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# THE NUMBER OF KEKULÉ STRUCTURES FOR RECTANGLE-SHAPED BENZENOIDS: PART VII

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*Abstract:* The studies of the number of Kekulé structures for oblate rectangle-shaped benzenoids are continued. A general method - the matrix method - is developed in order to obtain the recurrence relations discussed in PART III. Furthermore, a conjecture concerning the coefficients of the recurrence relations is rigorously proved.

## 1. INTRODUCTION

This is a continuation of the studies of the number of Kekulé structures ($K$) of oblate rectangles, $R^j(m,n)$. In PART III of this article series[1] it was found that there exist recurrence relations for the $K$ number of $R^j(m,n)$ with fixed $n$ values. The recurrence relations for $K\{R^j(m,n)\}$ are given up to $n$=6 in PART II and PART III.

In the present paper we develop a matrix approach to the recurrence relations for $K\{R^j(m,n)\}$. By this method direct calculations of determinants lead to the special cases of $n$ = 2, 3, ...., 10. There is no hindrance against an extension to still higher values of $n$.

## 2. THE MATRIX METHOD

We use the same notation as in PART III.[1] In particular, the following abbreviations are used.

$$R_n(m) = K\{R^j(m,n)\}$$

$$R_n^{(t)}(m) = K\{B(n,\ 2m-2,\ t)\}$$

A basic formula in PART III reads:

$$R_n(m+j) = \sum_{i=0}^{n} R_n^{(-i)}(j+1) R_n^{(-i)}(m)$$

It yields together with the symmetry property $R_n^{(-i)} = R_n^{(i-n)}$:

$$\begin{bmatrix} R_n(m) \\ R_n(m+1) \\ \vdots \\ R_n(m+j') \end{bmatrix} = \mathrm{J}_n \begin{bmatrix} R_n^{(0)}(m) \\ R_n^{(-1)}(m) \\ \vdots \\ R_n^{(-j')}(m) \end{bmatrix} \tag{2.1}$$

where $j' = [n/2]$, and $\mathrm{J}_n$ is a $(j'+1)\times(j'+1)$ square matrix with the general element equal to

$(\mathrm{J}_n)_{rs} = 2R_n^{(1-s)}(r)$ if $n$ is odd, or if $n$ is even and $1 \le s \le j'$;

$(\mathrm{J}_n)_{rs} = R_n^{(1-s)}(r)$ if $n$ is even and $s = j'+1$.

Therefore, when $n$ is even

$$\mathrm{J}_n = \begin{bmatrix} 2R_n^{(0)}(1) & 2R_n^{(-1)}(1) & \cdots & 2R_n^{(-j'+1)}(1) & R_n^{(-j')}(1) \\ 2R_n^{(0)}(2) & 2R_n^{(-1)}(2) & \cdots & 2R_n^{(-j'+1)}(2) & R_n^{(-j')}(2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2R_n^{(0)}(j') & 2R_n^{(-1)}(j') & \cdots & 2R_n^{(-j'+1)}(j') & R_n^{(-j')}(j') \\ 2R_n^{(0)}(j'+1) & 2R_n^{(-1)}(j'+1) & \cdots & 2R_n^{(-j'+1)}(j'+1) & R_n^{(-j')}(j'+1) \end{bmatrix}$$

and when $n$ is odd

$$\mathrm{J}_n = \begin{bmatrix} 2R_n^{(0)}(1) & 2R_n^{(-1)}(1) & \cdots & 2R_n^{(-j'+1)}(1) & 2R_n^{(-j')}(1) \\ 2R_n^{(0)}(2) & 2R_n^{(-1)}(2) & \cdots & 2R_n^{(-j'+1)}(2) & 2R_n^{(-j')}(2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2R_n^{(0)}(j') & 2R_n^{(-1)}(j') & \cdots & 2R_n^{(-j'+1)}(j') & 2R_n^{(-j')}(j') \\ 2R_n^{(0)}(j'+1) & 2R_n^{(-1)}(j'+1) & \cdots & 2R_n^{(-j'+1)}(j'+1) & 2R_n^{(-j')}(j'+1) \end{bmatrix}$$

