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ON THE NUMBER OF KEKULÉ STRUCTURES FOR RECTANGLE-SHAPED BENZENOIDS - PART III

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The number of Kekulé structures (K) for benzenoid classes of oblate rectangles, $R^j(m,n)$, with fixed values of n, are studied. A systematic method is developed for determining (a) K numbers of the auxiliary classes $B(n, 2m-2, t)$ expressed linearly by $K\{R^j(m-j, n)\} = R_n(m-j)$, and (b) the recurrence relation for $R_n(m-j)$. The example with n=5 is treated in detail for the first time. The recurrence relation reads:

$$R_5(m) = 42R_5(m-1) - 245R_5(m-2) + 343R_5(m-3); \qquad m > 3.$$

Some results for six additional classes related to $R^j(m,5)$ are summarized.

1. INTRODUCTION

The importance of rectangle-shaped benzenoids (or simply rectangles) was recognized from the beginning of the systematic enumerations of Kekulé structures.[1]

Rectangles with indentation inwards, viz. $R^i(m,n)$, are $(2m-1)$-tier strips referred to as prolate rectangles.[2] They would in modern terms be called essentially disconnected,[3,4] consisting of m linear chains (polyacenes) joined by fixed single bonds. Hence their number of Kekulé structures is given by $K = (n+1)^m$. The case of m=2 was solved by Gordon and Davison[1] and also considered later.[5] The general formula was first given by Yen,[6] and re-derived in different ways by others.[7,8]

Rectangles with indentation outwards, viz. $R^j(m,n)$, are also $(2m-1)$-tier strips; they are referred to as oblate rectangles.[2] The problems of enumerating Kekulé structures for these benzenoid systems are considerably more difficult than for the prolate rectangles. The problem was solved for the 3-tier strip (m=2) by the early investigators,[1,6] and also considered later.[5] Also the formula for the number of Kekulé structures (K) has been derived for $R^j(3,n)$, the 5-tier strip.[1,3,6,8] The cases of[2] m=4 and[9] m=5 were solved much later, and quite recently for[10] m=6 and[11] m=7.

So far the studies of oblate rectangles with fixed values of m have been reviewed. Gutman[12] attacked the problem of K number enumeration for oblate rectangles with fixed values of n. He solved this problem for n=1 and n=2 by introducing classes of auxiliary benzenoids and treating systems of coupled recurrence relations. This work stimulated the present authors, who independently solved the enumeration problem for n=3.[2,13] The studies have been extended to related systems derived from the n=2 and n=3 oblate rectangles.[14] Very recently the Kekulé structures of the n=4 oblate rectangles, $R^j(m,4)$, and related classes were enumerated by Su.[15]

In the present work we indicate a general method of K enumeration for oblate rectangles with fixed values of n and report a contribution to the case of n=5.

2. AUXILIARY BENZENOID CLASSES

Recall the definition of the auxiliary classes $B(n,\ 2m-2,\ t)$, which may be interpreted as $R^j(m,n)$ rectangles modified at one end; cf. Fig. 1. Notice that $l=n$ gives the rectangle itself; $B(n,\ 2m-2,\ n) = R^j(m,n)$.

Let us introduce the abbreviated notation for K numbers as

$$K\{B(n,\ 2m-2,\ t)\} = R_n^{(t)}(m) \tag{1}$$

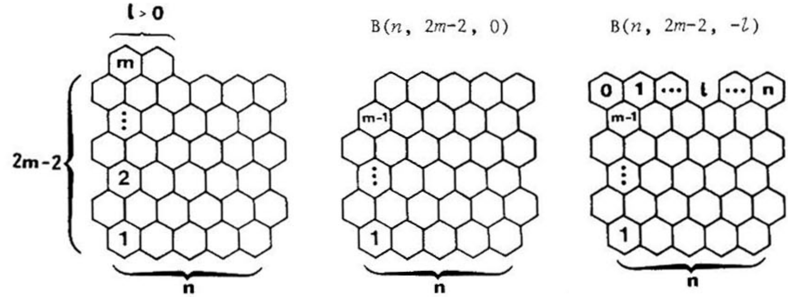
for all values of t (positive, zero and negative). Especially for the oblate rectangles themselves:

