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On the classification and generation of two- and higher-dimensional regular patterns.

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§ 0 Introduction

In this paper an n-dimensional regular pattern (M,D,G) consists of an n-dimensional manifold M , which is decomposed by a "polytopial" decomposition D into a disjoint union of (0-dimensional) vertices, (1-dimensional) edges, (2-dimensional) polygons, (3-dimensional) polyhedra and 4-, 5-..., k-,... up to n-dimensional "polytops" (i.e. subsets, which together with their closure, consisting of lower dimensional polytops, are homeomorphic to k-dimensional convex polytops in $\mathbb{R}^n$) together with a group G of transformations of M which respects the decomposition D and acts sharply transitive on the set of vertices of the polytopial decomposition D . We assume each polytop to be uniquely determined by the set of vertices, contained in its closure.

Thus the only 1-dimensional regular patterns are (up to isomorphism) the real line $\mathbb{R}$ with its "natural" decomposition with $\mathbb{Z} \subseteq \mathbb{R}$ as the set of vertices (and the open intervalls $(k,k+1)$ as edges, $k \in \mathbb{Z}$) and either the additive group $\mathbb{Z}$ as group G of "translational" transformations or the infinite dihedral group, generated by the reflections at the points $k+\frac{1}{2}$, $k \in \mathbb{Z}$ as group G of transformations, as well as the quotientspaces of this space with respect to the various fixed point free (i.e. translational) subgroups of G , i.e. the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with its decomposition, derived from $\mathbb{R}$ via the various maps $\mathbb{R} \to S^1 : x \mapsto e^{\frac{2\pi i}{n}x}$, and the corresponding transformations, consisting of either the multiplicative and cyclic group of the n-th roots of unity or - only in case n is even - of the dihedral group $D_{n/2}$, generated by the reflections at $e^{\frac{2\pi i}{n}(k+\frac{1}{2})}$, $k \in \mathbb{Z}$.

Since the cartesian product of an n-dimensional and an m-dimensional regular pattern is easily seen to be an (n+m)-dimensional pattern, we get easily higher dimensional regular patterns, e.g. $\mathbb{R}^n$ together with its "cubic" decomposition and $\mathbb{Z}^n$ as group of transformations. Further examples are the dual complexes of the semisimplical complexes, usually associated to Coxeter-groups (cf [1]).

The following remark, concerning n-dimensional regular patterns, may be useful: For any vertex $x \in M$ the set $E_x = \{g \in G \mid x$ and gx are "edge-connected", i.e. are the endpoints of a common edge$\}$ generates G , if M is connected. Geometrically, E_x may be identified with the set $\mathcal{E}_x$ of edges e, having x as one of its two boundary points, i.e. with $x \in \partial e$. Moreover E_x is closed with respect to taking inverses, i.e. $g \in E_x \Longleftrightarrow g^{-1} \in E_x$, since, if the edge e connects x and gx , then $g^{-1}e$ connects $g^{-1}x$ and x . Finally, for any two-dimensional polygon p in D, containing x among its vertices, we may order its vertices according to some orientation: $x = x_o, x_1, x_2, \ldots, x_{k-1}$, such that x_o and x_1, x_2 and $x_2, \ldots, x_{k-1}$ and x_o are edge-connected, in which case we can write $x_1 = g_1 x_o$ with $g_1 \in E_x$ and thus $g_1^{-1} x_2 = g_2 x_o$ with $g_2 \in E_x$ and thus $g_2^{-1} g_1^{-1} x_3 = g_3 x_o$ with $g_3 \in E_x$ and so on, i.e. we have $x_i = g_1 g_2 \ldots g_i\, x_o$ with $g_1, g_2, \ldots, g_i \in E_x$ and $g_1 g_2 \ldots g_k = 1$, since $x_k = x_o$.

Now consider the free group, generated by the symbols $\{X_g | g \in E_x\}$ and its normal subgroup generated by the relations $X_g \cdot X_h = 1$ if $g,h \in E_x$, $g = h^{-1}$ and $X_{g_1} \cdot X_{g_2} \cdot \ldots \cdot X_{g_k} = 1$ for each polygon containing x among its vertices. Let G^* denote the factor group. Thus one has a well-defined map $G^* \to G : X_g \mapsto g$, whose kernel is well known to be canonically isomorphic to the fundamental group $\Pi_1(M)$ (cf.[2]). In particular $G^* \cong G$, if M is simply connected.

Since a non simply-connected manifold M can always be derived from its simply connected universal covering $\hat{M}$ (as S^1 from $\mathbb{R}$), which inherits any regular pattern structure from M, we may henceforth assume M to be connected and simply connected and thus G to be the group generated by E_x modulo the obvious relations, coming from the pattern structure.

It is the purpose of this paper (a) to describe a method by which one can classify as well as systematically and explicitely generate all two-dimensional simply connected regular patterns, and (b) to discuss some open problems related to its higherdimensional generalization. The solution of these open problems - which would lead to a far reaching generalization of the concept of Coxeter groups - can almost certainly be facilitated by the explicit study of those 3-dimensional regular patterns, which are associated with the classical crystallographic groups.

§ 1 The local type of a two-dimensional regular pattern

Let (M,D,G) be a two-dimensional regular simply connected pattern. Then the number of edges ending at a given vertex x is dependent of x; more precisely, if $gx = y$ ($g \in G$; x,y vertices in M), then g maps the set $\mathcal{E}_x$ of edges ending at x bijectively onto the set $\mathcal{E}_y$ of edges, ending at y. The first local invariant, associated to (M,D,G) is, of course, the number n of edges in any $\mathcal{E}_y$. Obviously $n \geq 3$.