The following shows $\mathrm{J}_n$ for $n = 2, 3, \ldots, 10$. Here the number above the $i$-th column (or on the right-hand side of the $j$-th row) is the common factor extracted from the $i$-th column (or $j$-th row).

$$\mathrm{J}_2 = \begin{array}{cc} & 2 \\ \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} & \begin{matrix} \\ 2 \end{matrix} \end{array} \qquad \mathrm{J}_3 = \begin{array}{cc} & \begin{matrix} 2 & 2 \end{matrix} \\ \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} & \begin{matrix} \\ 5 \end{matrix} \end{array} \qquad \mathrm{J}_4 = \begin{array}{cc} & \begin{matrix} 2 & 2 & \\ 1 & 1 & 1 \end{matrix} \\ \begin{bmatrix} 5 & 8 & 9 \\ 35 & 60 & 69 \end{bmatrix} & \begin{matrix} 3 \\ 3^2 \end{matrix} \end{array} \qquad \mathrm{J}_5 = \begin{array}{cc} & \begin{matrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{matrix} \\ \begin{bmatrix} 3 & 5 & 6 \\ 14 & 25 & 31 \end{bmatrix} & \begin{matrix} 7 \\ 7^2 \end{matrix} \end{array}$$

$$J_6 = \begin{array}{c} \begin{matrix} 2 & 2 & 2 & \phantom{2848} \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 7 & 12 & 15 & 16 \\ 84 & 154 & 200 & 216 \\ 1092 & 2016 & 2632 & 2848 \end{bmatrix} \end{array} \begin{matrix} \\ 4 \\ 4^2 \\ 4^3 \end{matrix} \qquad J_7 = \begin{array}{c} \begin{matrix} 2 & 2 & 2 & 2 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 7 & 9 & 10 \\ 30 & 56 & 75 & 85 \\ 246 & 462 & 622 & 707 \end{bmatrix} \end{array} \begin{matrix} \\ 9 \\ 9^2 \\ 9^3 \end{matrix}$$

$$J_8 = \begin{array}{c} \begin{matrix} 2 & 2 & 2 & 2 & \phantom{220045} \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 9 & 16 & 21 & 24 & 25 \\ 165 & 312 & 427 & 500 & 525 \\ 3333 & 6336 & 8715 & 10240 & 10765 \\ 68013 & 129360 & 178035 & 209280 & 220045 \end{bmatrix} \end{array} \begin{matrix} \\ 5 \\ 5^2 \\ 5^3 \\ 5^4 \end{matrix} \qquad J_9 = \begin{array}{c} \begin{matrix} 2 & 2 & 2 & 2 & 2 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 5 & 9 & 12 & 14 & 15 \\ 55 & 105 & 146 & 175 & 190 \\ 671 & 1287 & 1798 & 2163 & 2353 \\ 8272 & 15873 & 22187 & 26703 & 29056 \end{bmatrix} \end{array} \begin{matrix} \\ 11 \\ 11^2 \\ 11^3 \\ 11^4 \end{matrix}$$

$$J_{10} = \begin{array}{c} \begin{matrix} 2 & 2 & 2 & 2 & 2 & \phantom{939060} \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 11 & 20 & 27 & 32 & 35 & 36 \\ 286 & 550 & 774 & 944 & 1050 & 1086 \\ 8294 & 16016 & 22638 & 27712 & 30898 & 31984 \\ 243100 & 469612 & 664092 & 813296 & 907076 & 939060 \\ 7133412 & 13780624 & 19488612 & 23868416 & 26621628 & 27560688 \end{bmatrix} \end{array} \begin{matrix} \\ 6 \\ 6^2 \\ 6^3 \\ 6^4 \\ 6^5 \end{matrix}$$

We now express (2.1) in another form as follows.

$$\begin{bmatrix} R_n^{(0)}(m) \\ R_n^{(-1)}(m) \\ \vdots \\ R_n^{(-j')}(m) \end{bmatrix} = J_n^{-1} \begin{bmatrix} R_n(m) \\ R_n(m+1) \\ \vdots \\ R_n(m+j') \end{bmatrix} \tag{2.2}$$

where $J_n^{-1}$ is the inverse matrix of $J_n$.