$$K\{R^j(m,n)\} = R_n^{(n)}(m) = R_n(m) \tag{2}$$

A basic formula reads[2,10]

$$R_n(m) = \sum_{i=0}^{n} R_n^{(-i)}(m) \tag{3}$$

B(n, $2m$−2, l) B(n, $2m$−2, 0) B(n, $2m$−2, −l)



$K = R_n^{(l)}(m)$ $K = R_n^{(0)}(m)$ $K = R_n^{(-l)}(m)$

Fig. 1. Definition of the auxiliary classes B(n, $2m$−2, t).

A more general form is obtained by the known methods[2,10] as

$$R_n(m+j) = \sum_{i=0}^{n} R_n^{(-i)}(j+1)\ R_n^{(-i)}(m) \tag{4}$$

Eqn. (3) is actually the special case of (4) for j=0 when we define for all n:

$$R_n^{(-l)}(1) = 1\ ; \qquad l \geq 0 \tag{5}$$

3. A SET OF LINEAR EQUATIONS

3.1. *General formulation*

Let eqn. (4) be applied for j = 0, 1, 2, ..., n. Then the obtained set of linear equations may be written

$$\begin{bmatrix} R_n(m) \\ R_n(m+1) \\ R_n(m+2) \\ \cdot \\ \cdot \\ \cdot \\ R_n(m+n) \end{bmatrix} = \mathsf{M} \begin{bmatrix} R_n^{(0)}(m) \\ R_n^{(-1)}(m) \\ R_n^{(-2)}(m) \\ \cdot \\ \cdot \\ \cdot \\ R_n^{(-n)}(m) \end{bmatrix} \tag{6}$$

where M is the square $(n+1)\times(n+1)$ matrix with the general element equal to

$$(\mathsf{M})_{rs} = R_n^{(1-s)}(r) \tag{7}$$

This set of equations makes it feasible to express all the $R_n^{(-l)}(m)$ quantities in terms of $R_n(m+j)$.

The number of equations is drastically reduced by virtue of the symmetry properties of the auxiliary benzenoids: $B(n,\ 2m-2,\ -l) = B(n,\ 2m-2,\ l-n)$. Hence

$$R_n^{(-l)} = R_n^{(l-n)}; \qquad l \geq 0 \tag{8}$$

The cases with even and odd n behave slightly differently. We will therefore exemplify both of these cases.

3.2. *The case of n=4*

Here we are faced with the three unknowns $R_4^{(0)}$, $R_4^{(-1)}$ and $R_4^{(-2)}$, while $R_4^{(-3)} = R_4^{(-1)}$ and $R_4^{(-4)} = R_4^{(0)}$. The set of linear equations (6) then reduces to three, viz.

$$\begin{bmatrix} R_4(m) \\ R_4(m+1) \\ R_4(m+2) \end{bmatrix} = \begin{bmatrix} 2R_4^{(0)}(1) & 2R_4^{(-1)}(1) & R_4^{(-2)}(1) \\ 2R_4^{(0)}(2) & 2R_4^{(-1)}(2) & R_4^{(-2)}(2) \\ 2R_4^{(0)}(3) & 2R_4^{(-1)}(3) & R_4^{(-2)}(3) \end{bmatrix} \begin{bmatrix} R_4^{(0)}(m) \\ R_4^{(-1)}(m) \\ R_4^{(-2)}(m) \end{bmatrix} \tag{9}$$

or with numerical values inserted:

$$\begin{bmatrix} R_4(m) \\ R_4(m+1) \\ R_4(m+2) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 30 & 48 & 27 \\ 630 & 1080 & 621 \end{bmatrix} \begin{bmatrix} R_4^{(0)}(m) \\ R_4^{(-1)}(m) \\ R_4^{(-2)}(m) \end{bmatrix} \qquad (10)$$

We will not pursue this example further since the problem with n=4 already is solved.