Secondly, we may identify $\mathcal{E}_x$ with the corners of a regular polygon, connecting two "corners" (edges) by a line if and only if the edges bound a common face, i.e. can be connected by a line in M, not crossing any other edge or vertex.

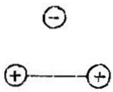
Thirdly, using the above identification of $\mathcal{E}_x$ and E_x , we can define an involution $\sigma_x : \mathcal{E}_x \to \mathcal{E}_x$ of $\mathcal{E}_x$ into itself, which corresponds to the involution $E_x \to E_x : g \mapsto g^{-1}$. In other words, if $e \in \mathcal{E}_x$ joins x and gx , then $\sigma_x(e) = g^{-1}e$.

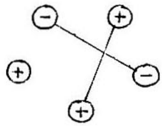
We may indicate the action of σ_x by drawing lines, connecting the corners e and $\sigma_x(e)$ in the regular polygon representing $\mathcal{E}_x$.

Finally we can define an augmentation $\varepsilon_x : E_x \to \{\pm 1\}$ by putting $\varepsilon_x(e) = +1$, if the corresponding element $g \in E_x$ with $\partial e = \{x,gx\}$ preserves any given orientation of M , and $\varepsilon(e) = -1$, if it reverses any given orientation of M . Obviously $\varepsilon_x(e) = \varepsilon_x(\sigma_x(e))$.

We may indicate the augmentation by drawing the corners of our representing polygon as small circles, filled with a "+", if $\varepsilon(e) = +1$ for the corresponding edge e , and filled with a "-" otherwise.

Thus the local structure of (M,D,G) at a vertex x can be represented by symbols of the following type:



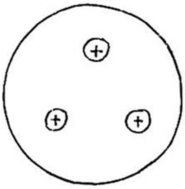


Obviously, if $y = hx (h \in G)$, then the map $E_x \to E_y : e \to he$ preserves not only the polygon-structure (though it may change its orientation) , but also the involution and augmentation, i.e. we have $\sigma_y(he) = h\,\sigma_x(e)$ and $\varepsilon_y(he) = \varepsilon_x(e)$, since $\partial e = \{x,gx\}$ if and only if $\partial he = \{hx,hgx\} = \{y,hgh^{-1}y\}$.

So we may define the local type of (M,D,G) to be represented by such symbols (i.e. regular polygons with an involution σ and a σ-invariant augmentation ε defined on its corners) or - rather - by the isomorphism classes of such symbols.

The first fundamental result, whose proof will become obvious in the following pages, is, that any such symbol, i.e. any polygon with an arbitrary involution σ and an arbitrary σ-invariant augmentation ε represents indeed the local type of a regular two-dimensional pattern (M,D,G) and moreover, one can construct any regular two-dimensional simply-connected pattern (M,D,G) from its local type in a rather straight-forward manner.

So let us list all local types for $n = 3$:



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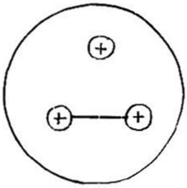

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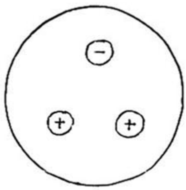

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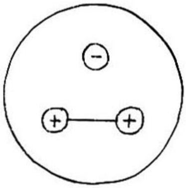

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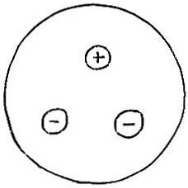

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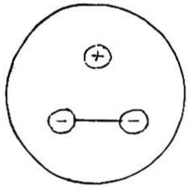

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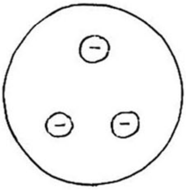

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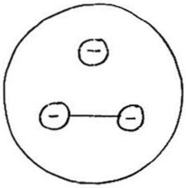

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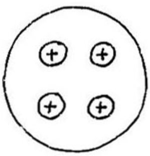
and for $n = 4$:



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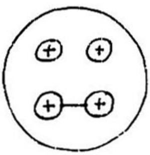

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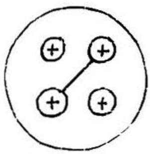

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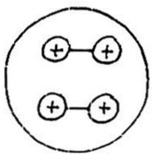

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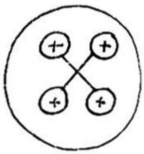

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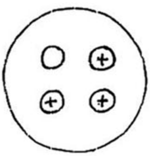

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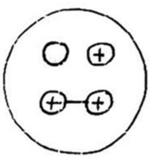

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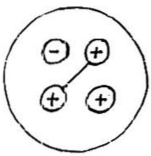

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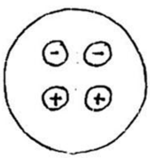

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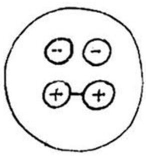