Let $J_n(i,j)$ denote the submatrix of $J_n$ obtained from $J_n$ by deleting the $i$-th row and the $j$-th column. Denote

$$u(i,j) = (-1)^{i+j}\det J_n(i,j) \tag{2.3}$$

By the standard methods of matrix calculus one obtains

$$(J_n^{-1})_{ij} = u(j,i)/\det J_n \tag{2.4}$$

Thus from (2.2) we have

$$R_n^{(0)}(m) = (1/\det J_n)[u(1,1)R_n(m) + u(2,1)R_n(m+1) + \dots + u(j'+1,\ 1)R_n(m+j')]$$

which together with

$$R_n^{(0)}(m) = (1/2)(n+2)R_n(m-1)$$

according to eqn. (21) of PART III yields

$$c_j = -u(j'-j,\ 1)/u(j'+1,\ 1)\ ; \qquad j = 0,\ 1,\ \ldots,\ j'-1$$
$$c_{j'} = (1/2)(n+2)\det J_n/u(j'+1,\ 1) \tag{2.5}$$

Therefore, when $J_n$ is given, the recurrence relation for $R_n(m)$ is available by direct calculation of determinants. By formulas (2.5) it was arrived at

$$R_2(m) = 8R_2(m-1) - 8R_2(m-2)$$
$$R_3(m) = 15R_3(m-1) - 25R_3(m-2)$$
$$R_4(m) = 27R_4(m-1) - 108R_4(m-2) + 108R_4(m-3)$$
$$R_5(m) = 42R_5(m-1) - 245R_5(m-2) + 343R_5(m-3)$$
$$R_6(m) = 64R_6(m-1) - 640R_6(m-2) + 2048R_6(m-3) - 2048R_6(m-4)$$
$$R_7(m) = 90R_7(m-1) - 15\cdot 9^2R_7(m-2) + 7\cdot 9^3R_7(m-3) - 9^4R_7(m-4)$$
$$R_8(m) = 125R_8(m-1) - 4\cdot 5^4R_8(m-2) + 28\cdot 5^4R_8(m-3) - 16\cdot 5^5R_8(m-4) + 16\cdot 5^5R_8(m-5)$$
$$R_9(m) = 15\cdot 11R_9(m-1) - 35\cdot 11^2R_9(m-2) + 28\cdot 11^3R_9(m-3) - 9\cdot 11^4R_9(m-4) + 11^5R_9(m-5)$$
$$R_{10}(m) = 6^3R_{10}(m-1) - 35\cdot 6^3R_{10}(m-2) + 448\cdot 6^3R_{10}(m-3) - 2\cdot 6^7R_{10}(m-4) + 32\cdot 6^6R_{10}(m-5) - 32\cdot 6^6R_{10}(m-6)$$

## 3. THE DETERMINANT OF $J_n$

From the above section one sees that the determinant $J_n$ is the denominator of each entry of $J_n^{-1}$. Now we investigate the explicit formula for $\det J_n$.

In order to simplify the derivation of $\det J_n$, we want to extract some common factors from each column and row. By the definition of $J_n$ it is not difficult to see that 2 is a common factor of each column except the $(j'+1)$-th column in the case when $n$ is even. Moreover, besides the factor 2, the $i$-th row ($i \geq 2$) of $J_n$ has the common factor $(n-2)^{i-1}$. This is evident for $i=2$ from the known formula

$$R_n^{(-t)}(2) = (1/2)(n+2)(n-t+1)(t+1)$$

Now suppose that each element of the $(i-1)$-th row, viz. $R_n^{(-t)}(i-1)$, has the factor $(n+2)^{i-2}$. Thus we can write

$$R_n^{(-t)}(i-1) = (n+2)^{i-2}f(t,n)$$

where $f(t,n)$ is a polynomial in $t$ and $n$.