3.3. *The case of* n=5

The following treatment of the case with n=5 is an original contribution and shows simultaneously the virtue of the present methods. There are again three unknowns, viz. $R_5^{(0)}$, $R_5^{(-1)}$ and $R_5^{(-2)}$, while $R_5^{(-3)} = R_5^{(-2)}$, $R_5^{(-4)} = R_5^{(-1)}$, and $R_5^{(-5)} = R_5^{(0)}$. The set of linear equations (6) may be written:

$$\frac{1}{2}\begin{bmatrix} R_5(m) \\ R_5(m+1) \\ R_5(m+2) \end{bmatrix} = \begin{bmatrix} R_5^{(0)}(1) & R_5^{(-1)}(1) & R_5^{(-2)}(1) \\ R_5^{(0)}(2) & R_5^{(-1)}(2) & R_5^{(-2)}(2) \\ R_5^{(0)}(3) & R_5^{(-1)}(3) & R_5^{(-2)}(3) \end{bmatrix} \begin{bmatrix} R_5^{(0)}(m) \\ R_5^{(-1)}(m) \\ R_5^{(-2)}(m) \end{bmatrix} \qquad (11)$$

or

$$\frac{1}{2}\begin{bmatrix} R_5(m) \\ R_5(m+1) \\ R_5(m+2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 21 & 35 & 42 \\ 686 & 1225 & 1519 \end{bmatrix} \begin{bmatrix} R_5^{(0)}(m) \\ R_5^{(-1)}(m) \\ R_5^{(-2)}(m) \end{bmatrix} \qquad (12)$$

The equations yield:

$$R_5^{(0)}(m) = \frac{1}{98} R_5(m+2) - \frac{3}{7} R_5(m+1) + \frac{5}{2} R_5(m) \qquad (13)$$

$$R_5^{(-1)}(m) = -\frac{3}{98} R_5(m+2) + \frac{17}{14} R_5(m+1) - \frac{9}{2} R_5(m) \qquad (14)$$

$$R_5^{(-2)}(m) = \frac{1}{49} R_5(m+2) - \frac{11}{14} R_5(m+1) + \frac{5}{2} R_5(m) \qquad (15)$$

4. CONNECTION BETWEEN R_n AND $R_n^{(0)}$

A relation for $R_n^{(0)}$ similar to eqn. (3) reads:

$$R_n^{(0)}(m) = \sum_{i=0}^{n} (i+1)R_n^{(-i)}(m-1) \tag{16}$$

It is derived by the known method of fragmentation.[2] Figure 2 shows an exemplification for $R_4^{(0)}(3)$. We can add the terms for which $R_n^{(-i)}$ coincide by virtue of the symmetry property (8). Again we distinguish between the cases of even and odd n. The legend of Fig. 2 exemplifies a case with an even n, viz. n=4. The depicted example gives $3R_4(2) = R_4^{(0)}(3)$, and more generally

$$3R_4(m-1) = R_4^{(0)}(m) \tag{17}$$

The situation is similar for odd n. As an example one obtains for n=5:

$$R_5^{(0)}(m) = (1+6)R_5^{(0)}(m-1) + (2+5)R_5^{(-1)}(m-1) + (3+4)R_5^{(-2)}(m-1) \tag{18}$$

while

$$R_5(m-1) = 2R_5^{(0)}(m-1) + 2R_5^{(-1)}(m-1) + 2R_5^{(-2)}(m-1) \tag{19}$$

Consequently

$$7R_5(m-1) = 2R_5^{(0)}(m) \tag{20}$$

The pattern of eqns. (17) and (20) is quite general:

$$R_n(m-1) = \frac{2}{n+2} R_n^{(0)}(m) \tag{21}$$

5. NOMINAL VALUES OF $R^{(-l)}$

Nominal values of $R_n^{(-l)}(m)$ should fit the systems of equations, but are extrapolated to m values for which no benzenoid system can be visualized.

