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# ON CIRCUIT CHARACTERIZATIONS OF NEARLY REGULAR GRAPHS

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## Abstract

It is shown that if a graph is cocircuit with a nearly regular graph , then it must also be nearly regular and have the same valency sequence . This result is then used to establish the characterizations of several families of nearly regular graphs by the circuit polynomial .

## Keywords and Phrases

circuit polynomial; nearly regular graph ; theta graph ; cyclomatic number ; dumbbell graph ; circuit characterization ; cocircuit graph ; circuit unique graph ; circuit equivalent; fascigraph ; rotagraph ; tadpole graph ; figure-eight graph .

## 1. Basic Definitions

We consider only graphs that are finite ,loopless and containing no multiple edges . Let G be a graph . We define a circuit (cycle) with one and two nodes in G to be a node and an edge respectively . Circuits with more than two nodes are called **proper circuits** . A **circuit cover** of G is a spanning subgraph of G in which all the components are circuits .

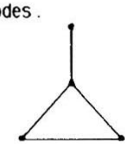
Let us associate an indeterminate or weight $w_\alpha$ with each circuit in G and the monomial $w(S) = \Pi_\alpha w_\alpha$ , with each circuit cover S ; where the product is taken over all the components in S . Then the **circuit polynomial** of G is

$$C(G;\mathbf{w}) = \Sigma\, w(S) ,$$

where the summation is taken over all the circuit covers of G , and **w** ( called the **weight vector** ) is a vector of the indeterminates $w_\alpha$ .

For example , let G be the following graph . We will assign the weight $w_r$ to each cycle with r nodes .

G :



G has one cover consisting or four isolated nodes . This cover has weight $w_1{}^4$ . G has four edges and therefore four covers consisting of an edge together with two isolated nodes . The weight of each such cover is $w_1{}^2w_2$ . Therefore the contribution of these covers to the circuit polynomial is $4w_1{}^2w_2$. G has one cover consisting of a triangle , together with an isolated node .The weight of this cover is $w_1w_3$ . Finally , G has one cover consisting of a pair of independent edges . The weight of this cover is $w_2^2$. Hence the circuit polynomial of G is

$$C(G;\mathbf{w}) = w_1^4 + 4w_1^2w_2 + w_1w_3 + w_2^2 .$$

The circuit polynomial was introduced in Farrell [2] . It has been shown in Farrell [3] , that both the characteristic polynomial and the matching polynomial are special cases of the circuit polynomial . Thus , the circuit polynomial is an interesting combinatorial tool . In this paper , we assign the weight $w_r$ to each cycle with r nodes . Therefore

$\mathbf{w} = (w_1, w_2, \ldots, w_p)$, where p is the number of nodes in G .

The most basic result about circuit polynomials is the following lemma which is taken from Farrell [3] .

**Lemma 1** (The Fundamental Edge Theorem)

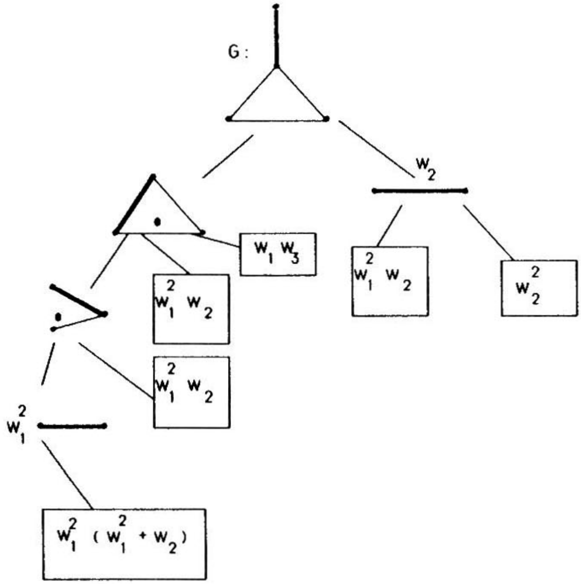
Let G be a graph and xy an edge in G . Then

$$C(G;\mathbf{w}) = C(G';\mathbf{w}) + w_2\, C(G-\{x,y\};\mathbf{w}) + C(G^*;\mathbf{w}),$$

where G' is the graph obtained from G by deleting xy , G-{x,y} is the graph obtained from G by removing the nodes x and y and G* is the graph whose covers are restricted to always contain the edge xy .