We now apply formula (1) of PART IB to the elements of the $i$-th row. We have

$$R_n^{(-j)}(i) = \sum_{t=0}^{j} (n+1-j)(t+1)R_n^{(-t)}(i-1) + \sum_{r=j+1}^{n} (n+1-r)(j+1)R_n^{(-r)}(i-1)$$

$$= \sum_{t=0}^{j} [(n+2) - (j+1)](t+1)(n+2)^{i-2}f(t,n)$$

$$+ \sum_{r=j+1}^{n} [(n+2) - (r+1)](j+1)(n+2)^{i-2}f(r,n)$$

$$= (n+2)^{i-1}\left[\sum_{t=0}^{j} (t+1)f(t,n) + \sum_{r=j+1}^{n} (j+1)f(r,n)\right]$$

$$- (n+2)^{i-2} \sum_{t=0}^{n} (j+1)(t+1)f(t,n)$$

As seen in PART IB the summation $\sum_{t=0}^{n} (t+1)$ has the factor $(n+2)$. One can verify that the last summation in the above formula has the factor $(n+2)$ in a similar way as in Section 6 of PART IB. Therefore, $R_n^{(-j)}(i)$ has the factor $(n+2)^{i-1}$.

Now we are in the position to derive the explicit formula for $\det J_n$. Since the cases with even and odd $n$ behave slightly differently, we distinguish two cases.

(a) <u>$n$ is odd</u>

Introduce $e_{ij} = (1/2)(J_n)_{ij}/(n+2)^{i-1}$; $i,j = 1, 2, \ldots, j', j'+1$. Define $J_n^*$ to be the matrix with the general element equal to

$$(J_n^*)_{ij} = e_{ij}$$

In other words, $J_n^*$ is the matrix obtained from $J_n$ by extracting the factor 2 from each column and the factor $(n+2)^{i-1}$ from the $i$-th row.

Let $n = 2s-1$. By formula (1) of PART IB and the symmetry property $R_n^{(-p)} = R_n^{(p-n)}$ one has

$$(J_n)_{ij} - (J_n)_{i\,j-1} = 2[R_n^{(-j+1)}(i) - R_n^{(-j+2)}(i)]$$

$$= 2\left[-\sum_{t=0}^{j-2} (t+1)R_n^{(-t)}(i-1) + \sum_{t=j-1}^{n} (n-t+1)R_n^{(-t)}(i-1)\right]$$

$$= 2\sum_{t=j-1}^{n-j+1} (n-t+1)R_n^{(-t)}(i-1) = 2\sum_{t=j-1}^{s-1} (n+2)R_n^{(-t)}(i-1)\Big]$$

Inserting of $s = (n+1)/2 = j'+1$ gives

$$(J_n)_{ij} - (J_n)_{i\ j-1} = 2(n+2) \sum_{t=j-1}^{j'} R_n^{(-t)}(i-1)$$

Therefore,

$$(J_n^*)_{ij} - (J_n^*)_{i\ j-1} = \sum_{t=j}^{j'+1} e_{i-1\ t}\,; \qquad j \geq 2 \tag{3.1}$$

Now we can derive the explicit formula for $\det J_n^*$ as follows.

$$\det J_n^* = \det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ e_{21} & e_{22} & \cdots & e_{2\ j'} & e_{2\ j'+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{j'-1\ 1} & e_{j'-1\ 2} & \cdots & e_{j'-1\ j'} & e_{j'-1\ j'+1} \\ e_{j'\ 1} & e_{j'\ 2} & \cdots & e_{j'\ j'} & e_{j'\ j'+1} \\ e_{j'+1\ 1} & e_{j'+1\ 2} & \cdots & e_{j'+1\ j'} & e_{j'+1\ j'+1} \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ e_{21} & j' & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ e_{j'-1\ 1} & e_{j'-2\ 2}+e_{j'-2\ 3}+\cdots+e_{j'-2\ j'+1} & \cdots & e_{j'-2\ j'+1} \\ e_{j'\ 1} & e_{j'-1\ 2}+e_{j'-1\ 3}+\cdots+e_{j'-1\ j'+1} & \cdots & e_{j'-1\ j'+1} \\ e_{j'+1\ 1} & e_{j'\ 2}+e_{j'\ 3}+\cdots+e_{j'\ j'+1} & \cdots & e_{j'\ j'+1} \end{bmatrix}$$