For m=1 we refer to eqn. (5). It can still be interpreted as pertaining to the "trivial cases of no hexagons" or a single acyclic chain (polyene) with one Kekulé structure.

We wish also the $R_n^{(-l)}$ values for m=0, without worrying about a pos-

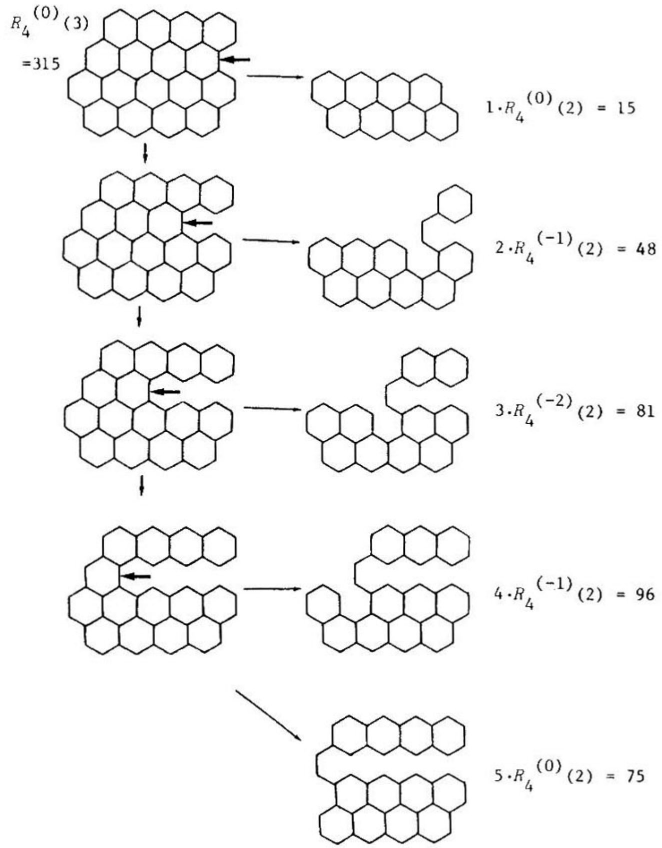


Fig. 2. Exemplification of eqn. (16) for n=4, m=3;

$$R_4^{(0)}(3) = 1\cdot R_4^{(0)}(2) + 2\cdot R_4^{(-1)}(2) + 3\cdot R_4^{(-2)}(2) + 4\cdot R_4^{(-3)}(2) + 5\cdot R_4^{(-4)}(2)$$

$$= (1+5)R_4^{(0)}(2) + (2+4)R_4^{(-1)}(2) + 3R_4^{(-2)}(2).$$

Notice also:

$R_4(2) = 2R_4^{(0)}(2) + 2R_4^{(-1)}(2) + R_4^{(-2)}(2)$ in accord with eqn. (3).

sible interpretation in terms of degenerate benzenoid systems. Eqn. (4) with j = -1 gives

$$R_n(m-1) = \sum_{i=0}^{n} R_n^{(-i)}(0)\; R_n^{(-i)}(m) \tag{22}$$

Since this is an identity, valid for all m values, it is clear from eqn. (21) that all terms must vanish except the first and the last one with $R_n^{(0)}(0) = R_n^{(-n)}(0)$. One obtains