This result is essentially the Fundamental Theorem given in [2]. It is quite useful for finding circuit polynomials of arbitrary graphs . We can apply the lemma recursively to smaller and smaller graphs until we obtain graphs whose circuit polynomials can be immediately written down. We will refer to this algorithm as the **edge reduction process .**

The following example illustrates the edge reduction process . The "pivotal" edge used in each application of Lemma 1 is highlighted . By summing the final contributions given in the boxes in the diagram , the circuit polynomial of G is obtained .




If we restrict the covers to those containing nodes and edges only , then the resulting circuit polynomial is called the **matching polynomial** of G . Thus every circuit polynomial can be written as

$$C(G;\mathbf{w}) = M(G;\mathbf{w}) + C(G^*;\mathbf{w}) \quad ,$$

where $M(G;\mathbf{w})$ is the matching polynomial of G and $C(G^*;\mathbf{w})$ is a polynomial over $\mathbf{w}$ , containing all the monomials corresponding to circuit covers of G with at least one proper cycle . If G has no proper cycles ,then $C(G^*;\mathbf{w}) = 0$.

Let G be a graph . We say that $C(G;\mathbf{w})$ **characterizes** G if and only if $C(G;\mathbf{w}) = C(H;\mathbf{w})$ implies that $H \cong G$ . In this case , we also say that G is **circuit unique** . It has been shown (Farrell and Guo [5] ) that many of the well known families of graphs are circuit unique . These include

**chains** (trees with nodes of valencies 1 and 2 only ) , cycles , wheels , complete graphs and regular complete bipartite graphs . In this article , we extend the set of circuit unique graphs to include unions of chains , the basic graphs with cyclomatic number 2 and various kinds of polygonal chains .

It has been shown [2,3] that the circuit polynomial is a generalization of the characteristic polynomial of a graph, and recently (see [6]), that it is a generalization of the $\mu$-polynomial of a graph. The $\mu$-polynomial has found applications in chemistry (see Gutman and Polansky [7] ) . Thus the circuit polynomial is related to some other important graph polynomials . For any graph polynomial , it is of interest to determine its ability to characterize graphs . In the case of the circuit polynomial,characterization implies among other things, that the graph is unique as far as the number of different circuit covers is concerned . This information could be vital to the use of the graph in applications where the cycle covers are important .

Since the circuit polynomial is a generalization of the characteristic polynomial, it is easy to show that any graph which is characterized by its characteristic polynomial must also be characterized by its circuit polynomial i.e. characteristic uniqueness implies circuit uniqueness. Many families of graphs are characterized by their characteristic polynomials (see Cvetkovic et al [1]) . Such graphs will therefore be circuit unique .We consider here, graphs which have not been shown to be characteristically unique, viz the basic graphs with cyclomatic number 2.

**Definition**

Let G be a graph . We say that G is **nearly regular** if and only if the modulus of the difference between the valencies of any two nodes in G does not exceed 1 .

From the above definition , it follows that regular graphs are also nearly regular . The results given in [5] pertain mainly to regular graphs .In the material which follows , we denote the chain , cycle and complete graph with p nodes , by $P_p$ , $C_p$ and $K_p$ respectively . The complete m by n bipartite graph is denoted by $K_{m,n}$ . We denote the graph consisting of components H and K by $\mathbf{H \cup K}$. The notation $\Pi(G)$ is used for the valency sequence of G . In $\Pi(G)$ , $d^r$ means d ,d, d , ... , d (r times). For brevity , C(G) is written for C(G;**w**) . Finally , we omit limits of summations when they are clear from the context of the summand .

## 2. Preliminary Results

Let G be a graph and $d_i$ the valency of node i in G . The following result can be easily proved .

### Lemma 2

Let G be a graph with p nodes and q edges . Then

(i) The highest power of $w_1$ in C(G;**w**) is $w_1{}^p$ and this occurs with coefficient 1 .

(ii) The coefficient of $w_1{}^{p-2} w_2$ is the number of edges in G .

(iii) The coefficient of $w_1{}^{p-4} w_2{}^2$ is

$$\binom{q}{2} - \sum_{i=1}^{p} \binom{d_i}{2} = a_2 .$$

The following lemma is crucial to the main result in the paper.