$$= \det \begin{bmatrix} j' & j'-1 & \cdots & 1 \\ e_{22}+e_{23}+\cdots e_{2\ j'+1} & e_{23}+e_{24}+\cdots+e_{2\ j'+1} & \cdots & e_{2\ j'+1} \\ \vdots & \vdots & \vdots & \\ e_{j'\ 2}+e_{j'\ 3}+\cdots+e_{j'\ j'+1} & e_{j'\ 3}+e_{j'\ 4}+\cdots+e_{j'\ j'+1} & \cdots & e_{j'\ j'+1} \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e_{22} & e_{23} & \cdots & e_{2\ j'+1} \\ e_{32} & e_{33} & \cdots & e_{3\ j'+1} \\ \vdots & \vdots & \vdots & \vdots \\ e_{j'\ 2} & e_{j'\ 3} & \cdots & e_{j'\ j'+1} \end{bmatrix}$$

Note that the above determinant is a subdeterminant obtained from $\det J_n^*$ by deleting the first column and the last row. Repeating the above process will lead to

$\det J_n^* =$

$$= \det\begin{bmatrix} 1 & 1 & \cdots & 1 \\ e_{22} & e_{23} & \cdots & e_{2\,j'+1} \\ e_{32} & e_{33} & \cdots & e_{3\,j'+1} \\ \vdots & \vdots & \vdots & \vdots \\ e_{j'\,2} & e_{j'\,3} & \cdots & e_{j'\,j'+1} \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & \cdots & 1 \\ e_{23} & e_{24} & \cdots & e_{2\,j'+1} \\ e_{33} & e_{34} & \cdots & e_{3\,j'+1} \\ \vdots & \vdots & \vdots & \vdots \\ e_{j'-1\,3} & e_{j'-1\,4} & \cdots & e_{j'-1\,j'+1} \end{bmatrix}$$

$$= \cdots = \det\begin{bmatrix} 1 & 1 \\ e_{2\,j'} & e_{2\,j'+1} \end{bmatrix} = \det[1] = 1$$

Consequently, by the relation between $J_n$ and $J_n^*$ we have

$$\det J_n = 2^{j'+1}(n+2)^{1+2+\cdots+j'}\det J_n^* = 2^{(n+1)/2}(n+2)^{(n-1)/8} \tag{3.2}$$

An illustration for $n=7$ is given below.

$$J_7^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 7 & 9 & 10 \\ 30 & 56 & 75 & 85 \\ 246 & 462 & 622 & 707 \end{bmatrix}, \quad \det J_7^* = \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 \\ 30 & 7+9+10 & 9+10 & 10 \\ 246 & 56+75+85 & 75+85 & 85 \end{bmatrix}$$

$$= \det\begin{bmatrix} 3 & 2 & 1 \\ 7+9+10 & 9+10 & 10 \\ 56+75+85 & 75+85 & 85 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & 1 \\ 7 & 9 & 10 \\ 56 & 75 & 85 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 \\ 9 & 10 \end{bmatrix} = 1$$

(b) <u>$n$ is even</u>

Let $n = 2s$. Then $j' = s$. Introduce $e_{ij} = (1/2)(J_n)_{ij}/[(n+2)/2]^{i-1}$ for $i = 1, 2, \ldots, j'+1$; $j = 1, 2, \ldots, j'$ and $e_{i\,j'+1} = (J_n)_{i\,j'+1}/[(n+2)/2]^{i-1}$ for $i = 1, 2, \ldots, j'+1$. Define $J_n^*$ to be the matrix satisfying $(J_n^*)_{ij} = e_{ij}$. Denote by $J_n^*(h)$ the submatrix of $J_n^*$ obtained by deleting the $i$-th columns for $i = 1, 2, \ldots, j'+1-h$ and the $j$-th rows for $j = h+1, h+2, \ldots, j'+1$. In a similar way as in Case (a) we have when $2 \le j \le j'$