$$R_n^{(0)}(0) = \frac{1}{n+2} \tag{23}$$

$$R_n^{(-l)}(0) = 0; \qquad n > l > 0 \tag{24}$$

6. RECURRENCE RELATION

6.1. *General*

It is of interest to deduce the recurrence relation for R_n, i.e. the linear dependence between the quantities $R_n(m+j)$. The set of linear equations (Section 3) is independent when the n+1 equations (6) are reduced by virtue of the symmetry properties. Their number then becomes $\left[\frac{n+2}{2}\right]$, i.e. 1 for n=1, 2 for n = 2 and 3, 3 for n = 4 and 5, 4 for n = 6 and 7, etc. The dependence between the R_n quantities is introduced by adding one equation more to the set. Consequently we can predict at once the number of terms in the recurrence relation. It is $\left[\frac{n+4}{2}\right]$, i.e. 2 for n=1, 3 for n = 2 and 3, 4 for n = 4 and 5, 5 for n = 6 and 7, etc.

An obvious way to derive the recurrence relation would be to add an equation by increasing j in $R_n(m+j)$ with one unit. The computation becomes substantially easier, however, when the matrix M is augmented by a row on top of it, i.e. assuming j = -1. Here we take advantage of the nominal values introduced in the preceding section.

6.2. *The case of n=4*

The case of n=4 has basically been solved before; see Paragraph 3.2. A part of the solution is the recurrence relation[2,15]

$$R_4(m+2) = 27R_4(m+1) - 108R_4(m) + 108R_4(m-1) \tag{25}$$

6.3. *The case of* n=5

Eqn. (12) when augmented in the way it was described in Paragraph 6.1 assumes the form

$$\frac{1}{2}\begin{bmatrix} R_5(m-1) \\ R_5(m) \\ R_5(m+1) \\ R_5(m+2) \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & 0 & 0 \\ 1 & 1 & 1 \\ 21 & 35 & 42 \\ 686 & 1225 & 1519 \end{bmatrix} \begin{bmatrix} R_5^{(0)}(m) \\ R_5^{(-1)}(m) \\ R_5^{(-2)}(m) \end{bmatrix} \tag{26}$$

The first row yields

$$R_5^{(0)}(m) = \frac{7}{2} R_5(m-1) \tag{27}$$

in consistence with eqns. (20) and (21). On equating (13) and (27) the desired recurrence relation is readily obtained as:

$$\boxed{R_5(m+2) = 42R_5(m+1) - 245R_5(m) + 343R_5(m-1)} \tag{28}$$

By virtue of the linear dependencies of the quantities R this relation applies to all of the quantities $R_5^{(t)}(m)$.

7. INCORPORATION OF THE QUANTITIES $R^{(l)}$

In general:

$$R_n^{(n-k)}(m) = R_n^{(n-k-1)}(m) + R_n^{(-k)}(m); \quad k = 0, 1, 2, \dots, n-1 \tag{29}$$

In supplement of eqn. (5) for m=1, and (23), (24) for m=0, we have:

$$R_n^{(l)}(1) = l+1 \; ; \qquad l \geq 0 \tag{30}$$

$$R_n^{(l)}(0) = \frac{1}{n+2}\ ; \qquad 0 \leq l < n \tag{31}$$

$$R_n(0) = \frac{2}{n+2} \qquad (l=n) \tag{32}$$

It is again expedient to employ the symmetry properties (8) in practical applications of (29). In the case of n=5 we have:

$$R_5(m) = R_5^{(4)}(m) + R_5^{(0)}(m) \tag{33}$$

$$R_5^{(4)}(m) = R_5^{(3)}(m) + R_5^{(-1)}(m) \tag{34}$$

$$R_5^{(3)}(m) = R_5^{(2)}(m) + R_5^{(-2)}(m) \tag{35}$$

$$R_5^{(2)}(m) = R_5^{(1)}(m) + R_5^{(-2)}(m) \tag{36}$$

$$R_5^{(1)}(m) = R_5^{(0)}(m) + R_5^{(-1)}(m) \tag{37}$$

8. FINAL RELATIONS FOR n=5

8.1. *Explicit formula for* $R_5(m)$

The recurrence relation (28) gives the solution of $R_5(m)$ in a closed form by standard mathematical methods,[14,15] which imply the solution of the cubic equation