### Lemma 3

Let p be a positive integer and d and r non-negative integers with $0 \leq r < p$ . Let $d_i$ (i=1, 2, ,..,p) be non-negative integers such that $\Sigma_i d_i$ = pd+r . Then $\Sigma_i \binom{d_i}{2}$ is minimum if and only if

$d_{j_1} = d_{j_2} = d_{j_3} = \ldots = d_{j_r} = d+1$ and $d_{k_1} = d_{k_2} = \ldots = d_{k_{p-r}} = d$ for some positive integers $j_i$ ( i = 1 , 2 , ..., r) and $k_i$ ( i = 1 , 2 , ..., , p-r ) .

**Proof**

Put $S = \Sigma \binom{d_i}{2}$ and $Q = \Sigma_i d_i$ . Then $2S = \Sigma_i d_i^2 - Q$ . Then S is minimum if and only if $\Sigma_{i=1} d_i^2$ is minimum . Since $\Sigma_i d_i = pd+r$ ,we can write $d_i = d + \sigma_i$ , where $\sigma_i$ is an integer . Then

$\Sigma_i d_i^2 = \Sigma_i( d + \sigma_i )^2 = \Sigma_i( d^2 + 2d\,\sigma_i + \sigma_i^2 ) = pd^2 + 2d\, \Sigma_i \sigma_i + \Sigma_i \sigma_i^2$ .

Since $\Sigma_i( d + \sigma_i ) = pd+r$ , it follows that $\Sigma_i \sigma_i = r$ . Hence we get

$\Sigma_i d_i^2 = pd^2 + 2dr + \Sigma_i \sigma_i^2$ . $\Sigma_i d_i^2$ is minimum if and only if $\Sigma_i \sigma_i^2$ is minimum . Since $\Sigma_i \sigma_i = r$ and $r < p$ , it is clear that $\Sigma_i \sigma_i^2$ is minimum if and only if r of the $\sigma_i$'s is 1 and the others , zero . Hence the result . □

The following theorem is immediate from Lemma 3 .

**Theorem 1**

Any graph that is cocircuit with a nearly regular graph is itself nearly regular with the same valency sequence .

**Proof**

From Lemma 3 , for any cocircuit graph , the value of $a_2$ is unique . Hence the result follows . □

Lemma 2 and therefore Theorem 1, also hold for matching polynomials . Thus we have the following corollary

**Corollary 1.1**

Any graph that is comatching with a nearly regular graph is itself nearly regular with the same valency sequence .

The above corollaries show that the set of nearly regular graphs is closed with respect to the properties of comatching and cocircuit . This phenomenon is quite useful for investigating matching uniqueness and circuit uniqueness of nearly regular graphs .

We denote by $P_n$ , the chain with n nodes .

**Lemma 4**

A chain cannot be cocircuit with the union of two non-zero chains .i.e. If $C(P_p) = C(P_r)\, C(P_s)$ , then either $r=0$ and $s=p$ ; or $s=0$ and $r=p$.

**Proof**

Let us assume that the lemma is false . Then $\exists$ non-zero integers r and s such that $C(P_p) = C(P_r)\, C(P_s)$ . Now , the highest power of $w_1$ on the left-hand side is ${w_1}^p$ while the highest power of $w_1$ on the right-hand side . is ${w_1}^{r+s}$. $\Rightarrow p = r+s$ . The coefficient of ${w_1}^{p-2} w_2$ on the left-hand side is $p-1$ , the number of edges in $P_p$ . The coefficient of ${w_1}^{r+s-2} w_2$ on the right-hand side is $r+s-2 \Rightarrow p-1 = r+s-2 . \Rightarrow p = r+s-1$ . This is a contradiction , Hence our assumption is false . Hence the lemma is true . □

**Lemma 5**

The union of two chains cannot be cocircuit with a different union of two chains. i.e. $C(P_p)\, C(P_q) = C(P_p)\, C(P_q)$ if and only if $p = p'$ and $q = q'$ ; or (what is the same ) $p=p'$ and $q = q'$ .

