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On the Arithmetic–Geometric Index of Graphs

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Abstract

Very recently, the first geometric–arithmetic index GA and arithmetic–geometric index AG were introduced in mathematical chemistry. In the present paper, we first obtain some lower and upper bounds on AG and characterize the extremal graphs. We also establish various relations between AG and other topological indices, such as the first geometric–arithmetic index GA, atom–bond connectivity index ABC, symmetric division deg index SDD, chromatic number χ and so on. Finally, we present some sufficient conditions of $GA(G) > GA(G-e)$ or $AG(G) > AG(G-e)$ for an edge e of a graph G. In particular, for the first geometric–arithmetic index, we also give a refinement of Bollobás–Erdős–type theorem obtained in [3].

1. Introduction

We consider only finite, undirected and simple graph throughout this paper. Let $G = (V, E)$ be a simple graph of order n and size m, with vertex set $V(G) = \{v_1, v_2, \cdots v_n\}$ and edge set $E(G)$. Denote an edge $e \in E(G)$ with end vertices v_i and v_j by v_iv_j, simply by $i \sim j$. Let d_i be the degree of vertex v_i for $i = 1, 2, \ldots, n$. If an edge $e = v_iv_j$ satisfying $d_i = 1$, we say that e is a pendent edge and v_i is a pendent vertex. The maximum and

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minimum degrees of G are denoted by Δ and δ, respectively. Let $\overline{d}$ be the average degree of G. The minimum non–pendent vertex degree of G is written by δ_1. Also let p denote number of pendent vertices in G.

If the vertex set $V(G)$ is the disjoint union of two nonempty subsets V_1 and V_2, such that every vertex in V_1 has degree s and every vertex in V_2 has degree r, then G is said to be (s,r)*–semiregular*. In particular, if $s = r$, then G is said to be r*–regular*. As usual, the complete bipartite graph, the complete graph and the star on n vertices are denoted by $K_{p,q}$, K_n and $K_{1,n-1}$, respectively.

Topological indices are graph invariant under graph isomorphisms and reflect some structural properties of the corresponding molecule graph. It is found that these indices have some chemical applications in chemical graph theory, for example, see [4, 8–10, 13, 17–22, 30, 31, 33] and the references cited therein. Recently, Vukičević and Furtula [28] proposed a newly graph invariant, namely the *first geometric–arithmetic index*, which is defined as follows:

$$GA(G) = \sum_{v_iv_j \in E(G)} \left(\frac{2\sqrt{d_id_j}}{d_i + d_j} \right).$$

They also obtained some bounds on GA index and determined the trees with maximum and minimum GA indices, which are the star and the path, respectively. In [6], Das et al. gave some lower and upper bounds on GA index in terms of the order n, the size m, the minimum degree δ, maximum degree Δ and the other topological index. In [1], several further inequalities, involving GA index and several other graph parameters, were obtained. Aouchiche and Hansen [2] presented some bounds on GA index in terms of the order n, the chromatic number χ, the minimum degree δ, maximum degree Δ and average degree $\overline{d}$. At the same time, some conjectures were proposed in [2]. Very recently, Chen and Wu [3] disprove four of these conjectures. In addition, they also presented a sufficient condition with $GA(G) > GA(G - e)$ when an edge e is deleted from a graph G. For a comprehensive survey and more details on this area, we refer the reader to [7] and references therein.

In 2015, Shegehall and Kanabur [23] introduced the *arithmetic–geometric index* AG of G. It is defined as follows:

$$AG(G) = \sum_{v_iv_j \in E(G)} \frac{1}{2} \left(\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \right).$$

The AG index of path graph with pendent vertices attached to the middle vertices was discussed in [23, 24]. In addition, the AG index of graphene, which is most conductive and effective material for electromagnetic interference shielding [26], was computed in [25]. Using this newly topological index, Zheng and the present two authors [32] studied spectrum and energy of arithmetic–geometric matrix, in which the sum of all elements is equal to $2AG$. Other bounds of the arithmetic–geometric energy of graphs were offered in [5, 14]. Very recently, as one of the referees said, Vujošević et al. [27] characterized chemical trees that maximize the value of arithmetic–geometric index. At the same time, they also obtained some lower and upper bounds on AG and elaborated the relations between the AG and GA. Motivated by these papers, we further consider bounds on the AG index and discuss the effect on GA and AG indices of deleting an edge from a graph.

For the sake of convenience, here we list some degree–based topological indices, which will be used in subsequent sections.

- The forgotten index [15], $F(G) = \sum\limits_{i=1}^{n} d_i^3 = \sum\limits_{v_iv_j \in E(G)} \left(d_i^2 + d_j^2\right)$.
- The first Zagreb index [15], $M_1(G) = \sum\limits_{i=1}^{n} d_i^2 = \sum\limits_{v_iv_j \in E(G)} (d_i + d_j)$.
- The second Zagreb index [16], $M_2(G) = \sum\limits_{v_iv_j \in E(G)} d_i d_j$.
- The symmetric division deg index [29], $SDD(G) = \sum\limits_{v_iv_j \in E(G)} \left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)$.
- The atom–bond connectivity index [12], $ABC(G) = \sum\limits_{v_iv_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}$.

