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CRYSTALLOGRAPHIC GROUPS OF FOUR-DIMENSIONAL SPACE

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ABSTRACT

Of the groups that play a role in chemical and physical research, perhaps the best known ones are the three-dimensional space groups. The detailed information on them, tabulated in the "INTERNATIONAL TABLES FOR X-RAY CRYSTALLOGRAPHY" is widely used by practising chemists, solid state physicists, crystallographers, mineralogists etc.. It is less well known that shortly after the determination of the (230 classes of) three-dimensional space groups in 1890 by Fedorov and Schoenflies a dimension-independent theory on crystallographic groups was started by several mathematicians. In particular Bieberbach and Frobenius proved in 1910/11 that for each dimension n there are only finitely many classes of n-dimensional space groups under a classification completely analogous to the one established by Fedorov and Schoenflies for n=3.

In spite of this rather general result, however, such classifications were not explicitly done for any dimension bigger than 3 until recently. The volume under review here presents for the first time a complete classification for n=4. There are e.g. 4895 classes of 4-dimensional space groups under the classification that yields the famous 230 ones for dimension 3.

It seems that this explicit listing is not just proof of the fact that progress of theoretical insight and the use of computers makes such determination feasable, but also that it comes in time for some physical applications of these groups that start to emerge.

The book is opened by a short section "Some remarks on history" about the development of the subject. Apart of this it consists of 4 parts.

1. Chapter 1, "Basic concepts", gives (on 23 pages) a concise series of definitions of concepts and terms needed. The authors try to use as little mathematical background as possible here, and they do not give any proofs of facts used. However, they define the main concepts occuring in a dimension-independent way, which in several cases makes them clearer than they are when restricted to a casewise description in three dimensions, where accidental facts obscure certain general features.

The starting point is the concept of an n-dimensional crystal structure, defined as a function on the n-dimensional Euclidean space that admits an n-dimensional lattice of translations. This concept covers all physical applications and can be generalized in a natural way for an analogous treatment of partially periodic and "coloured" crystallographic groups. From the notion of a crystal structure they derive the concepts of space group, affine and crystallographic space group class, enantiomorphism, arithmetic and geometric crystal class, Bravais class of lattices, crystal system and crystal family.

2. Chapter 2, "Guide to the tables", gives (on 23 pages) a very detailed description of all the entries in the tables and of the hierarchy of concepts used in the ordering of the material in the tables. It also contains recipes and a fully worked out example how to derive easily more detailed information (analogous to the one in the International Tables) from the necessarily more condensed data listed here.

3. Chapter 3, "Lists and Tables", occupies the main bulk of the book. The number of groups listed and the hierarchy of their classification is best seen from the following

"Overall statistics: Comparison of crystallographic symmetries in different dimensions"

that is copied from the opening paragraph of Chapter 3:

Dimension:	1	2	3	4	
Number of arithmetic crystal classes:	2	13	73	710	(70)
Number of geometric crystal classes:	2	10	32	227	(44)
Number of space group types :	2	17	219(11)	4783	(112)
Number of Bravais flocks :	1	5	14	64	(10)
Number of crystal systems :	1	4	7	33	(7)
Number of crystal families :	1	4	6	23	(6)

Here numbers in parenthesis count enantiomorphic pairs.

Following the summary lists of statistics there are nine tables, each split into three parts according to the dimensions 2, 3 and 4. They cover different aspects of the groups and lattices listed and are cross-referenced in various ways:

Table 0	:	Point symmetry operations
" 1	:	Space group types (this is the main table)
" 2	:	Bravais types of lattices
" 3	:	Bravais flocks
" 4	:	Centerings
" 5	:	Normalizers
" 6	:	Isomorphism types and Character tables
" 7	:	Enantiomorphic pairs
" 8	:	Types of fixed-point-free space groups

In addition the Preface of the book states that further tables concerning the subgroup lattices of the maximal finite subgroups of GL(4,$\mathbb{Z}$) (i.e. of the arithmetical classes of highest symmetry) are available from the third of the authors on microfiche by special request.

4. The book closes with an appendix (of 23 pages), "The role of space groups in the theory of discrete groups of motions of Euclidean spaces", which is both in concept and background-requirement directed to the mathematician who may consider this as a large collection of infinite groups that also have a purely mathematical interest. A bibliography and an Index are provided.

An errate sheet for the book will be available from the third of the authors. He has asked the reviewer to point out in particular the following necessary correction:

In the definition of the conventional concept of a crystal system (p. 16, last paragraph) the following statement is made: "However, for any $\mathbb{Q}$-class C there is a unique holohedry H such that each f.u. group in C is a subgroup of some f.u. group in H but is not a subgroup of any f.u. group belonging to a holohedry of smaller order". This statement is correct for dimensions 1, 2, 3, and 4. However the statement does not hold independently of the dimension. In fact there is a counterexample to it for dimension 7. The definition of "crystal system" given on p. 16/17 therefore is known to be unambiguous for the dimensions 1, 2, 3, and 4 for which it is used in the tables of the book but the concept "crystal system" is not well-defined for higher dimensions.

It should be noted that in contrast the concept of crystal family is well-defined independently of the dimension if one uses the "more natural" second definition of it at the end of p. 17, which avoids the use of crystal systems altogether.

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