**Proof**

Suppose that $p = p'$ and $q = q'$ . Then clearly , the equation follows . Conversely , let us assume that

$$C(P_p)\, C(P_q) = C(P_{p'})\, C(P_{q'}) \; . \qquad \ldots (1)$$

By considering the highest power of $w_1$ on both sides of the equation , we get $p + q = p' + q'$ . Let us write $a = p + q$ . Apply the reduction process to the chain $P_a$ in two ways (i) by deleting the p th. edge and (ii) by deleting the p' th. edge . This yields the respective equations [ with $P_r$ written for $C(P_r)$ ]

$$P_a = P_p\, P_q + w_2\, P_{p-1}\, P_{q-1}$$

and $$P_a = P_{p'}\, P_{q'} + w_2\, P_{p'-1}\, P_{q'-1}\,.$$

It follows that

$$P_p\, P_q + w_2\, P_{p-1}\, P_{q-1} = P_{p'}\, P_{q'} + w_2\, P_{p'-1}\, P_{q'-1}\,.$$

From Equation (1), we get that

$$P_{p-1}\, P_{q-1} = P_{p'-1}\, P_{q'-1}\,.$$

By repeating the argument for $b = p+q-2$ etc, we get

$$P_{p-m}\, P_0 = P_{p'-m}\, P_{q'-m} \quad ; \text{ i.e } \quad P_{p-m} = P_{p'-m}\, P_{q'-m}\,,$$

where $m = \min(p, p', q, q')$ ( Here we assume that $m=q$ , without loss in generality ) But from Lemma 3, this implies that either $p'-m = p-m$ and $q'-m = 0$ or $q'-m = p-m$ and $p'-m = 0$ . This further implies that either $p' = p$ and $q' = q$ ( since $q'-m = q-m$ ) or $q' = p$ and $p'=q$ . Hence the result follows . □

**Theorem 2**

The circuit polynomial characterizes $P_m \cup P_n$, where $m, n \geq 2$.

**Proof**

Let G be a graph such that $C(G) = C(P_m \cup P_n)$ . Since $P_m \cup P_n$ is nearly regular , it follows from Theorem 1 , that G is also nearly regular and that $\pi(G) = (2^{m+n-4}, 1^4)$ . Since $C(P_m \cup P_n)$ has no term in the weight $w_r$ for $r > 2$ , it follows that G is a forest . Also , from the terms $w_1^{m+n}$ and $w_1^{m+n-2} w_2$ in C(G) , it follows that G has m+n nodes and m+n-2 edges . Therefore G has two components . The only trees with nodes of valencies 1 and 2 only are chains ; and so , G is the union of chains . From Lemma 4 , we conclude that $G \cong P_m \cup P_n$. Hence the result follows .

□

### 3. The basic graphs with cyclomatic Number 2

The basic graphs with cyclomatic number 2 are shown below in Figure 1.

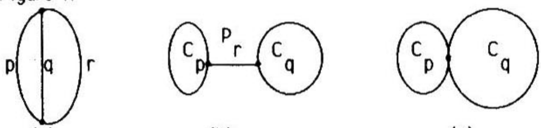



**Figure 1**

We call the graphs in Figure 1(a), (b) and (c), the **theta graph** $\theta_{p,q,r}$ $(p,q \geq 2\ ,\ r>0)$ , the **dumbbell** $D_{p,q,r}$ $(p,q > 2\ ,\ r>0)$ and the **figure-eight graph** $E_{p,q}$ $(p,q > 2)$. Theorem 1 can be used to establish the characterizations of the theta graph and the dumbbell . In the case of the $\theta$-graph , any graph H such that $C(H) = C(\theta_{p,q,r})$ must have the property that $\Pi(H) = \Pi(\theta_{p,q,r}) = (2^{p+q+r-3},\ 3^2)$ . This restriction is sufficient to yield $H \cong G$ . Similarly , it can be easily shown that $D_{p,q,r}$ is characterized by its circuit polynomial .

Notice that we can also define $E_{p,q}$ to be the graph obtained from $\theta_{p,q,r}$ by "shrinking" the chain $P_r$ to a node i.e. $E_{p,q} \cong \theta_{p,q,1}$ .

Also , $E_{p,q}$ can be obtained from $D_{p,q,r}$ by shrinking the chain $P_r$ to a node . i.e. $E_{p,q} \cong D_{p,q,1}$ Theorem 1 can be used to establish the characterization of $E_{p,q}$ .The only basic graph with cyclomatic number 1 is the cycle $C_p$ ; and this has been shown ( [5] ) to be circuit unique . Hence all the basic graphs with cyclomatic numbers 1 and 2 are characterized by their circuit polynomials . Hence we have the following result .