$$\alpha^3 - 42\alpha^2 + 245\alpha - 343 = 0 \tag{38}$$

In this case the expressions are somewhat awkward since (38) has no integer solution. One has

$$\alpha_1 = \frac{7}{6}(a + b + 12) \tag{39}$$

$$\alpha_2 = \frac{7}{6}(a\omega + b\omega^2 + 12) \tag{40}$$

$$\alpha_3 = \frac{7}{6}(a\omega^2 + b\omega + 12) \tag{41}$$

where

$$a = \sqrt[3]{756 + 84i\sqrt{3}}\ , \qquad b = \sqrt[3]{756 - 84i\sqrt{3}} \tag{41}$$

and

$$\omega = \frac{1}{2}\ (-1 + i\sqrt{3}), \qquad \omega^2 = \frac{1}{2}\ (-1 - i\sqrt{3}) \tag{43}$$

Here $i = \sqrt{-1}$, and $\omega^3 = 1$. Next we employ the solution of the following set of linear equations,

$$p_1 + p_2 + p_3 = R_5(2) = 196 \tag{44}$$

$$\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = R_5(3) = 6860 \tag{45}$$

$$\alpha_1{}^2 p_1 + \alpha_2{}^2 p_2 + \alpha_3{}^2 p_3 = R_5(4) = 242158 \tag{46}$$

viz.

$$p_1 = \frac{1}{9}[588 - 21i\sqrt{3}a^2 + 21i\sqrt{3}b^2 - 191ia\sqrt{3} + 191ib\sqrt{3}] \tag{47}$$

$$p_2 = \frac{1}{18}[1176-21(3-i\sqrt{3})a^2-21(3+i\sqrt{3})b^2+191(3+i\sqrt{3})a+191(3-i\sqrt{3})b] \tag{48}$$

$$p_3 = \frac{1}{18}[1176+21(3+i\sqrt{3})a^2+21(3-i\sqrt{3})b^2-191(3-i\sqrt{3})a-191(3+i\sqrt{3})b] \tag{49}$$

Then the explicit formula for $R_5(m)$ reads

$$R_5(m) = p_1\alpha_1{}^{m-2} + p_2\alpha_2{}^{m-2} + p_3\alpha_3{}^{m-2} \tag{50}$$

8.2. *Relations for* $R_5{}^{(t)}(m)$

By means of the relation (28) the equations (13)-(15) were rendered into the form which is presented in CHART 1. Also the quantities $R_5{}^{(l)}$ for $l > 0$ were coupled to the system of linear equations through (33)-(37).

In view of the preceding paragraph all the quantities $R_5{}^{(t)}(m)$ are now in principle known as explicit formulas of m, although the expressions are rather awkward.

CHART 1 - $R_5^{(t)}$ expressed by linear combinations of $R_5(m-j)$

$$R_5^{(4)}(m) = R_5(m) - \frac{7}{2} R_5(m-1)$$

$$R_5^{(3)}(m) = R_5(m) - \frac{21}{2} R_5(m-1) + \frac{49}{2} R_5(m-2)$$

$$R_5^{(2)}(m) = \frac{1}{2} R_5(m)$$

$$R_5^{(1)}(m) = \frac{21}{2} R_5(m-1) - \frac{49}{2} R_5(m-2)$$

$$R_5^{(0)}(m) = \frac{7}{2} R_5(m-1)$$

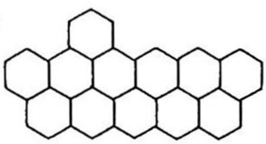
$$R_5^{(-1)}(m) = 7 R_5(m-1) - \frac{49}{2} R_5(m-2)$$

$$R_5^{(-2)}(m) = \frac{1}{2} R_5(m) - \frac{21}{2} R_5(m-1) + \frac{49}{2} R_5(m-2)$$

9. ADDITIONAL BENZENOID CLASSES

An infinite number of benzenoid classes is compatible with the recurrence relation (28); those pertaining to $R_5^{(t)}$ are just a few examples. Below we give six selected additional classes which follow (28) and are modifications of the oblate rectangle $R^j(m,5)$ at one end. The depicted figures have m=2.