This paper is organized as follows. In Section 2, we present some lower and upper bounds on AG and characterize the extremal graphs. In Section 3, we establish various relations between AG and other topological indices, such as the first geometric–arithmetic index GA, atom–bond connectivity index ABC, symmetric division deg index SDD, chromatic number and so on. In Section 4, we obtain some sufficient conditions of $GA(G) > GA(G - e)$ or $AG(G) > AG(G - e)$ for an edge e of a graph G. In particular, we give a refinement of Bollobás–Erdős–type theorem obtained in [3] for the first geometric–arithmetic index. Many examples show that there are considerable differences between GA and AG indices of graphs.

2. Upper and lower bounds on arithmetic–geometric index

Throughout this section, we always assume that G is a graph with p pendent vertices.

Theorem 1. If G is a connected graph of order n with size m, maximum degree Δ, minimum non–pendent vertex degree δ_1, then

$$AG(G) \leq \frac{p(\Delta+1)}{2\sqrt{\Delta}} + \frac{1}{2\delta_1}\sqrt{(m-p)(F+2M_2-p(\delta_1+1)^2)}, \tag{1}$$

with equality if and only if G is isomorphic to $K_{1,n-1}$ or G is isomorphic to a regular graph or G is isomorphic to a $(\Delta,1)$–semiregular graph.

Proof. By the Cauchy–Schwarz inequality, one has

$$\begin{aligned}
AG(G) &= \sum_{i\sim j}\frac{1}{2}\left(\sqrt{\frac{d_i}{d_j}}+\sqrt{\frac{d_j}{d_i}}\right)\\
&= \frac{1}{2}\sum_{i\sim j,\, d_j=1}\left(\sqrt{\frac{d_i}{d_j}}+\sqrt{\frac{d_j}{d_i}}\right)+\frac{1}{2}\sum_{i\sim j, d_i,d_j>1}\left(\sqrt{\frac{d_i}{d_j}}+\sqrt{\frac{d_j}{d_i}}\right)\\
&\leq \frac{p(\Delta+1)}{2\sqrt{\Delta}}+\frac{1}{2}\sqrt{(m-p)}\sqrt{\sum_{i\sim j, d_i,d_j>1}\left(\sqrt{\frac{d_i}{d_j}}+\sqrt{\frac{d_j}{d_i}}\right)^2}\\
&\leq \frac{p(\Delta+1)}{2\sqrt{\Delta}}+\frac{1}{2\delta_1}\sqrt{(m-p)}\sqrt{\sum_{i\sim j, d_i,d_j>1}(d_i+d_j)^2} \qquad (2)\\
&= \frac{p(\Delta+1)}{2\sqrt{\Delta}}+\frac{1}{2\delta_1}\sqrt{(m-p)}\sqrt{\sum_{i\sim j}(d_i+d_j)^2-\sum_{i\sim j, d_j=1}(d_i+d_j)^2}\\
&\leq \frac{p(\Delta+1)}{2\sqrt{\Delta}}+\frac{1}{2\delta_1}\sqrt{(m-p)(F+2M_2-p(\delta_1+1)^2)} \quad \text{as} \quad d_i\geq\delta_1, \qquad (3)
\end{aligned}$$

implying the required result (1).

Now assume that the equality holds in (1). Then all inequalities in above proof must be equalities. From the equality in (2), we have $d_i=\Delta$ and $d_j=1$ for any pendent edge $i\sim j$, and $d_i=d_j=\delta_1$ for any non–pendent edge $i\sim j$. From the equality in (3), we have $d_i=\delta_1$ and $d_j=1$ for any pendent edge $i\sim j$. In particular, if G has no pendent edge, that is, $p=0$, then G is isomorphic to a Δ–regular graph. If every edge of G is pendent edge, that is, $m=p$, then G is isomorphic to $K_{1,n-1}$. Otherwise, $0<p<m$, which implies that G is isomorphic to a $(\Delta,1)$–semiregular graph as G is connected.

Conversely, it is easy to check that the equality holds in (1) for $K_{1,n-1}$ or a regular graph or a $(\Delta,1)$–semiregular graph. ■

Corollary 1. If G is a connected graph of order n with size m, minimum degree δ, then

$$AG(G) \le \frac{1}{2\delta}\sqrt{m(F+2M_2)}, \tag{4}$$

with equality if and only if G is isomorphic to a regular graph.

