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# PARITY OF THE DISTANCE NUMBERS AND WIENER NUMBERS OF BIPARTITE GRAPHS

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Abstract. Simple rules for the parity of the distance numbers and Wiener numbers of bipartite graphs are deduced and some of their consequences pointed out.

There exists nowadays a quite extensive literature on the mathematical properties and chemical applications of topological indices based on the distance matrix of the molecular graph; for details see[1-5] and the references cited therein. In the present note we establish certain properties of the distance numbers and Wiener numbers of bipartite graphs. These results, although elementary, seem not to be observed in the chemical literature[1-5].

We follow the notation and terminology of the book[4]. Thus, let G denote a molecular graph whose vertex set is $V$. The distance between two vertices u and v of G is the length of the shortest path connecting u and v. The distance between u and v is denoted by d(u,v).

Obviously, d(u,v) = d(v,u) and d(u,v) = 0 if and only if u = v.

The *distance number* of the vertex u (of the graph G) is

$$d(u) = d(u|G) = \sum_{v \in V} d(u,v) \quad . \tag{1}$$

The *Wiener number* of the graph G is the sum of all distances in G:

$$W = W(G) = \sum_{u,v \in V} d(u,v) \quad . \tag{2}$$

Of course,

$$W = \frac{1}{2} \sum_{u \in V} d(u) \quad . \tag{2'}$$

The graph G is said to be bipartite[4] if its vertex set can be partitioned into two parts, $V_a$ and $V_b$, such that two adjacent vertices belong neither both to $V_a$ nor both to $V_b$. The cardinality of the sets $V_a$ and $V_b$ is denoted by $\underline{a}$ and $\underline{b}$, respectively.

The above definition has the following straightforward consequence.

Lemma 1. If $u,v \in V_a$ or $u,v \in V_b$, then all paths connecting u and v have even length. If $u \in V_a$ and $v \in V_b$ or vice versa, all paths connecting u and v have odd length.

Proof. Consider the case when $u,v \in V_a$. Suppose that the vertices $v_o, v_1, v_2, \ldots, v_{k-1}, v_k$ form a path in the graph G and that $u \equiv v_o$, $v \equiv v_k$. This means that the vertices $v_{i-1}$ and $v_i$, $i = 1,2,\ldots,k$, are adjacent in G.

The length of this path is k.

From $v_o, v_k \in V_a$ it follows $v_1, v_3, \ldots, v_{k-1} \in V_b$. This implies that k-1 is odd. Therefore k, the length of the path between u and v, $u,v \in V_a$ must be even.

The proof for the remaining two cases: $u,v \in V_b$ and $u \in V_a, v \in V_b$ is fully analogous. □

Bearing in mind that the distance is just the length of the shortest path, we arrive at an immediate corollary of Lemma 1.

Lemma 2. If $u,v \in V_a$ or $u,v \in V_b$, then the distance between u and v is even. If $u \in V_a$ and $v \in V_b$ or vice versa, the distance between u and v is odd.

Suppose now that G is a (connected) bipartite graph on a+b vertices. Let u be a vertex from the set $V_a$. Then by (1) the distance number d(u) is the sum of a even and b odd terms. Clearly, d(u) will be even (odd) if and only if b is even (odd). Thus we arrive at our first theorem.

Theorem 1. If $u \in V_a$, then d(u) and b have equal parity. If $v \in V_b$, then d(v) and a have equal parity.

A similar result holds also for the Wiener number. In order to see it we rewrite eq. (2) in the form:

$$W = A + B + C$$

where

$$A = \sum_{u,v \in V_a} d(u,v) \quad (3)$$

$$B = \sum_{u,v \in V_b} d(u,v) \quad (4)$$

$$C = \sum_{\substack{u \in V_a \\ v \in V_b}} d(u,v) \quad (5)$$

The quantities A and B are necessarily even since by Lemma 2 all distances appearing on the r.h.s. of (3) and (4) are even. The quantity C is the sum of a·b terms, all of which are odd numbers. Therefore C is even (odd) if a·b is even (odd).

With these observations we can formulate our second theorem.

Theorem 2. W(G) is odd if and only if both a and b are odd. If a or/and b is even, then W(G) is also even.

A few direct consequences of Theorems 1 and 2 are worth mentioning.

Corollary 1.1. If H is a (connected) spanning subgraph of a bipartite graph G, then for all vertices u of G and H, $d(u|G)$ and $d(u|H)$ have equal parity. In particular, the distance number of a vertex of G and the distance number of the same vertex in a spanning tree of f G have equal parity.

Corollary 2.1. Using the notation of Corollary 1.1, W(G) and W(H) have equal parity. In particular, the Wiener numbers of all spanning trees of G have the same parity as W(G).

Corollary 1.2. Let u and v be two adjacent vertices of a bipartite graph G and let z be a third vertex. Then $d(z|G\text{-}u\text{-}v)$ and $d(z|G)$ have opposite parity.

Corollary 2.2. Let G be a bipartite graph on $\underline{a}+\underline{b}$ vertices. Let u and v be adjacent vertices of G. Then W(G-u-v) and W(G) have equal (opposite) parity if $\underline{a}$ and $\underline{b}$ have opposite (equal) parity.

Corollary 1.3. Let $u,v \in V_a$ or $u,v \in V_b$ and let z be a third vertex. Then $d(z|G-u-v)$ and $d(z|G)$ have equal parity.

Corollary 2.3. Using the notation of Corollary 1.3, W(G-u-v) and W(G) have equal parity.

Corollary 2.4. The Wiener number of a bipartite graph with odd number of vertices is even.

Corollary 2.5. If G possesses a perfect matching (i.e. the corresponding molecule possesses a Kekulé structure), then W(G) has the same parity as n/2, where n is the number of vertices of G.

Corollary 2.6. If G is a molecular graph of a catacondensed benzenoid hydrocarbon, then W(G) is odd.

We conclude the present note with a result on compound graphs. Let $G_1$ be a (connected, bipartite) graph and u and v its two vertices. Let $G_2$ be another (connected, bipartite) graph and z its vertex. Construct the graph $H_1$ from $G_1$ and $G_2$ by joining or identifying u and z. Construct another graph $H_2$ from $G_1$ and $G_2$ by joining or identifying v and z.

Theorem 3. If $G_2$ has even number of vertices, then $W(H_1)$ and $W(H_2)$ have equal parity. If $G_2$ has odd number of vertices then $W(H_1)$ and $W(H_2)$ have equal (opposite) parity if $d(u|G_1)$ and $d(v|G_1)$ have equal (opposite) parity.

Proof. A simple combinatorial reasoning gives[6,7]

$$W(H_1) = W(G_1) + W(G_2) + n_2 d(u|G_1) + n_1 d(z|G_2) + g n_1 n_2$$

$$W(H_2) = W(G_1) + W(G_2) + n_2 d(v|G_1) + n_1 d(z|G_2) + gn_1n_2$$

where $n_1$ and $n_2$ denote the number of vertices of $G_1$ and $G_2$, respectively. If $H_{1,2}$ is obtained by joining the vertices u,v with z, then $g = 1$; if the vertices u,v are identified with z, then $g = 0$. Therefore

$$W(H_1) - W(H_2) = n_2[d(u|G_1) - d(v|G_1)]$$

and Theorem 3 follows immediately. ¤

Corollary 3.1. If $u,v \in V_a(G_1)$ or $u,v \in V_b(G_1)$, then $W(H_1)$ and $W(H_2)$ have equal parity.

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