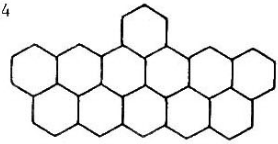
$A = 77$



$$A(m) = R_5^{(-1)}(m) + R_5^{(-2)}(m)$$

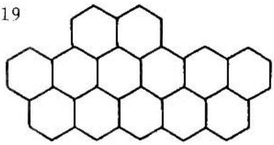
$$= \frac{1}{2} R_5(m) - \frac{7}{2} R_5(m-1)$$

$B = 84$



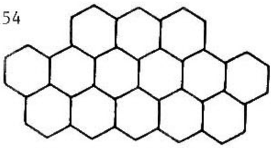
$$B(m) = 2R_5^{(-2)}(m)$$
$$= R_5(m) - 21R_5(m-1) + 49R_5(m-2)$$

$C = 119$



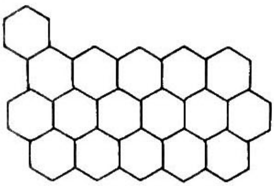
$$C(m) = B(m) + R_5^{(-1)}(m)$$
$$= R_5(m) - 14R_5(m-1) + \frac{49}{2}R_5(m-2)$$

$D = 154$



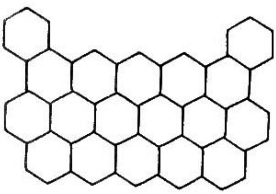
$$D(m) = C(m) + R_5^{(-1)}(m)$$
$$= R_5(m) - 7R_5(m-1)$$

$E = 371$



$$E(m) = R_5(m) + R_5^{(4)}(m)$$
$$= 2R_5(m) - \frac{7}{2}R_5(m-1)$$

$F = 700$



$$F(m) = E(m) + R_5^{(4)}(m) + D(m)$$
$$= 4R_5(m) - 14R_5(m-1)$$

10. UNSOLVED PROBLEMS

10.1. *Notation*

Write the recurrence relation

$$R_n(m+1) = \sum_{j=0}^{j'} c_j R_n(m-j) \tag{51}$$

where

$$j' = \left[\frac{n}{2}\right] \tag{52}$$

Here the form (51) was chosen so that the j values also indicate the different quantities (reduced by virtue of symmetry) $R_n^{(-l)}$ for $l \geq 0$. They are in other words $R_n^{(0)}$, $R_n^{(-1)}$, $R_n^{(-2)}$, ..., $R_n^{(-j')}$.

10.2. *Conjecture A*

$$c_0 = R_n^{(-j')}(2) \tag{53}$$

We know that $R_n^{(-j')}(1) = 1$ and $R_n^{(-j')}(0) = 0$; cf. eqns. (5) and (24), respectively.

10.3. *Conjecture B*

$$c_j > 0 \; ; \qquad j = 0, 2, 4, \ldots. \tag{54}$$

$$c_j < 0 \; ; \qquad j = 1, 3, 5, \ldots. \tag{55}$$

10.4. *Conjecture C*

$$c_{j'} = -c_{j'-1} \; ; \qquad n = 2, 4, 6, \ldots. \tag{56}$$

11. NUMERICAL VALUES

The following tables show numerical values for $R_n^{(l)}(m) = K\{B(n,\ 2m-2,\ l)\}$ with $n = 1, 2, 3, 4, 5, 6$. Apart from the information in themselves these values are supposed to be useful in the further studies of Kekulé structures for oblate rectangles, which are in progress.