Proof. If G has no pendent edge, then $p = 0$ and $\delta = \delta_1$. By Theorem 1, we get the required result. Now assume that G has at least a pendent edge, that is $\delta = 1$. We need only to prove that $AG(G) \le \frac{1}{2}\sqrt{m(F+2M_2)}$. Indeed, from the Cauchy–Schwarz inequality, one has

$$(2AG(G))^2 = \left(\sum_{i\sim j}\left(\sqrt{\frac{d_i}{d_j}}+\sqrt{\frac{d_j}{d_i}}\right)\right)^2 \le \left(\sum_{i\sim j}(d_i+d_j)\right)^2 \le m\sum_{i\sim j}(d_i+d_j)^2 = m(F+2M_2),$$

with equality if and only if $d_i = d_j = 1$ for a pendent edge $i\sim j$, equivalently, G is isomorphic to K_2 as G is connected. ■

Theorem 2. If G is a connected graph of order n with size m, maximum degree Δ, minimum non–pendent vertex degree δ_1, then

$$AG(G) \le \frac{p(\Delta+1)}{2\sqrt{\Delta}} + \frac{1}{2\delta_1}\sqrt{F+2M_2-p(\delta_1+1)^2+4\Delta^2(m-p)(m-p-1)}, \tag{5}$$

with equality if and only if G is isomorphic to $K_{1,n-1}$ or G is isomorphic to a regular graph or G is isomorphic to a $(\Delta,1)$–semiregular graph.

Proof. For the sake of convenience, we first estimate

$$\sum_{i\sim j,d_i,d_j>1}\left(\sqrt{\frac{d_i}{d_j}}+\sqrt{\frac{d_j}{d_i}}\right)$$

$$= \sqrt{\sum_{i\sim j,d_i,d_j>1}\left(\sqrt{\frac{d_i}{d_j}}+\sqrt{\frac{d_j}{d_i}}\right)^2 + 2\sum_{i\sim j,k\sim l,(v_i,v_j)\neq(v_k,v_l),d_i,d_j,d_k,d_l>1}\left(\sqrt{\frac{d_i}{d_j}}+\sqrt{\frac{d_j}{d_i}}\right)\left(\sqrt{\frac{d_k}{d_l}}+\sqrt{\frac{d_l}{d_k}}\right)}$$

$$\le \frac{1}{\delta_1}\sqrt{\sum_{i\sim j,d_i,d_j>1}(d_i+d_j)^2 + 2\sum_{i\sim j,k\sim l,(v_i,v_j)\neq(v_k,v_l),d_i,d_j,d_k,d_l>1}(d_i+d_j)(d_k+d_l)} \tag{6}$$

$$\le \frac{1}{\delta_1}\sqrt{F+2M_2-p(\delta_1+1)^2+8\Delta^2\binom{m-p}{2}} \tag{7}$$

$$= \frac{1}{\delta_1}\sqrt{F+2M_2-p(\delta_1+1)^2+4\Delta^2(m-p)(m-p-1)}.$$

It is easy to see that the function $f(x) = x + \frac{1}{x}$ is an increasing function for $x \ge 1$. Then, we have

$$AG(G) = \frac{1}{2}\sum_{i\sim j,d_j=1}\left(\sqrt{\frac{d_i}{d_j}}+\sqrt{\frac{d_j}{d_i}}\right) + \frac{1}{2}\sum_{i\sim j,d_i,d_j>1}\left(\sqrt{\frac{d_i}{d_j}}+\sqrt{\frac{d_j}{d_i}}\right)$$

$$\leq \frac{p(\Delta+1)}{2\sqrt{\Delta}} + \frac{1}{2\delta_1}\sqrt{F + 2M_2 - p(\delta_1+1)^2 + 4\Delta^2(m-p)(m-p-1)}. \qquad (8)$$

Now assume that the equality holds in (5). Then all inequalities in above argument must be equalities. From the equality in (6), we have $d_i = d_j = \delta_1$ for any non–pendent edge $i \sim j$. The equality in (7) implies that $d_i = \delta_1$ and $d_j = 1$ for any pendent edge $i \sim j$. At the same time, it follows from the equality in (8) that $d_i = \Delta$ and $d_j = 1$ for any pendent edge $i \sim j$. We have to keep in mind that G is connected. Similar to the argument of Theorem 1, then G is isomorphic to $K_{1,n-1}$, or G is isomorphic to a regular graph, or G is isomorphic to a $(\Delta, 1)$–semiregular graph.

Conversely, it is easy to check that the equality holds in (5) for $K_{1,n-1}$ or a regular graph or a $(\Delta, 1)$–semiregular graph. ■

Corollary 2. If G is a connected graph of order n with size m, minimum degree δ and maximum degree Δ, then

$$AG(G) \leq \frac{1}{2\delta}\sqrt{F + 2M_2 + 4m(m-1)\Delta^2}, \qquad (9)$$

with equality if and only if G is isomorphic to a regular graph.

Proof. The proof is similar to that of Corollary 1, omitted. ■

Theorem 3. If G is a connected graph of order n with size m, then

$$AG(G) \leq \frac{1}{2}p\left(\frac{n}{\sqrt{n-1}}\right) + \frac{1}{2}(m-p)\left(\sqrt{\frac{n-1}{2}} + \sqrt{\frac{2}{n-1}}\right), \qquad (10)$$

with equality if and only if G is isomorphic to a star $K_{1,n-1}$ or G is isomorphic to a complete graph K_3.