$$(J_n)_{ij} - (J_n)_{i\,j-1} = 2[R_n^{(-j+1)}(i) - R_n^{(-j+2)}(i)]$$

$$= 2\sum_{t=j-1}^{n-j+1}(n-t+1)R_n^{(-t)}(i-1) = 2\Big[\sum_{t=j-1}^{s-1}(n+2)R_n^{(-t)}(i-1) - (n-s+1)R_n^{(-s)}(i-1)\Big]$$

$$= (n+2)\Big[\sum_{t=j-1}^{j'-1}2R_n^{(-t)}(i-1) + R_n^{(-j')}(i-1)\Big]$$

and when $j = j'$:

$$(J_n)_{i\ j'} - (1/2)(J_n)_{i\ j'-1} = (1/2)(n+2)R_n^{(-j')}(i-1)$$

Therefore

$$(J_n^*)_{ij} - (J_n^*)_{i\ j-1} = 2\sum_{t=j}^{j'} e_{i-1\ t} + e_{i-1\ j'+1}$$

In a similar way as in Case (a) we reach at

$$\det J_n^* = 2^{j'-1}\det J_n^*(j') = 2^{j'-1}2^{j'-2}\det J_n^*(j'-1)$$
$$= 2^{j'-1}2^{j'-2}2^{j'-3}\cdot\ \ldots\ \cdot 2^2 2\det J_n^*(1) = 2^{(n^2-2n)/8}$$

Consequently,

$$\det J_n = 2^{j'}[(n+2)/2]^{1+2+\ldots+j'}\det J_n^* = (n+2)^{(n^2+2n)/8} \tag{3.3}$$

In the following we give an illustration for $n$=6.

$$J_6^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 7 & 12 & 15 & 16 \\ 84 & 154 & 200 & 216 \\ 1092 & 2016 & 2632 & 2848 \end{bmatrix}, \quad \det J_6^* = \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 7 & 2(1+1)+1 & 2+1 & 1 \\ 84 & 2(12+15)+16 & 2\times15+16 & 16 \\ 1092 & 2(154+200)+216 & 2\times200+216 & 216 \end{bmatrix}$$

$$= \det\begin{bmatrix} 2 & 2 & \\ 1 & 1 & 1 \\ 12 & 15 & 16 \\ 154 & 200 & 216 \end{bmatrix} = 2^2\det\begin{bmatrix} 1 & 0 & 0 \\ 12 & 2\times1+1 & 1 \\ 154 & 2\times15+16 & 16 \end{bmatrix} = 2^2\det\begin{bmatrix} 2 & & \\ 1 & 1 & \\ 15 & 16 & \end{bmatrix} = 2^{2+1} = 2^3$$

Therefore, $\det J_6 = 2^3 4^6 \det J_6^* = 2^6 4^6 = 8^6$.

## 4. THE PROOF OF CONJECTURE A OF PART III[1]

By eqn. (2.5) we can rewrite Conjecture A as

$$u(j'+1,\ 1)R_n^{(-j')}(2) = -u(j',1) \tag{4.1}$$

We need to derive the formulas for $u(j'+1,\ 1)$ and $u(j',1)$.

(a) <u>$n$ is odd</u>

By the definition of $u(j'+1,\ 1)$ we have

$$u(j'+1,\ 1) = (-1)^{(j'+2)}2^{j'}(n+2)^{1+2+\ldots+(j'-1)}\det J_n^{**}$$

where $J_n^{**}$ is the submatrix of $J_n^*$ obtained by deleting the first column and the last row. We already know from Section 3 that $\det J_n^{**} = 1$. Thus

$$u(j'+1,1) = (-1)^{j'}2^{(n-1)/2}(n+2)^{(n^2-4n+3)/8} \tag{4.2}$$

Now we concentrate on $u(j',1)$. Bearing in mind eqn. (3.1) we can derive this quantity as follows.