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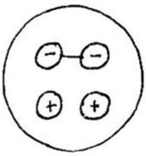
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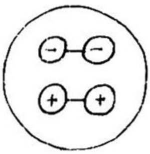
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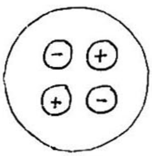
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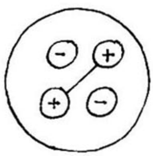
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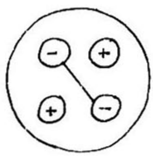
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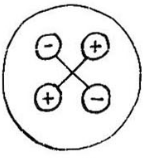
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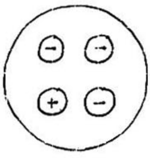
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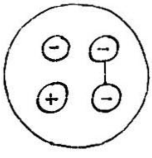
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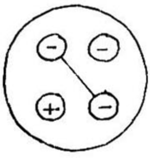
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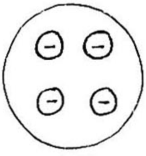
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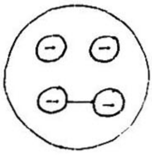
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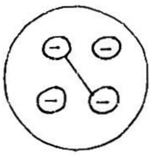
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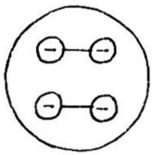
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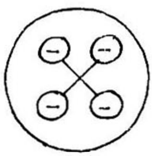
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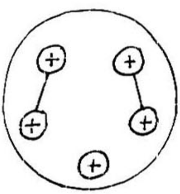
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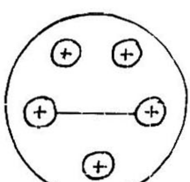
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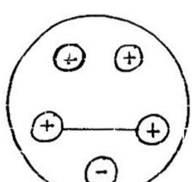
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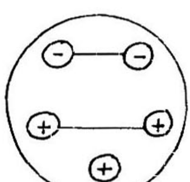
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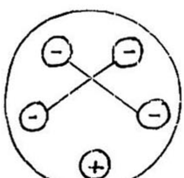
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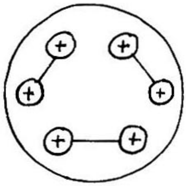

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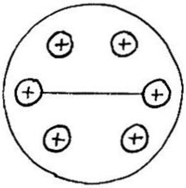

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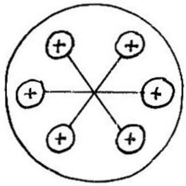

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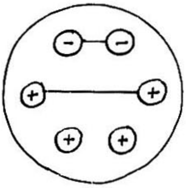

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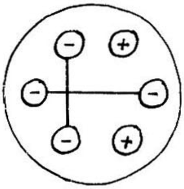

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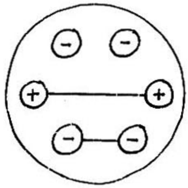

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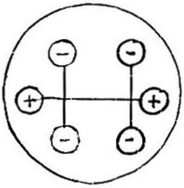


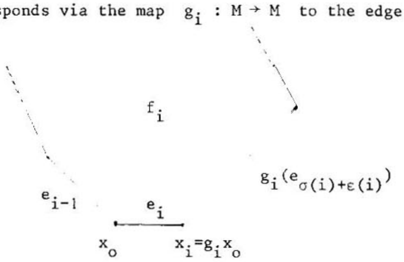
6.7

§ 2 How to construct two-dimensional regular patterns from their local type?

Let us start again with a two-dimensional regular pattern (M,D,G) and a fixed vertex $x_o \in M$. Let us choose an orientation of M and let us index the edges in $E_x = \{e_1, e_2, \ldots, e_n\}$ in such a way, that e_{i+1} follows e_i (modulo n) according to the sense of direction around x_o, defined by the given orientation. Let us transfer the involution σ and the augmentation ε, defined on E_x, in such a way on $\{1, \ldots n\}$, that $\sigma(e_i) = e_{\sigma(i)}$ and $\varepsilon(i) = \varepsilon(e_i)$. Let us denote the endpoints of e_i by x_o and x_i and the element $g \in E_x$ with $\partial e_i = \{x_o, gx_o\}$ by g_i.

Now let us look at the faces of M, relative to D. If f is such a face, having x_o among its vertices, there exists precisely one i such that e_{i-1} <u>and</u> e_i are among the edges of f. Vice versa, for any $i = 1,2,\ldots,n$ there exists precisely one face f_i having e_{i-1} and e_i among its edges. Let us follow the boundary of f_i along the sequence of edges $e_{i-1}, e_i, \ldots$ corresponding to the given orientation of M. Depending on wether $\varepsilon(i) = +1$ or $\varepsilon(i) = -1$, i.e. depending on wether g_i changes the orientation of M or not, the edge following

e_i along the boundary line of f_i corresponds via the map $g_i : M \to M$ to the edge following or preceding the edge $e_{\sigma(i)}$ connecting $g_i^{-1} x_o$ and $g_i^{-1} x_i = x_o$, i.e. this edge is precisely the edge $g_i(e_{\sigma(i)+\varepsilon(i)})$, which connects $g_i x_o = x_i$ with $g_i(x_{\sigma(i)+\varepsilon(i)}) = g_i\, g_{\sigma(i)+\varepsilon(i)}\, x_o$.



In general, if the edge $g\, e_i$ connecting $g\, x_o$ and $g\, x_i = g\, g_i\, x_o$ occurs in the boundary of the face f, then the edge, following $g\, e_i$ in the boundary of f with respect to the given orientation of M, is precisely the edge $g\, g_i(e_{\sigma(i)+\varepsilon(i)\eta})$ with $\eta = \eta(g) = +1$, if g preserves the orientation of M, and $\eta(g) = -1$ otherwise. To formalize this fact, we consider the set $\{1,\dots,n\} \times \{\pm 1\}$ and the map

$$\{1,\dots,n\} \times \{\pm 1\} \to \{1,\dots,n\} \times \{\pm 1\} : (i,\eta) \mapsto (i,\eta)^*$$

defined by $(i,\eta)^* = (\sigma(i) + \varepsilon(i)\ ,\ \varepsilon(i) \cdot \eta)$.