Proof. Since the function $f(x) = x + \frac{1}{x}$ is an increasing function for $x \geq 1$. Then, for any pendent edge $i \sim j$ and $d_j = 1$,

$$\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \leq \sqrt{n-1} + \frac{1}{\sqrt{n-1}} = \frac{n}{\sqrt{n-1}}, \qquad (11)$$

with equality if and only if $d_i = n-1$. Similarly, for any non–pendent edge $i \sim j$, one has

$$\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \leq \sqrt{\frac{n-1}{2}} + \sqrt{\frac{2}{n-1}}, \qquad (12)$$

with equality if and only if $d_i = n-1$ and $d_j = 2$ for $d_i \geq d_j$. Therefore,

$$AG(G) = \frac{1}{2}\sum_{i\sim j, d_j=1}\left(\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}}\right) + \frac{1}{2}\sum_{i\sim j, d_i, d_j>1}\left(\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}}\right)$$

$$\leq \frac{p}{2}\left(\frac{n}{\sqrt{n-1}}\right)+\frac{1}{2}(m-p)\left(\sqrt{\frac{n-1}{2}}+\sqrt{\frac{2}{n-1}}\right).$$

Now assume that the equality holds in (10). Then all inequalities in above argument must be equalities. In the following, without loss of generality, assume that $d_i \geq d_j$ for every edge $i \sim j$. First if G has no pendent edge, equivalently, $p = 0$. Then the equality in (12) implies that there is a common neighbor between the end vertices of every edge of G. This shows that G is isomorphic to a complete graph K_3 as G is connected. Clearly, G is isomorphic to $K_{1,n-1}$ when $p = m$. Finally, assume that $0 < p < m$ and $d_i = n-1$, $d_k = 1$ for some pendent edge $i \sim k$. Then, there must exist a non–pendent edge $i \sim j$ of G such that $d_j = 2$ as $m > p$. Thus, the vertices i and j have must a common neighbor l. Also, from the equality in (12), we have $d_l = n-1$. Therefore, $l \sim k$, which implies that $d_k \geq 2$ contradicting to our assumption.

Conversely, it is easy to check that the equality holds in (10) for a complete graph K_3 or a star $K_{1,n-1}$. ■

Corollary 3. If G is a connected graph of order n with size m, minimum degree $\delta \geq 2$, then

$$AG(G) \leq \frac{1}{2}m\left(\sqrt{\frac{n-1}{2}}+\sqrt{\frac{2}{n-1}}\right), \tag{13}$$

with equality if and only if G is isomorphic to a complete graph K_3.

The following lemma comes from [11]. First let $(a_1, a_2, \ldots, a_n)$, $(b_1, b_2, \ldots, b_n)$ be two sequences of positive real numbers, such that there are positive numbers A, a, B, b satisfying, for any $i \in \{1, 2, \ldots, n\}$,

$$0 < a \leq a_i \leq A < \infty,\ 0 < b \leq b_i \leq B < \infty.$$

Lemma 1(Pólya-Szegö inequality [11]).

$$\frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{(\sum_{i=1}^{n} a_i b_i)^2} \leq \frac{(ab+AB)^2}{4abAB},$$

where the equality holds if and only if

$$p = n \cdot \frac{A}{a}\Big/\left(\frac{A}{a}+\frac{B}{b}\right),\ q = n \cdot \frac{B}{b}\Big/\left(\frac{A}{a}+\frac{B}{b}\right)$$

are integers and if p of the numbers $a_1, a_2, \ldots, a_n$ are equal to a and q of these numbers are equal to A, and if the corresponding numbers b_i are equal to B and b, respectively.

Theorem 4. If G is a connected graph of order n with size m, maximum degree Δ, minimum non–pendent vertex degree δ_1, then

$$AG(G) \geq \frac{p(\delta_1+1)}{2\sqrt{\delta_1}} + \frac{\sqrt{2(m-p)(\Delta+\delta_1)\sqrt{\Delta\delta_1}}}{\Delta(\Delta+\delta_1+2\sqrt{\Delta\delta_1})}\sqrt{F+2m\Delta^2-p(3\Delta^2+1)}, \tag{14}$$

with equality if and only if G is isomorphic to $K_{1,n-1}$ or G is isomorphic to a regular graph or G is isomorphic to a $(\Delta, 1)$–semiregular graph.

Proof. For any non–pendent edge $i \sim j$, we get

$$2 \leq \sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \leq \sqrt{\frac{\Delta}{\delta_1}} + \sqrt{\frac{\delta_1}{\Delta}}.$$

Then, take $a = 2$, $A = \sqrt{\frac{\Delta}{\delta_1}} + \sqrt{\frac{\delta_1}{\Delta}}$ and $b = B = 1$ in Lemma 1, one has

$$\left(\sum_{i\sim j, d_i, d_j>1}\left(\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}}\right)\right)^2 \geq \frac{8(m-p)(\Delta+\delta_1)\sqrt{\Delta\delta_1}}{(\Delta+\delta_1+2\sqrt{\Delta\delta_1})^2}\sum_{i\sim j, d_i, d_j>1}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}+2\right). \tag{15}$$

For the sake of convenience, let

$$\Gamma_1 = \left(\sum_{i\sim j, d_i, d_j>1}\left(\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}}\right)\right)^2$$

and

$$\Gamma_2 = \sum_{i\sim j, d_i, d_j>1}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}+2\right).$$

