and goes on to give (without proof) an interpretation to the coefficients appearing in the linear decomposition of these matrices.

Other material by Redfield

As well as material already mentioned, some letters from Percy A. MacMahon and Sir Thomas Muir were found in the Redfield family papers. Redfield had studied MacMahon's books [14] and he sent an offprint of his 1927 paper to him. In his reply (dated 19 November 1927), MacMahon suggested that Redfield might be able to prove a conjecture which he had posed at the Toronto meeting of the International Congress of Mathematicians held in 1924 and also at the Rouse Ball Memorial Lecture which he had delivered in Cambridge in 1927. Within a few weeks Redfield had indeed proved the conjecture - the details are included in a draft reply from Redfield to MacMahon dated December 26, 1927. MacMahon was delighted that his conjecture had been settled, but, for some reason, Redfield did not submit his proof for publication, even though, a few years later, he was encouraged to do so by Sir Thomas Muir. In another unpublished typescript. Redfield gives a different proof of the conjecture; further details, including both proofs, may be found in [12].

In his letter of submission to the *American Journal of Mathematics*, dated October 19, 1940, Redfield states that the accompanying paper [21] deals primarily with theory and that he plans to follow it with another concerned with applications which was ready for final revision and copying. This paper, entitled *Enumeration of distinguishable arrangements for general frame groups* has also been found together with a further untitled typescript in a much less finalised form. The former contains the second proof of MacMahon's conjecture mentioned above.

Editing of the lecture typescript

In his letter to Frank Harary (see pp. 81/82 of [6]), Oakley, who had heard Redfield lecture on a number of occasions, wrote "His board work, however, was impeccable. It could have been photographed and printed by photo offset, it was so perfect". It is not clear how much of the lecture text would have been written on the blackboard, but presumably the tables and figures were, and certainly the typescript has required minimal editing. The tables and figures were variously placed in the main typescript and on two manuscript pages at the end (with some in both places); these have all been placed at appropriate points in the text in the published version. At various points Redfield typed "(INDICATE)" to remind himself when to draw attention to a table, figure, *etc.* Most of these reminders have been left in (and it should be clear to what they refer), but where they refer to an item in the references they have been replaced by the number of the reference. Several footnotes have been added: in some cases they refer to later manuscript notes added by Redfield; in other cases they are intended to clarify details in certain tables. With a manual typewriter, Redfield could only emphasise text by underlining it; this has been replaced by the use of italic type. Details of publisher, *etc*, have been added to the first reference and a very small number of typing errors has been corrected.

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