Obviously (with $i,j,k \in \{1,\dots,n\}$ and $\eta,\zeta,\xi \in \{\pm 1\}$) :

$$(i,\eta)^* = (j,\zeta) \Leftrightarrow j = \sigma(i)+\varepsilon(i)\eta \text{ and } \varepsilon(i)\eta = \zeta \Leftrightarrow \sigma(i) = j-\zeta$$

$$= \sigma(\sigma(j)) - \varepsilon(\sigma(j)) \cdot \varepsilon(j) \cdot \zeta \text{ and } -\varepsilon(\sigma(j)) \cdot \varepsilon(j) \cdot \zeta = -\zeta = -\varepsilon(i)\eta \Leftrightarrow$$

$$(\sigma(j),-\varepsilon(j)\zeta)^* = (i,-\varepsilon(i)\eta) .$$

Thus, in particular, $(i,\eta)^* = (k,\xi)^* = (j,\zeta) \Rightarrow \sigma(i) = j-\zeta = \sigma(k)$ and $\varepsilon(i) \cdot \eta = \varepsilon(k) \cdot \xi \Rightarrow i = k$ and $\eta = \xi$, i.e. $(i,\eta) = (k,\xi)$. So the map

$$^* : \{1,\dots,n\} \times \{\pm 1\} \to \{1,\dots,n\} \times \{\pm 1\} : (i,\eta) \mapsto (i,\eta)^*$$

is injective and therefore bijective and we may therefore split $\{1,\dots,n\} \times \{\pm 1\}$ into equivalence classes or orbits with respect to *, putting two elements (i,η) and (j,ξ) into one orbit or equivalence class, if the sequence (i,η), $(i,\eta)^*$, $(i,\eta)^{**}$,... contains (j,ξ). For any *-orbit (i_1,η_1), $(i_2,\eta_2) = (i_1,\eta_1)^*,\dots,(i_1,\eta_1) = (i_{1-1},\eta_{1-1})^*$ with $(i_1,\eta_1)^* = (i_1,\eta_1)$ we have the "dual" *-orbit $(\sigma(i_1),-\varepsilon(i_1)\cdot\eta_1),(\sigma_{i_1},-\varepsilon(i_1)\cdot\eta_1) = (\sigma(i_1),-\varepsilon(i_1)\cdot\eta_1)^*,\dots,(\sigma(i_2),-\varepsilon(i_2)\cdot\eta_2)$ with $(\sigma(i_2),-\varepsilon(i_2)\cdot\eta_2)^* = (\sigma(i_1),-\varepsilon(i_1)\cdot\eta_1)$.

Now let's come back to the faces of D. Our above analysis can now be phrased in the following way: if f is a face, if $g\, e_i$, connecting $g\, x_o$ and $g\, x_i = g\, g_i\, x_o$ is an edge, occuring in the boundary ∂f of f such that $g\, x_o$ and $g\, x_i$ are vertices of ∂f, following each other with respect to the orientation of ∂f, induced from the given orientation of M, and if $(i,\eta(g))^* = (j,\zeta)$, then the edge, following $g\,e_i$ in ∂f is precisely the edge $g\,g_i\,e_j$ connecting $g\,g_i\,x_o = g\,x_i$ and $g\,g_i\,x_j = g\,g_i\,g_j\,x_o$, and we have $\eta(g\,g_i) = \eta(g)\cdot\varepsilon(i) = \zeta$.

Thus, if we consider $f = f_i$ and start with the edge e_i, connecting x_o and x_i, we may define by induction $(i_1,\eta_1) = (i,+1)$ and $(i_{\nu+1},\eta_{\nu+1}) = (i_\nu,\eta_\nu)^*$ $(\nu = 2,3,\ldots)$ and get $x_o, x_{i_1} = g_{i_1}\,x_o,\ g_{i_1}\,g_{i_2}\,x_o,\ g_{i_1}\,g_{i_2}\,x_o,\ldots,$ $g_{i_1}\,g_{i_2}\cdots g_{i_\nu}\,x_o,\ldots$ with $\eta(g_{i_1}\,g_{i_2}\cdots g_{i_{\nu-1}}) = \eta_\nu$ as the sequence of vertices occuring **in** the boundary ∂f of f, one after another. Of course, if f_i has $N = n_i \geq 3$ vertices and edges, then $g_{i_1}\,g_{i_2}\cdots g_{i_N}\,x_o = x_o, (i_N,\eta_N)^* = (i,1) = (i_1,\eta_1)$ and thus $g_{i_1}\,g_{i_2}\cdots g_{i_N} = 1$, but $g_{i_1}\cdot\ldots\cdot g_{i_\nu} \neq 1$ for $\nu < N = n_i$. So, if l_i is the length of the orbit $(i,1),(i,1)^*,(i,1)^{**},\ldots,$ then $N = n_i$ must be an integral multiple of l_i, say $l_i\cdot k_i = n_i$. We define k_i to be the orbit-parameter of the orbit, containing $(i,1)$. Obviously $n_i = n_j$ and $l_i = l_j$, so $k_i = k_j$, if $(j,1)$ occurs either in the orbit, containing $(i,1)$, or in the "dual" orbit, containing $(\sigma(i),-\varepsilon(i))$, since in both cases there exists $g \in G$ with $g\cdot f_i = f_j$.

From the foregoing it is obvious, that the local type of (M,D,G) together with the orbit parameters determines (M,D,G) up to isomorphism and it is almost obvious, that for any local type (n,σ,ε) and any set of orbit-parameters k_i, satisfying $k_i l_i \geq 3$ and $k_i = k_j$ if $(j,1)$ occurs either in the orbit of $(i,1)$ or in the orbit of $(\sigma(i), -\varepsilon(i))$, one can construct a simply connected two-dimensional regular pattern (M,D,G) which is of the given local type and has the given orbit-parameters.