We first estimate the value of Γ_2,

$$\begin{aligned}\Gamma_2 &= \sum_{i\sim j, d_i, d_j>1}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right) + 2(m-p)\\ &\geq \frac{1}{\Delta^2}\left(\sum_{i\sim j}(d_i^2+d_j^2) - \sum_{i\sim j, d_j=1}(d_i^2+d_j^2)\right) + 2(m-p) \qquad (16)\\ &\geq \frac{1}{\Delta^2}(F - p(\Delta^2+1)) + 2(m-p). \qquad (17)\end{aligned}$$

Plugging (17) into (15), one gets

$$\Gamma_1 \geq \frac{8(m-p)(\Delta+\delta_1)\sqrt{\Delta\delta_1}}{\Delta^2(\Delta+\delta_1+2\sqrt{\Delta\delta_1})^2}(F+2m\Delta^2-p(3\Delta^2+1)),$$

which implies that

$$AG(G) = \frac{1}{2}\sum_{i\sim j, d_j=1}\left(\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}}\right) + \frac{1}{2}\sum_{i\sim j, d_i, d_j>1}\left(\sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}}\right)$$

$$\geq \frac{p(\delta_1+1)}{2\sqrt{\delta_1}} + \frac{\sqrt{2(m-p)(\Delta+\delta_1)\sqrt{\Delta\delta_1}}}{\Delta(\Delta+\delta_1+2\sqrt{\Delta\delta_1})}\sqrt{F+2m\Delta^2-p(3\Delta^2+1)}. \qquad (18)$$

Now assume that the equality holds in (14). Then all inequalities in above proof must be equalities. From the equality in (16), we have $d_i = d_j = \Delta$ for any non–pendent edge $i \sim j$. It follows from the equality in (18) that $d_i = \delta_1$ and $d_j = 1$ for any pendent edge $i \sim j$ with pendent vertex j.

Next, one has to keep in mind that G is connected. If $p = 0$, that is, G has no pendent edge, then G is isomorphic to a Δ–regular graph. If $m = p$, that is, each one of edges in G is pendent edge, then G is isomorphic to $K_{1,n-1}$. Otherwise, $0 < p < m$, which implies that G is isomorphic to a $(\Delta, 1)$–semiregular graph.

Conversely, it is easy to see that the equality holds in (14) for $K_{1,n-1}$ or a regular graph or a $(\Delta, 1)$–semiregular graph. ■

Corollary 4. Let G be a connected graph of order n with size m, maximum degree Δ, minimum degree δ. If G has no pendent vertices, then

$$AG(G) \geq \frac{\sqrt{2m(\Delta+\delta_1)\sqrt{\Delta\delta_1}}}{\Delta(\Delta+\delta_1+2\sqrt{\Delta\delta_1})}\sqrt{F+2m\Delta^2},$$

with equality if and only if G is isomorphic to a regular graph.

Similar to the proof of Theorem 4, we may obtain the following theorem.

Theorem 5. If G is a connected graph of order n with size m, maximum degree Δ, minimum non–pendent vertex degree δ_1, then

$$AG(G) \geq \frac{p(\delta_1+1)}{2\sqrt{\delta_1}} + \frac{\sqrt{2(m-p)(\Delta+\delta_1)\sqrt{\Delta\delta_1}}}{\Delta+\delta_1+2\sqrt{\Delta\delta_1}}\sqrt{SDD-p(\Delta+\frac{1}{\Delta})+2(m-p)},$$

with equality if and only if G is isomorphic to $K_{1,n-1}$ or G is isomorphic to a regular graph or G is isomorphic to a $(\Delta, 1)$–semiregular graph.

Clearly, $AG(G) \geq \frac{M_1}{2\Delta}$. Here we shall give a minor improvement on this lower bound as follow.

Theorem 6. If G is a connected graph of order n with size m, maximum degree Δ, minimum non–pendent vertex degree δ_1, then

$$AG(G) \geq \frac{p(\delta_1+1)}{2\sqrt{\delta_1}} + \frac{1}{2\Delta}(M_1 - p(\Delta+1)),$$

with equality if and only if G is isomorphic to $K_{1,n-1}$ or G is isomorphic to a regular graph or G is isomorphic to a $(\Delta, 1)$–semiregular graph.

Proof. It is easy to verify that

$$\begin{aligned} AG(G) &= \tfrac{1}{2} \sum_{i\sim j, d_j=1} \left(\sqrt{\tfrac{d_i}{d_j}} + \sqrt{\tfrac{d_j}{d_i}}\right) + \tfrac{1}{2} \sum_{i\sim j, d_i, d_j>1} \left(\sqrt{\tfrac{d_i}{d_j}} + \sqrt{\tfrac{d_j}{d_i}}\right) \\ &\geq \tfrac{p}{2}\left(\sqrt{\delta_1} + \tfrac{1}{\sqrt{\delta_1}}\right) + \tfrac{1}{2\Delta}\left(\sum_{i\sim j}(d_i+d_j) - \sum_{i\sim j, d_j=1}(d_i+d_j)\right) \\ &\geq \tfrac{p(\delta_1+1)}{2\sqrt{\delta_1}} + \tfrac{1}{2\Delta}(M_1 - p(\Delta+1)), \end{aligned}$$

with equality if and only if G has same degree for all non–pendent vertex. The rest of the proof is similar to that of Theorem 4, omitted. ■

3. Comparison between arithmetic–geometric index and other topological indices

Theorem 7. Let G be a connected graph of order n, with minimum degree δ. Then

$$GA(G) \leq AG(G) \leq \frac{(\delta+n-1)^2}{4\delta(n-1)} GA(G) \tag{19}$$

with left–hand side of equality if and only if G is a regular graph, and right–hand side of equality if and only if G is isomorphic to $K_{1,n-1}$ or G is isomorphic to K_n.