**Theorem 3**

The circuit polynomial characterizes all the basic graphs with cyclomatic numbers 1 and 2 .

**Definition**

We define the **tadpole** $\mathbf{T_{p,q}}$ to be the graph shown below in Figure 2 .

$T_{p,q}$

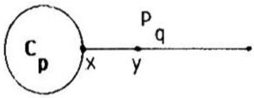



**Figure 2**

We note that the tadpole is not a nearly regular graph . However , if we extend our definition of the dumbbell to include the cases in which $C_p$ and $C_q$ are improper cycles ( nodes and edges ) then we get that $T_{p,q}$ is the dumbbell $D_{p,1,r}$ . It is therefore appropriate to consider this graph in our discussions

The following result is taken from Farrell & Grell [4] . It will be useful for the material which follows .

**Lemma 6**

$$\frac{\partial^n C(G)}{\partial w_i^n} = n! \sum C(G - \alpha_i^n) ,$$

where $\alpha_i{}^n$ is a set of n node-disjoint i-cycles in G ; and the summation is taken over all such sets of n node-disjoint cycles in G .

Let us apply the reduction process to $T_{p,q}$ by deleting the edge xy . This yields

$$C(T_{p,q}) = C(C_p)\, C(P_{q-1}) + w_2\, C(P_{p-1})\, C(P_{q-2})$$

$$= [M(C_p) + w_p]\, C(P_{q-1}) + w_2 C(P_{p-1})\, C(P_{q-2})$$

$$= M(C_p)\, C(P_{q-1}) + w_p\, C(P_{q-1}) + w_2\, C(P_{p-1})\, C(P_{q-2}) \;. \ldots \quad (2)$$

Let G be a graph such that $C(G) = C(T_{p,q})$ . Then G has the following properties :

(1) G has p+q-1 nodes and p+q-1 edges .

(2) G has a unique cycle $C_p$ [ from the term $w_1^{q-1} w_p$ in the polynomial $w_p \quad C(P_{q-1})$ ] .

Also, by applying Lemma 6 to Equation (2) , we get

$$\frac{\partial C(G)}{\partial w_p} = C(G-C_p) = C(P_{q-1}) \;.$$

$$\Rightarrow G-C_p \cong P_{q-1} \;. \Rightarrow G \text{ has a spanning subgraph } C_p \cup P_{q-1} \;.$$

Hence G can be obtained by adding [ p+q-1 -(p+q-2)] an edge to $C_p \cup P_{q-1}$ subject to the restrictions above .There are two possible graphs ; (1) the tadpole $T_{p,q}$ and (2) the graph H formed by joining a node of valency 2 in $P_{q-1}$ to a node of valency 2 in G . Now $\pi(H) = (1^2, 2^{p+q-5}, 3^2)$ . Let $d_i$ be the valency of node i in H . Then

$$\sum_{i=1}^{p+q+1} \binom{d_i}{2} = (p + q - 5) + 2\binom{3}{2} = p + q + 1 \;.$$

From Lemma 2 , this sum for G is p+q . $\Rightarrow$ $a_2(G) \neq a_2(H)$ .

Therefore $G \ncong H$ . It follows that $G \cong T_{p,q}$ . We therefore have the following theorem .

**Theorem 4**

The circuit polynomial characterizes tadpoles .

**4. Some Other Families of Nearly Regular Graphs**

The graph $G = K_m \cup K_{m+1}$ $(m>2)$ is nearly regular. Let H be a graph such that $C(H) = C(G)$. Since $C(G)$ (and therefore $C(H)$) contains the term $w_m w_{m+1}$ with coefficient unity, it follows that H has a cover consisting of cycles $C_m$ and $C_{m+1}$. Suppose that $\exists$ "link" edges joining nodes of $C_m$ to nodes of $C_{m+1}$ in H. Then we consider the possible ways in which these edges can occur.