The group G is in this case given by the generators $g_1,\ldots,g_n$, corresponding to the corners of the polygon representing the local type, and the relations

$$g_i\,g_{\sigma(i)} = 1\,,\quad i = 1,\ldots,n$$

$$(g_i\cdot g_{i_2}\cdot\ldots\cdot g_{i_{l_i}})^{k_i} = 1\,,\quad i = 1,\ldots,n$$

if $(i,1) = (i_1,\eta_1),(i_2,\eta_2) = (i_1,\eta_1)^*,\ldots,(i_{l_1},\eta_{i_1}) = (i_{l-1},\eta_{i_{l-1}})^*$ is the *-orbit of $(i,1)$.

Rather then giving an abstract proof of this fact, we will discuss a few examples. But at first let us remark, that by using regular polygons for the faces f of D, all with edges of the same length, the two edges at each corner of the face f_i with its $n_i = k_i l_i$ vertices and edges will span an angle of arc length

$\pi(1-2/_{k_i l_i})$. Thus the sum $\sum_{i=1}^{n} \pi(1-2/_{k_i l_i})$ represents the sum of the arc length's of all the angles surrounding x_o . This leads to the following definition: of the metric type of $(M,D,G):(M,D,G)$ or $(n,\sigma,\varepsilon;\ k_1,\ldots,k_n)$ is defined to be spherical or elliptic, if $\sum_{i-1}^{n} \pi(1-2/_{k_i l_i}) < 2\pi$; euclidean or parabolic, if $\sum_{i=1}^{n} \pi(1-\frac{2}{k_i l_i}) = 2\pi$, and hyperbolic, if $\sum_{i=1}^{n} \pi(1-\frac{2}{k_i l_i}) > 2\pi$. Morecver we define $2-\sum_{1}^{n}(1-\frac{2}{k_i l_i})$ be the curvature of (M,D,G) or $(n,\sigma,\varepsilon;k_1,\ldots,k_n)$.

Because $k_i l_i \geq 3$, one gets easily, that elliptic patterns can occur only if $n \leq 5$ and parabolic patterns only if $n \leq 6$. We'll give a complete list of all elliptic and parabolic patterns later on.

One can prove, that for (M,D,G), corresponding to $(n,\sigma,\varepsilon;\ k_1,\ldots,k_n)$, the following statements are equivalent:

(i) M is homeomorphic to the 2-sphere S^2,

(ii) G is of finite order,

(iii) (M,D,G) is elliptic,

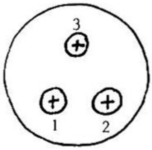
in which case the order $|G|$ of G equals $4(2-\sum_{1}^{n}(1-\frac{2}{k_i l_i}))^{-1}$ and (M,D,G) is isomorphic to (S^2,D',G') with $G' \subseteq O^3(R)$ consisting of finitely many (proper or improper) isometries of S^2, i.e. G' is a point group, and D' a metrically regular pattern on S^2 .

Similarly (M,D,G) is euclidean if and only if (M,D,G) is isomorphic to $(\mathbb{E}^2,D',G')$ with $\mathbb{E}^2$ the euclidean plane, G' one of the 17 two-dimensional crystallographic groups acting by isometrics on $\mathbb{E}^2$, and D' a metrically regular pattern on $\mathbb{E}^2$, whereas any hyperbolic pattern (M,D,G) is isomorphic to some pattern $(\mathbb{H}^2,D',G')$ with $\mathbb{H}^2$ the hyperbolic plane, G' a discrete group of isometries of $\mathbb{H}^2$ with a compact fundamental domain, and D' a metrically regular pattern on $\mathbb{H}^2$.

In any such case, a fundamental domain of G' is given by each face of the - metrically - dual decomposition of S^2, $\mathbb{E}^2$ or $\mathbb{H}^2$ and we can get, indeed, all discrete subgroups of $O(\mathbb{E}^2)$ and $O(\mathbb{H}^2)$ with a compact fundamental domain (or quotient space) in this way.

Let us now turn to our examples:

3.1



Orbits: $(1,+) \to (2,+) \to (3,+) \to (1,+)$

Orbit length: $l_1 = l_2 = l_3 = 3$

Orbit parameters: $k_1 = k_2 = k_3 =: k \geq 1$

Curvature: $2 - \sum_1^3 (1 - \frac{2}{k_i l_i}) = \frac{2}{k} - 1$

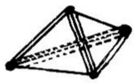
Relations: $g_1{}^2 = g_2{}^2 = g_3{}^2 = (g_1 g_2 g_3)^k = 1$

Metric type: elliptic: $k = 1$, $|G| = 4$

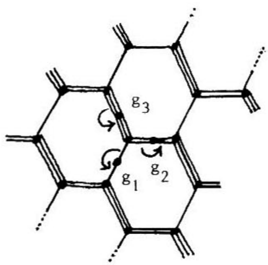
euclidean : $k = 2$

hyperbolic : $k \geq 3$

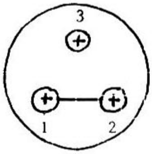
$(M,D)_{k\,=\,1} =$



$(M,D)_{k\,=\,2} =$



3.2



Orbits: $(1,+) \to (3,+) \to (1,+)$

$(2,+) \to (2,+)$

Orbit length: $l_1 = l_2 = 2$, $l_2 = 1$

Orbit parameters: $k_1 = k_3 \geq 2$, $k_2 \geq 3$

Curvature: $2 - \sum_1^3 (1 - \frac{2}{k_i l_i}) = \frac{2}{k_1} + \frac{2}{k_2} - 1$