Proof. Consider the following function

$$f(x,y) = \frac{\frac{1}{2}\left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}}\right)}{\frac{2\sqrt{xy}}{x+y}} = \frac{(x+y)^2}{4xy},$$

where $1 \leq \delta \leq x \leq y \leq n-1$. Now, by a simple computation, we get

$$\frac{\partial f}{\partial x} = \frac{4y(x^2-y^2)}{16x^2y^2} \leq 0,$$

which implies that $f(x,y)$ is decreasing in x. Thus, $f(x,y)$ attains the maximum at (δ, y) for some $\delta \leq y \leq n-1$. On the other hand, it is easy to verify that $f(\delta, y)$ is an increasing function for $y \geq \delta \geq 1$. Therefore,

$$f(x,y) \leq f(\delta, n-1) = \frac{(\delta+n-1)^2}{4\delta(n-1)},$$

which implies that

$$AG(G) \leq \frac{(\delta+n-1)^2}{4\delta(n-1)} GA(G)$$

with equality if and only if $(d_i, d_j) = (\delta, n-1)$ for every edge $i \sim j$ of G. If $\delta = 1$, then G is isomorphic to $K_{1,n-1}$. Otherwise, $\delta \geq 2$, this time G has no pendent edge. Without loss of generality, suppose that $d_i = \delta$, then the vertex i has at least two adjacent vertices with degree $n-1$. This implies that $\delta = n-1$. Therefore, G is isomorphic to K_n.

The left–hand side of inequality in (19) is clearly true (also see [27], Observation 1). Therefore, the required result follows. ■

Corollary 5 [27]. Let G be a connected graph of order $n \geq 2$. Then

$$AG(G) \leq \frac{n^2}{4(n-1)} GA(G),$$

with equality if and only if G is isomorphic to $K_{1,n-1}$.

Denote the chromatic number of a graph G by $\chi(G)$. It was proved in [1] that if G is a connected graph with $\delta \geq 2$, then $\chi(G) \leq \frac{2}{\delta} GA(G)$ with equality if and only if G is isomorphic to K_n. In [2], Aouchiche and Hansen proposed the following conjecture.

Conjecture 1 [2]. Let G be a connected graph of order n with m edges and average degree $\overline{d}$. Then

$$\chi(G) \leq \frac{2GA(G)}{\overline{d}}$$

with equality if and only if G is isomorphic to K_n.

It is easy to see that Conjecture 1 holds for a regular graph G or complete bipartite graph K_{n_1,n_2} of order $n = n_1 + n_2$. Denote the join of G_1 and G_2 by $G_1 \vee G_2$, we define $L(n,k) = K_k \vee \overline{K_{n-k}}$, where $\overline{K_{n-k}}$ is the complement of the complete graph K_{n-k}. Notice that $L(n,1) = K_{1,n-1}$ and $L(n,n-1) = K_n$. Next we assume that $2 \leq k \leq n-2$. Clearly, $\chi(L(n,k)) = k+1$. By a simple computation, we obtain

$$\frac{2GA(L(n,k))}{\overline{d}} = n \cdot \frac{\binom{k}{2} + k(n-k)\frac{2\sqrt{k(n-1)}}{n+k-1}}{\binom{k}{2} + k(n-k)} = O(\sqrt{n}).$$

Hence, we arrive at

Theorem 8. For a fixed number k and sufficiently large n, we have

$$\chi(L(n,k)) \leq \frac{2GA(L(n,k))}{\overline{d}}.$$

As we all know, Conjecture 1 is still open. However, if the first geometric–arithmetic index $GA(G)$ is replaced by arithmetic–geometric index $AG(G)$ in above conjecture, then

$$\chi(G) \leq \frac{2m}{\overline{d}} \leq \frac{2AG(G)}{\overline{d}},$$

with equality if and only if G is isomorphic to K_n.

In [2], it is also pointed out that, there exist graphs with $\chi(G) > \frac{2GA(G)}{\Delta}$. But, similar to Theorem 8, It can be easily proved that, for a sufficiently large n,

$$\chi(L(n,k)) \leq \frac{2AG(L(n,k))}{n-1}.$$

Thus, the following problem arises: does there exist a graph G satisfying $\chi(G) > \frac{2AG(G)}{\Delta}$?