	$K\{B(1,\ 2m-2,\ l)\}$		$K\{B(2,\ 2m-2,\ l)\}$			
m	$l=1$	$l=\{^{0}_{-1}$	$l=2$	$l=1$	$l=\{^{0}_{-2}$	$l=-1$
0	2/3	1/3	1/2	1/4	1/4	0
1	2	1	3	2	1	1
2	6	3	20	14	6	8
3	18	9	136	96	40	56
4	54	27	928	656	272	384
5	162	81	6336	4480	1856	2624
6	486	243	43264	30592	12672	17920
7	1458	729	295424	208896	86528	122368
8	4374	2187	2017280	1426432	590848	835584
9	13122	6561	13774848	9740288	4034560	5705728
10	39366	19683	94060544	66510848	27549696	38961152
11	118098	59049	642285568	454164480	188121088	266043392
12	354294	177147	4385800192	3101229056	1284571136	1816657920
13	1062882	531441			8771600384	

	$K\{B(3,\ 2m-2,\ l)\}$				
m	$l=3$	$l=2$	$l=1$	$l=\{^{0}_{-3}$	$l=\{^{-1}_{-2}$
0	2/5	1/5	1/5	1/5	0
1	4	3	2	1	1
2	50	40	25	10	15
3	650	525	325	125	200
4	8500	6875	4250	1625	2625
5	111250	90000	55625	21250	34375
6	1456250	1178125	728125	278125	450000
7	19062500	15421875	9531250	3640625	5890625
8	249531250	201875000	124765625	47656250	77109375
9	3266406250	2642578125	1633203125	623828125	1009375000
10				8166015625	

	$K\{B(4,\ 2m-2,\ l)\}$						
m	$l=4$	$l=3$	$l=2$	$l=1$	$l=\{^{0}_{-4}$	$l=\{^{-1}_{-3}$	$l=-2$
0	1/3	1/6	1/6	1/6	1/6	0	0
1	5	4	3	2	1	1	1
2	105	90	66	39	15	24	27
3	2331	2016	1476	855	315	540	621
4	52137	45144	33048	19089	6993	12096	13959
5	1167291	1010880	740016	427275	156411	270864	312741
6	26137809	22635936	16570656	9567153	3501873	6065280	7003503
7	585284211	506870784	371055168	214229043	78413427	135815616	156826125
8				4797077337	1755852633	3041224704	3511703079

$K\{B(5,\ 2m-2,\ l)\}$

m	$l=5$	$l=4$	$l=3$	$l=2$	$l=1$	$l=\{^{0}_{-5}$	$l=\{^{-1}_{-4}$	$l=\{^{-2}_{-3}$
0	2/7	1/7	1/7	1/7	1/7	1/7	0	0
1	6	5	4	3	2	1	1	1
2	196	175	140	98	56	21	35	42
3	6860	6174	4949	3430	1911	686	1225	1519
4	242158	218148	174930	121079	67228	24010	43218	53851
5	8557164	7709611	6182575	4278582	2374589	847553	1527036	1903993
6	302425158	272475084	218507807	151212579	83917351	29950074	53967277	67295228
7		9629923597	7722598009	5344205825	2965813641	1058488053	1907325588	2378392184

$K\{B(6,\ 2m-2,l)\} = R_6^{(l)}(m)$

m	$l=6$	$l=5$	$l=4$	$l=3$	$l=2$	$l=1$	$l=\{^{0}_{-6}$	$l=\{^{-1}_{-5}$	$l=\{^{-2}_{-4}$	$l=-3$
0	1/4	1/8	1/8	1/8	1/8	1/8	1/8	0	0	0
1	7	6	5	4	3	2	1	1	1	1
2	336	308	260	200	136	76	28	48	60	64
3	17472	16128	13664	10464	7008	3808	1344	2464	3200	3456
4	916992	847104	718080	549632	367360	198912	69888	129024	168448	182272
5	48179200	44511232	37734400	28880896	19298304	10444800	3667968	6776832	8853504	9582592
6	2531688448	2338971648	1982881792	1517633536	1014054912	548806656	192716800	356089856	465248256	503578624

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