Relations: $g_1 g_2 = g_3{}^2 = (g_1 g_3)^{k_1} = g_2{}^{k_2} = 1$

Metric type: elliptic: $k_1 = 2, k_3 \geq 3$, $|G| = 2k_2$

$k_1 = 3, 3 \geq k_2 \leq 5$, $|G| = \frac{12k_2}{6-k_2}$

$k_1 = 4, k_2 = 3$, $|G| = 24$

$k_1 = 5, k_2 = 3$, $|G| = 60$

euclidean: $k_1 = 3,\ k_2 = 6$

$k_1 = 6,\ k_2 = 3$

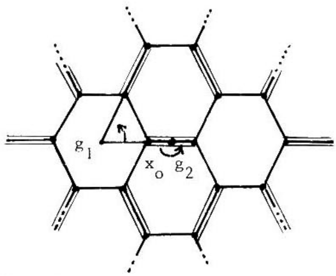
$k_1 = k_2 = 4$

hyperbolic: $k_1 = 3,\ k_2 > 6$

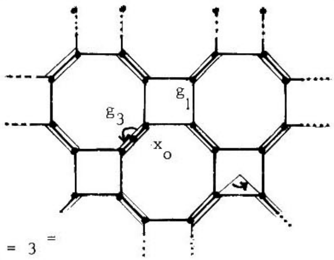
$k_1 > 3,\ k_2 > 3$

$(M,d)_{k_1 = 3,\ k_2 = 6}$

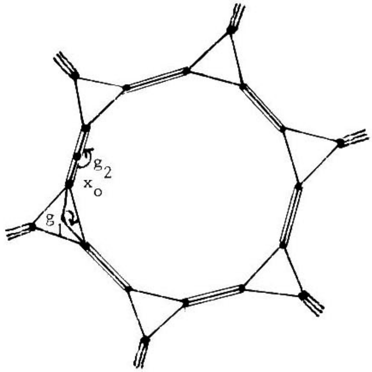
=



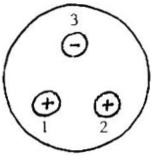
$(M,D)_{k_1 = k_2 = 4}$ =



$(M,D)_{k_1 = 6,\ k_2 = 3}$ =



3.3



Orbits: $(1,+) \to (2,+) \to (3,+) \to (2,-) \to (1,-) \to (3,-) \to (1,+)$

Orbit length: $l_1 = l_2 = l_3 = 6$

Orbit parameters: $k_1 = k_2 = k_3 \geq : k \geq 1$

Curvature: $2 - \sum_{1}^{3}(1 - \frac{2}{k_i l_i}) = \frac{1}{k} - 1$

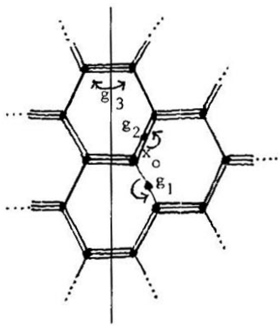
Relations: $g_1^{\ 2} = g_2^{\ 2} = g_3^{\ 2} = (g_1 g_2 g_3 g_2 g_1 g_3)^k = 1$

Metric type: elliptic: -

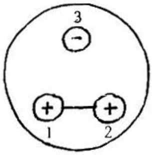
euclidean: $k = 1$

hyperbolic: $k \geq 2$

$(M,D)_{k=1} =$



3.4



Orbits: $(1,+) \to (3,+) \to (2,-) \to (3,-) \to (1,+)$

Orbit length: $l_1 = l_3 = 4,\ l_2 = 1$

Orbit parameters: $k_1 = k_3 \geq 1,\ k_2 \geq 3$

Curvature: $2 - \sum_{1}^{3} (1 - \frac{2}{l_i k_i}) = \frac{1}{k_1} + \frac{2}{k_2} - 1$

Relations: $g_1 g_2 = g_3^{\ 2} = (g_1 g_3 g_2 g_3)^{k_1} = g_2^{\ k_2} = 1$

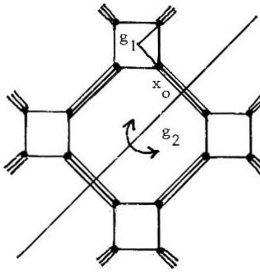
Metric type: elliptic: $k_1 = 1,\ k_2 \geq 3,\ |G| = 2k_2$

$k_1 = 2,\ k_2 = 3,\ |G| = 24$

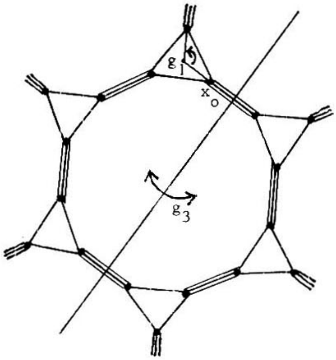
euclidean: $k_1 = 2,\ k_2 = 4$; $k_1 = 3, k_2 = 3$;

hyperbolic: $k_1 = 2,\ k_2 \geq 5; k_1 = 3, k_2 \geq 4; k_1 \geq 4$;

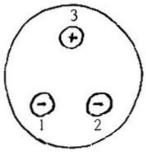
$(M,D)_{k_1 = 2,\ k_2 = 4}$



$(M,D)_{k_1 = k_2 = 3}$



3.5



Orbits: (1,+) (3,-) (2,-) (3,+) (1,+)

(2,+) (1,-) (2,+)

Orbit length: $l_1 = l_3 = 4$, $l_2 = 2$

Orbit parameters: $k_1 = k_3 \geq 1$, $k_2 \geq 2$

Curvature: $2 - \Sigma(1 - \frac{2}{l_i k_i}) = \frac{1}{k_1} + \frac{1}{k_2} - 1$

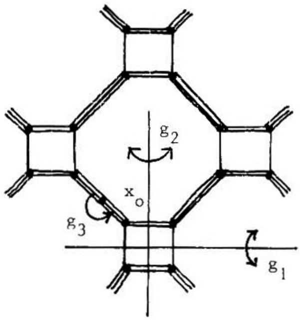
Relations: $g_1{}^2 = g_2{}^2 = g_3{}^2 = (g_1 g_3 g_2 g_3)^{k_1} = 1$