In the following, we shall consider relations between arithmetic–geometric index $AG(G)$ and atom–bond connectivity index $ABC(G)$. Let T^* denote the tree obtained by joining the central vertices of two copies of $K_{1,3}$ by an edge. Das and Trinajstić [8] proved that if G is a connected graph with $\Delta - \delta \leq 3$ and it is neither isomorphic to $K_{1,4}$ nor T^*, then $GA(G) > ABC(G)$. Note that $AG(K_{1,4}) > ABC(K_{1,4})$ and $AG(T^*) > ABC(T^*)$. Thus, combining these results with Theorem 7, one gets $AG(G) > ABC(G)$ for any connected graph G with $\Delta - \delta \leq 3$. Next, we give an improvement on this result.

Theorem 9. Let G be a connected graph of order n, with minimum degree $\delta \geq 2$. Then

$$\frac{\delta}{\sqrt{2\delta-2}} ABC(G) \leq AG(G) \leq \frac{n-1}{\sqrt{2n-4}} ABC(G). \tag{20}$$

Moreover, the left–hand side of equality holds in (20) if and only if G is a δ–regular graph, and right–hand side of equality holds in (20) if and only if G is isomorphic to K_n.

Proof. Consider the following function

$$f(x,y) = \left(\frac{\frac{1}{2}\left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}}\right)}{\frac{\sqrt{x+y-2}}{\sqrt{xy}}} \right)^2 = \frac{(x+y)^2}{4(x+y-2)},$$

where $2 \leq \delta \leq x \leq y \leq n-1$. Now, by a simple computation, we get

$$\frac{\partial f}{\partial x} = \frac{(x+y)(x+y-4)}{4(x+y-2)^2} \geq 0,$$

which implies that $f(x,y)$ is increasing in x. Thus, $f(x,y)$ attains the minimum at (δ, y_1) for some $\delta \leq y_1 \leq n-1$ and maximum at (y_2, y_2) for some $\delta \leq y_2 \leq n-1$. On the other hand, it is easy to verify that $f(\delta, y)$ is an increasing function for $y \geq \delta \geq 2$. Thus,

$$f(\delta,\delta) \leq f(x,y) \leq f(n-1,n-1),$$

which implies that

$$\frac{\delta}{\sqrt{2\delta-2}} ABC(G) \leq AG(G) \leq \frac{n-1}{\sqrt{2n-4}} ABC(G)$$

with left–hand side of equality if and only if $(d_i, d_j) = (\delta, \delta)$ for every edge $i \sim j$ of G, and right–hand side of equality if and only if $(d_i, d_j) = (n-1, n-1)$ for every edge $i \sim j$ of G. Hence, the required result follows. ■

Note that it follows from Theorem 9 that, for a graph G with $\delta \geq 2$, $AG(G) > \sqrt{2}ABC(G)$ unless G is isomorphic to C_n.

Using the similar technique to the proof in Theorem 9, we easily obtain the following bounds for the arithmetic–geometric index $AG(G)$ in terms of the symmetric division deg index $SDD(G)$ (the details is omitted).

Theorem 10. Let G be a connected graph of order n, with minimum degree δ. Then

$$\frac{(\delta+n-1)\sqrt{\delta(n-1)}}{2(\delta^2+(n-1)^2)}SDD(G) \leq AG(G) \leq \frac{1}{2}SDD(G). \tag{21}$$

Moreover, the left–hand side of equality holds in (21) if and only if G is isomorphic to $K_{1,n-1}$ or G is isomorphic to K_n, and right–hand side of equality holds in (21) if and only if G is a δ–regular graph.

4. Effect on GA and AG indices of deleting an edge from a graph

In this section, we mainly discuss the effect on GA and AG indices when an edge is deleted from a graph G. First we note that GA and AG indices will always decrease when an edge $e = v_iv_j$ with $d_i = d_j = 1$, is deleted from G. For the sake of convenience, assume that $e = v_iv_j$ is an edge with non–pendent vertex v_j throughout this section.

4.1. Effect on GA index of deleting an edge

In [6], Das et al. presented a sufficient condition with $GA(G+e) > GA(G)$ when a new edge e is inserted into the graph G. Recently, Chen and Wu [3] pointed out that the result obtained in [6] is not complete. Furthermore, they established Bollobás–Erdős–type theorem for the first geometric–arithmetic index of a graph G as follows.

Theorem 11 [3]. Let G be a simple graph with an edge $e = v_iv_j$. Also let $d_r = \max\{d_k | v_iv_k \in E(G)\}$ and $d_s = \max\{d_l | v_jv_l \in E(G)\}$. If one of the following conditions is satisfied, then $GA(G) > GA(G-e)$:

(i) $\max\{\frac{d_i}{d_r}, \frac{d_j}{d_s}\} \leq 1$, or

(ii) $\max\{\frac{d_i}{d_j}, \frac{d_j}{d_i}\} \leq \min\{\frac{d_i}{d_r}, \frac{d_j}{d_s}\}$.

Example 1. Let G be the graph as shown in Figure 1, where $d_i = 10$ and $d_j = d_r = d_s = 1000$. Clearly, G satisfies the condition (i) of Theorem 11, that is, $\max\{\frac{10}{1000}, \frac{1000}{1000}\} \leq 1$, but $GA(G) - GA(G - e) = -0.0447$.