Suppose that $\exists$ a pair of link edges which have no nodes in common ( i.e. independent link edges ). Let r and s be the number of nodes in $C_m$ and $C_{m+1}$ respectively, which separate the nodes at the end of the

independent edges. Then H contains the cycles of lengths $r+s+4$ and $2m+1-(r+s)$. But from $C(H)$, the largest cycle in H has length $m+1$. Therefore $r+s+4 \leq m+1$. So $r+s \leq m-3$ ; and $2m+1-(r+s) \geq m+3 > m+1$. This is impossible. Therefore no such link edges occur in H. The only other possibility is that all link edges are incident to a single node x either in $C_m$ or in $C_{m+1}$. But

$$\Pi(H) = \Pi(G) = (m^{m-1}, (m+1)^m),$$

since G is nearly regular. Suppose that node x belongs to $C_m$, then its valency is $m-1+t$, where t is the number of link edges. Therefore t must either be 1 or 0. If $t=1$, then the subgraph containing $C_{m+1}$ has a node of valency $m+1$ (since m nodes of $C_{m+1}$ must be adjacent to the node at the end of the link edge). This is impossible (by looking at $\Pi(H)$). Therefore $t=0$. It follows that $H \cong K_m \cup K_{m+1} = G$. Hence we have the following theorem.

**Theorem 5**

The circuit polynomial characterizes $K_m \cup K_{m+1}$ .

We refer to the graph below in Figure 3 as $\mathbf{D}_{n_1,n_2,\ldots,n_k}(n_i > 2)$ . It consists of the cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ linked by single edges .

$D_{n_1 n_2 \ldots n_k}$

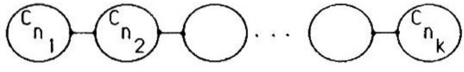



**Figure 3**

**Theorem 6**

The circuit polynomial characterizes $D_{n_1,n_2,\ldots,n_k}(n_i > 2)$ .

**Proof**

Let G denote the graph $D_{n_1,n_2,\ldots,n_k}(n_i > 2)$ . Let H be a graph such that C(H) = C(G) . Since G is nearly regular ,

$\Pi(H) = \Pi(G) = (2^{N-k+1}, 3^{k-1})$ , where $N = \Sigma_{i=1}\, n_i$ .

Since C(G) contains the term $\Pi_{i=1} w_{r_i}$ , it follows that H has a cover consisting of k $n_i$-gons (i=1,2,...,k) . Since H and G have the same number (N+k-1) of edges , H contains k-1 "link" (non -cycle) edges . It can be easily seen that if any of these link edges join two nodes of the same cycle ,then either H will contain covers which are not in G or the condition $\Pi(H) = \Pi(G)$ will be violated . Therefore the link edges cannot belong to any proper cycles . This restriction , together with Theorem 1 is sufficient to yield the graph $D_{n_1,n_2,\ldots,n_k}(n_i > 2)$ i.e $H \cong G$ .Hence the result follows . □

**Definition**

The **(regular) linear polygonal chain** $\mathbf{L_{n,k}}$ (n>3) is the graph consisting of k n-gons linked together as shown below in Figure 4 .

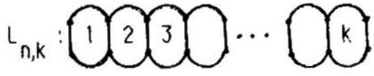



**Figure 4**

Let G be the graph $L_{n,k}$. Let H be a graph such that C(H) = C(G). Then C(H) contains the term $w_N$, where N = nk-2(k-1). It follows that contains a hamiltonian cycle $C_N$. Therefore the remaining k-1 edges of H are "chords" of $C_N$. Since G is nearly regular $\Pi(H) = \Pi(G) = (2^N, 3^{2(k-1)})$. It follows that the addition of the k-1 chords must create 2(k-1) nodes of valency 3. Therefore each chord must be incident to exactly two nodes of valency 3. C(G), and therefore C(H), contains a term in $w_n^{k'}$, where $k' = (1/2)(k+1)$, if k' is odd and $k'=k/2$, if k is even. Therefore H contains k' disjoint cycles. This is sufficient to yield the graph $L_{n,k}$ i.e. H $\cong$ G. Hence we have the following theorem.

**Theorem 7**

The graph $L_{n,k}$ (n >3) is circuit unique.

The graph $L_{n,k}$ belongs to a class of graphs which in mathematical chemistry are called **fascigraphs**. If we identify the corresponding nodes of valency two in cells 1 and k, then the resulting 'circular' graph $\mathbf{R_{n,k}}$ belongs to the so-called **rotagraphs**. This graph is also obtained by joining corresponding nodes of two equal cycles. We can use a similar argument to show that $R_{n,k}$ is also characterized by $C(R_{n,k})$. We leave this as an exercise for the reader. Thus we have the following theorem.

**Theorem 8**

The rotagraph $R_{n,k}$ (n >3) is circuit unique.

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