Metric type: elliptic: $k_1 = 1$, $k_2 \geq 2$, $|G| = 4k_2$

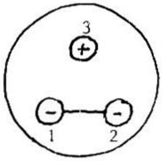
euclidean: $k_1 = k_2 = 2$

hyperbolic: $k_1 = 2$, $k_2 \geq 3$; $k_1 \geq 3$

$(M,D)_{k_1 = k_2 = 2}$



3.6



Orbits: (1,+) → (1,-) → (3,+) → (1,+)

(2,+) → (3,-) → (2,-) → (2,+)

The two orbits are dual to each other

Orbit length: $l_1 = l_2 = l_3 = 3$

Orbit parameters: $k_1 = k_2 = k_3 =: k \geq 1$

Curvature: $2 - \sum_{1}^{3} (1 - \frac{2}{l_i k_i}) = \frac{2}{k} - 1$

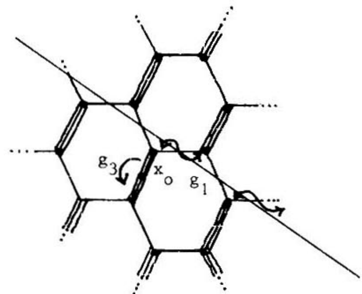
Relations: $g_1 g_2 = g_3^2 = (g_1^2 g_3)^k = 1$

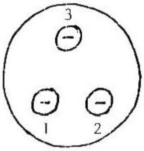
Metric type: elliptic : $k = 1$, $|G| = 4$

euclidean = $k = 2$

hyperbolic = $k > 2$

$(M,D)_{k = 2}$



3.7 

Orbits: $(1,+) \to (3,-) \to (1,+)$

$(2,+) \to (1,-) \to (2,+)$

$(3,+) \to (2,-) \to (3,+)$

Orbit length: $l_1 = l_2 = l_3 = 2$

Orbit parameters: $k_1,\ k_2,\ k_3 \geq 2$

Curvature: $2 - \sum_1^3 (1 - \frac{2}{l_i k_i}) = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} - 1$

Relations: $g_1^2 = g_2^2 = g_3^2 = (g_1 g_3)^{k_1} = (g_2 g_1)^{k_2} = (g_3 g_2)^{k_3} = 1$

By symmetry one may assume $k_1 \leq k_2 \leq k_3$

Metric type: elliptic : $k_1 = k_2 = 2,\ k_3 \geq 2,\ |G| = 4 \cdot k_3$

$k_1 = 2,\ k_2 = 3,\ 3 \leq k_3 \leq 5;\ |G| = \dfrac{24\ k_3}{6 - k_3}$

euclidean : $k_1 = 2,\ k_2 = k_3 = 4;$

$k_1 = 2,\ k_2 = 3,\ k_3 = 6;$

$k_1 = k_2 = k_3 = 3;$

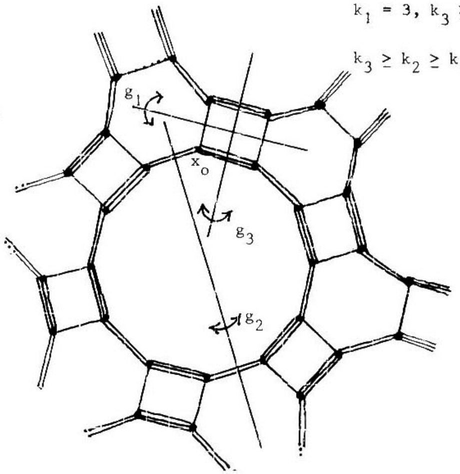
hyperbolic : $k_1 = 2,\ k_2 = 3,\ k_3 \geq 7$

