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With kind permission of H.S.M. Coxeter we reprint the abstract and the introduction of his paper

Higher-dimensional analogues of the tetrahedrite crystal twin

The complete paper may be found in the proceedings of the Coxeter Symposium (May, 1979), published by the University of Toronto Press

ABSTRACT

Consider 6 square mirrors facing inwards on the faces of a cube, and a flat pencil of light rays reflected from these mirrors. If the plane of the rays is carefully chosen, the reflected path may close so as to form a finite polyhedron. The simplest instance is a regular tetrahedron whose 6 edges are diagonals (one each) of the 6 faces of the cube. In 4 papers on '*Extremum problems for the motions of a billiard ball*', I.J. Schoenberg has shown that this instance maximizes the minimum distance from the centre of the cube to a face of the polyhedron. He has also generalized this problem to an (n-1)-dimensional 'path' inside an n-dimensional cube. It now appears that his non-convex polytopes can be thoroughly investigated by using their connection with P. H. Schoute's 'Simplex nets', such as the (n-1)-dimensional lattice whose points have n integral coordinates with a constant sum (say zero).

1. Introduction

On a square billiard table with corners $(\pm 1, \pm 1)$, the path of a ball is easily seen to be periodic if and only if it begins with a line

$$Xx + Yy = N ,$$

where X and Y are integers and $|N| < |X| + |Y|$ [König and Szücs 1913, p. 82]. Ignoring a trivial case, we shall assume $XY \neq 0$. We lose no generality by taking these integers to be positive and relatively prime. After any number of bounces, the path is still of the form

$$\pm Xx \pm Yy = N \pm 2k ,$$

where k is an integer. Among these paths for various values of k, those that come closest to the origin are of the form

$$\pm Xx \pm Yy = N' ,$$

where $0 \leq N' \leq 1$. The distance of such a path from the origin is

$$N' / \sqrt{X^2 + Y^2} .$$

Schoenberg [1975, p.8] was looking for the values of X, Y, N which will maximize this distance. For this purpose we must have

$$N' = X = Y = 1 ,$$

so that N is an odd integer. Since $|N| < |X| + |Y| = 2$, this implies $N = \pm 1$. The paths

$$\pm x \pm y = 1$$

form a square whose vertices are the midpoints of the edges of the billiard table.

Analogously, in a kaleidoscope whose mirrors are the bounding hyperplanes

$$x_\nu = \pm 1 \qquad (\nu = 1,2,\ldots,n)$$

of an n-cube γ_n, consider an $(n-1)$-dimensional pencil of light rays in the hyperplane

$$\sum X_\nu x_\nu = N ,$$

where the X_ν are positive integers with no common divisor greater than 1, and $|N| < \sum X_\nu$. The mirror $x_\mu = 1$ will reflect this hyperplane so as to yield

$$X_\mu (2-x_\mu) + \sum_{\nu \neq \mu} X_\nu x_\nu = N ,$$

and any number of such reflections will produce

$$\sum \pm X_\nu x_\nu = N \pm 2k ,$$

where k is an integer. Among these hyperplanes for various values of k, those nearest to the origin are of the form

$$\sum \pm X_\nu x_\nu = N' ,$$

where $0 \leq N' \leq 1$. The distance of such a hyperplane from the origin, namely

$$N' / \sqrt{\sum X_\nu^2} ,$$

attains its greatest possible value when

$$N' = X_1 = X_2 = \ldots = X_n = 1 ,$$

so that N is an odd integer. Since each reflection reverses the sign of one coordinate and changes by one unit the k in the equation

$$\sum \pm x_\nu = N \pm 2k ,$$

the number of minus signs on the left has the same parity as k. Thus, if we begin with

$$\sum x_\nu = 1 ,$$

all the hyperplanes are given by

$$\sum \epsilon_\nu x_\nu = (1 \pm 4m)\Pi \epsilon_\nu$$

where $\epsilon_\nu = \pm 1$ and $m = 0, 1, \ldots,$ the possible values of m being limited by the requirement that

$$|1 \pm 4m| \leq n .$$

Since such a hyperplane is unchanged when we reverse the signs on both sides of the equation, the list can be simplified to

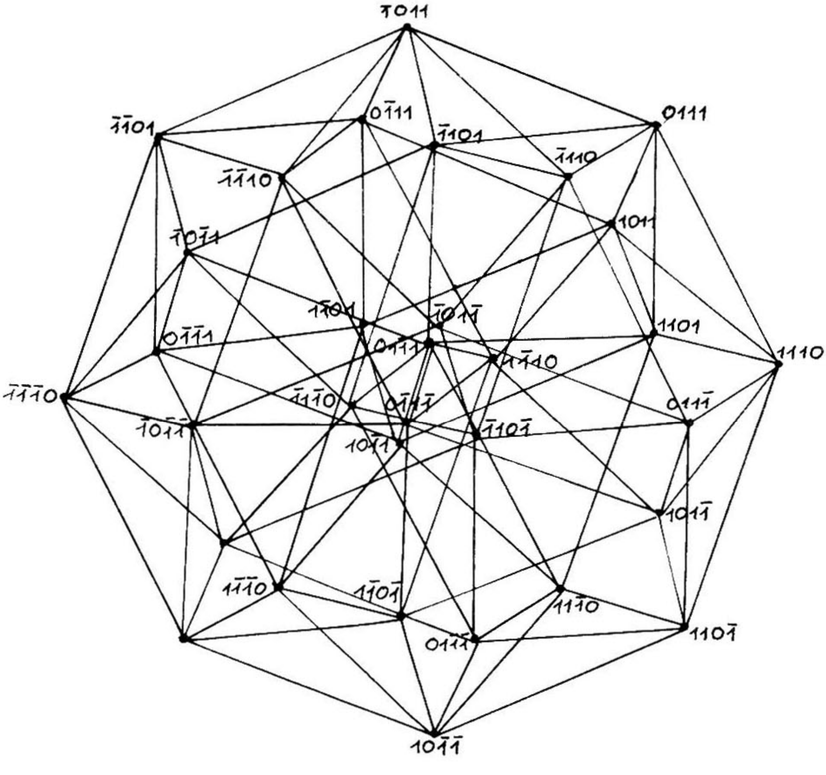
$$\sum \epsilon_\nu x_\nu = 1,3,5,\ldots,n-1 \qquad (\epsilon_\nu = \pm 1)$$

when n is even, and to

$$\sum \epsilon_\nu x_\nu = 1,-3,5,-7,\ldots,\pm n, \qquad \Pi \epsilon_\nu = 1$$

when n is odd.

The figure formed by all these hyperplanes is simply a square when $n = 2$ and a tetrahedron when $n = 3$ [König and Szücs 1913, p. 87]. When $n > 3$, the facets intersect one another internally, like the sides of a pentagram, so we shall call the figure *Schoenberg's star-polytope* [Schoenberg 1979, p.00]. His symbol for it is $\tilde{\Pi}_n^{n-1}$.



The 4-dimensional star-polytope $\alpha_{\underline{1}} h^2 (\mathrm{mod}\ 4)$

(This is Schoenberg's $\widetilde{\Pi}\,{}^{3}_{4}$)