$$u(j',1) = (-1)^{j'+1}(n+2)^{\Delta}2^{j'}\det J_n^*(j',1)$$

$$= (-1)^{j'+1}(n+2)^{\Delta}2^{j'}\det\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ e_{22} & e_{23} & \cdots & e_{2\,j'} & e_{2\,j'+1} \\ e_{32} & e_{33} & \cdots & e_{3\,j'} & e_{3\,j'+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{j'-1\;2} & e_{j'-1\;3} & \cdots & e_{j'-1\;j'} & e_{j'-1\;j'+1} \\ e_{j'+1\;2} & e_{j'+1\;3} & \cdots & e_{j'+1\;j'} & e_{j'+1\;j'+1} \end{bmatrix}$$

$$= (-1)^{j'+1}(n+2)^{\Delta}2^{j'}\det\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ e_{22} & 1 & \cdots & 1 & 1 \\ e_{32} & e_{23} & \cdots & e_{2\,j'} & e_{2\,j'+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{j'-1\;2} & e_{j'-2\;3} & \cdots & e_{j'-2\;j'} & e_{j'-2\;j'+1} \\ e_{j'+1\;2} & e_{j'\;3} & \cdots & e_{j'\;j'} & e_{j'\;j'+1} \end{bmatrix}$$

$$= (-1)^{j'+1}(n+2)^{\Delta}2^{j'}\det\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ e_{23} & 1 & \cdots & 1 & 1 \\ e_{33} & e_{23} & \cdots & e_{2\,j'} & e_{2\,j'+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{j'-2\;3} & e_{j'-3\;3} & \cdots & e_{j'-3\;j'} & e_{j'-3\;j'+1} \\ e_{j'\;3} & e_{j'-1\;3} & \cdots & e_{j'-1\;j'} & e_{j'-1\;j'+1} \end{bmatrix}$$

$$= \cdots = (-1)^{j'+1}(n+2)^{\Delta}2^{j'}\det\begin{bmatrix} 1 & 1 & 1 \\ e_{2\,j'-1} & e_{2\,j'} & e_{2\,j'+1} \\ e_{4\,j'-1} & e_{4\,j'} & e_{4\,j'+1} \end{bmatrix}$$

$$= (-1)^{j'+1}(n+2)^{\Delta}2^{j'}\det\begin{bmatrix} 1 & 1 \\ e_{3\,j'} & e_{3\,j'+1} \end{bmatrix} = (-1)^{j'+1}(n+2)^{\Delta}2^{j'}\det\begin{bmatrix} 1 & 0 \\ e_{3\,j'} & e_{2\,j'+1} \end{bmatrix}$$

$$= (-1)^{j'+1}(n+2)^{\Delta}2^{j'}e_{2\,j'+1}$$

where $\Delta = 1 + 2 + \cdots + (j'-2) + j' = (n^2 - 4n + 11)/8$.

Inserting of

$$e_{2\,j'+1} = (1/2)(J_n)_{2\,j'+1}/(n+2) = R_n^{(-j')}(2)/(n+2)$$

into the above equation yields

$$u(j',1) = (-1)^{j'+1}2^{(n-1)/2}(n+2)^{(n^2-4n+3)/8}R_n^{(-j')}(2) \tag{4.3}$$

An illustration is given for $n$=9 in the following.

$$u(4,1) = 2^4 11^7 \det\begin{bmatrix} 1 & 1 & 1 & 1 \\ 9 & 12 & 14 & 15 \\ 105 & 146 & 175 & 190 \\ 15873 & 22187 & 26703 & 29056 \end{bmatrix} = 2^4 11^7 \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 9 & 1 & 1 & 1 \\ 105 & 12 & 14 & 15 \\ 15873 & 1798 & 2163 & 2353 \end{bmatrix}$$

$$= 2^4 11^7 \det\begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 1 \\ 1798 & 175 & 190 \end{bmatrix} = 2^4 11^7 \det\begin{bmatrix} 1 & 0 \\ 175 & 15 \end{bmatrix} = 2^4 11^6 (11\times 15),$$

where $11\times 15 = R_9^{(-4)}$.

Conjecture A is a direct consequence of eqns. (4.2) and (4.3).

(b) <u>$n$ is even</u>

An analogous reasoning as in the case when $n$ is odd yields

$$u(j'+1,\ 1) = (n+2)^{(n^2-2n)/8}$$

$$u(j',1) = (n+2)^{(n^2-2n)/8} R_n^{(-j')}(2)$$

which indicates that Conjecture A is valid for even $n$.

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