$k_1 = k_2 = 3,\ k_3 \geq 4$

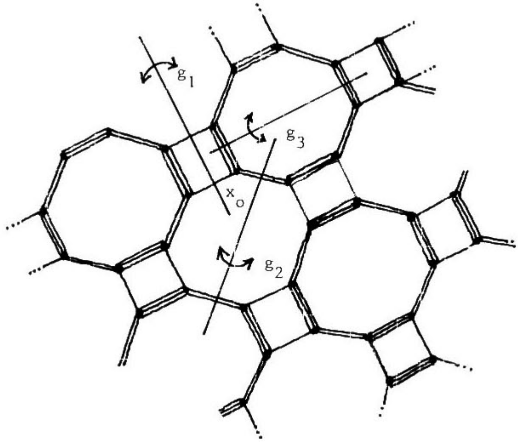
$k_1 = 3,\ k_3 \geq k_2 \geq 4$

$k_3 \geq k_2 \geq k_1 \geq 4$

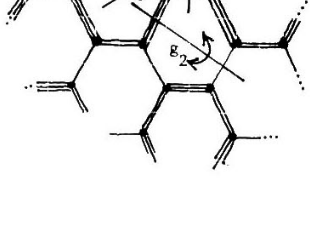
$(M,D)_{2,3,6} =$



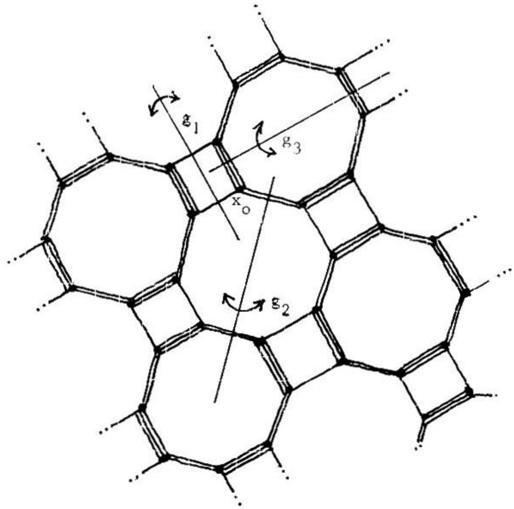
$^{(M,D)}k_1 = 2,\ k_2 = k_3 = 4$



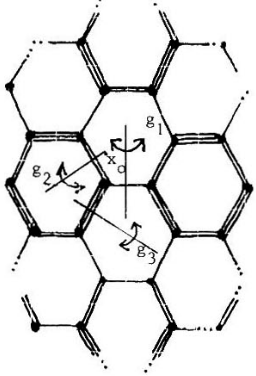
$^{(M,D)}k_1 = k_2 = k_3 = 3$



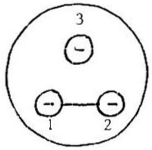
$(M,D)_{k_1 = 2,\ k_2 = k_3 = 4}$



$(M,D)_{k_1 = k_2 = k_3 = 3}$



3.8



Orbits: $(1,+) \to (1,-) \to (3,+) \to (2,-) \to (2,+) \to (3,-) \to (1,+)$

Orbit length: $l_1 = l_2 = l_3 = 6$

Orbit parameters: $k_1 = k_2 = k_3 =: k \geq 1$

Curvature: $2 - \Sigma(1 - \frac{2}{l_i k_i}) = \frac{1}{k} - 1$

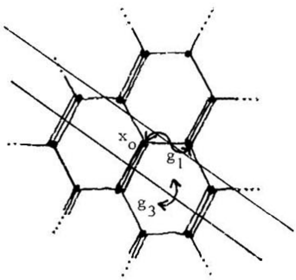
Relations: $g_1 g_2 = g_3{}^2 = (g_1{}^2 g_3 g_2{}^2 g_3)^k = 1$

Metric type: elliptic : -

euclidean : $k = 1$

hyperbolic : $k \geq 2$.

$(M,D)_{k=1} =$



Among the patterns associated to the local types with $n = 4$ there are precisely 2 infinite series and four individual patterns of elliptic metric type and 17 of euclidean metric type, for $n = 5$ we have one pattern of elliptic metric type, whose local structure is given by 5.1 , and four euclidean patterns, whose local structure is 5.2,5.3,5.4. or 5.5. For $n \geq 6$ there are no elliptic patterns and 7 euclidean patterns with local structure 6.1,6.2,... or 6.7 .

One observes easily that each 2-dimensional crystallographic group occurs among the groups, associated to euclidean patterns, some of them even several times (according to which group elements are chosen as generators). Similarly, each finite subgroup of the full isometry group of S^2 occurs among the groups, associated to elliptic patterns, except for the finite cyclic groups, generated by a proper rotation, and the group consisting of the inversion and the identity, only.

§ 3 Higher dimensional patterns.

In higher dimensions the local type consists of a polytopial decomposition D of the $(n-1)$-sphere S^{n-1}, an involution σ defined on the set $D_o(S^{n-1})$ of vertices of this decomposition, a σ-invariant augmentation ε , defined on $D_o(S^{n-1})$, and -

in addition - for each vertex $x \in D_o(S^{n-1})$ an orientation reversing or orientation preserving identification of the local structure around x, induced by D, and the local structure around $\sigma(x)$, depending on wether $\varepsilon(x) = +1$ or $\varepsilon(x) = -1$.

In case D is the standard decomposition of S^{n-1}, identifying S^{n-1} with the boundary $\partial \Delta^n$ of the n-dimensional standard simplex, and $\sigma(x) = x$, $\varepsilon(x) = -1$ for all x we may use the identity, to identify the local structure around x, orientation preserving, with the local structure around $x = \sigma(x)$ and thus get all the Coxeter groups.

In general, the canonical procedure, explained in the 2-dimensional case in the foregoing section, will lead to an orbit-structure on the set of pairs of connected edges in $D_1(S^{n-1})$ (i.e. edges $e_1, e_2 \in D_1(S^{n-1})$ with $|\partial e_1 \cap \partial e_2| = 1$), which in turn leads to a definition of orbit parameters, the associated relations of the group G and a prescription for constructing the space M together with its decomposition D.

The following questions remain to be answered:

- classify for each dimension n all local types, for which orbit parameters exist, which lead to a manifold M (for $n = 2$ there were no restrictions;I am not so sure, that this remains true in higher dimensions),
- classify for a given local type the various orbit parameters, which lead not only to a manifold M, but to a manifold of given metric and homöomorphic type (for $n = 2$ this was achieved by introducing the curvature),
- list all local types and orbit parameters, for which M is compact or isometric to the n-dimensional euclidean space E^n (in higher dimensions this will be possible only by the use of computers).

In any case, I believe that these problems can be solved in a satisfying way only by an explicit study of the well known three-dimensional and the (less well-known) four-dimensional crystallographic groups from this point of view. As a reward one could hope to get in a systematic way one (or several) geometrically interpretable sets of generators and relations for each of these groups.

References

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Remark

Only after finishing this note I have been made aware of the beautiful work of Branko Grünbaum and G.C. Shephard on planar patterns. An interesting report on this work and further references can be found in their memoir "Incidence Symbols and their application" in "Relations between combinatorics and other parts of mathematics", Proceedings of Symposia in Pure Mathematics, Vol. XXXIV, AMS, 1979.

Though much wider in its scope, it still does not seem to make the above note superfluous: whereas their incidence symbol does not necessarily generate a pattern, but, if it does so, classifies a much wider class of patterns, the symbol defined above always generates a pattern and lends itself more easily to higher dimensional generalizations. A detailed comparison of the two approaches will appear in a forthcoming paper.