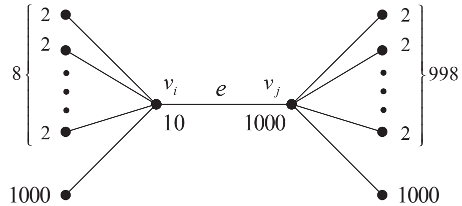

Figure 1. A counterexample to the (i) of Theorem 11.

Example 2. Let G be the graph as shown in Figure 2, where $d_i = 100$, $d_j = 500$, $d_r = 500$ and $d_s = 100$. By a simple calculation, one can see that $GA(G) - GA(G - e) = 0.5501$, in spite of $\max\{\frac{100}{500}, \frac{500}{100}\} > \min\{\frac{100}{500}, \frac{500}{100}\}$. Therefore, the (ii) of Theorem 11 is invalid for this graph G.

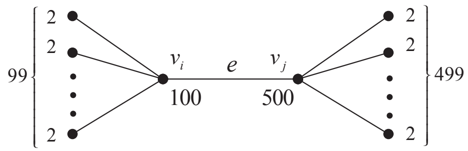

Figure 2. The graph G in Example 2.

Example 3. For two given graphs G_1, G_2 in Figure 3, one can see that $GA(G_1) - GA(G_1 - e) = 0.0652$, whereas $GA(G_2) - GA(G_2 - e) = -0.0363$. This example shows that GA index may either increase or decrease when a pendent edge e is deleted from a graph.

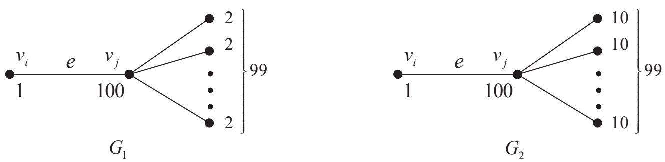

Figure 3. In the case of G_1 the GA index decreases, whereas in the case of G_2 the GA index increases.

Assume that G is a simple graph and $e = v_iv_j$ is an edge of G with non–pendent vertex v_j. For the sake of convenience, we define

$$d_{\min}^{(j)} = \min\{d_k|v_k \in N(v_j)\backslash\{v_i\}\} \text{ and } d_{\max}^{(j)} = \max\{d_k|v_k \in N(v_j)\backslash\{v_i\}\}.$$

Note that one may give similar definitions when v_i also is a non–pendent vertex of G. Next, by many helpful techniques provided in [3, 6] and some analysis, we first provide a sufficient condition for $GA(G) > GA(G - e)$ when $e = v_iv_j$ is a pendent edge of G.

Theorem 12. Assume that G is a simple graph and $e = v_iv_j$ is a pendent edge of G with non-pendent vertex v_j. If one of the following conditions is satisfied, then $GA(G) > GA(G - e)$:

(i) $d_{\min}^{(j)} \geq d_j$, or

(ii) $\frac{\sqrt{d_{\max}^{(j)}}}{2\sqrt{d_j - \frac{1}{2} + 6d_{\max}^{(j)}}} \leq \frac{\sqrt{d_j}}{d_j+1}$.

Proof. Since $e = v_iv_j$ is a pendent edge of G with non–pendent vertex v_j, then

$$GA(G) - GA(G - e) = 2 \sum_{v_k \in N(v_j)\backslash\{v_i\}} \left(\frac{\sqrt{d_jd_k}}{d_j + d_k} - \frac{\sqrt{(d_j - 1)d_k}}{d_j + d_k - 1} \right) + 2\frac{\sqrt{d_j}}{d_j + 1}. \quad (22)$$

If $d_{\min}^{(j)} \geq d_j$, then $d_k \geq d_j$ for any $v_k \in N(v_j)\backslash\{v_i\}$. Notice that $f(x) = \frac{1}{x+\frac{1}{x}}$ is an increasing function for $x \in (0, 1]$. Thus one can easily see that

$$\frac{\sqrt{(d_j - 1)d_k}}{d_j + d_k - 1} - \frac{\sqrt{d_jd_k}}{d_j + d_k} < 0, \quad (23)$$

which implies that $GA(G) > GA(G - e)$. Hence, the (i) follows.

Now suppose that $d_k \leq d_j - 1$ for some $v_k \in N(v_j)\backslash\{v_i\}$. Then

$$\begin{aligned}
&\frac{\sqrt{(d_j - 1)d_k}}{d_j + d_k - 1} - \frac{\sqrt{d_jd_k}}{d_j + d_k} \\
&\quad = \frac{\sqrt{d_k}}{(d_j + d_k)(d_j + d_k - 1)} \left(\sqrt{d_j - 1}(d_j + d_k) - \sqrt{d_j}(d_j + d_k - 1)\right) \\
&\quad \leq \frac{\sqrt{d_k}\left(\sqrt{d_j - 1}(d_j + d_k) - (d_j + d_k - 1)(\sqrt{d_j - 1} + \frac{1}{2\sqrt{d_j-1}} - \frac{1}{8(d_j-1)^{3/2}})\right)}{(d_j + d_k)(d_j + d_k - 1)} \\
&\quad = \frac{\sqrt{d_k}}{(d_j + d_k)(d_j + d_k - 1)} \left(d_j - d_k - 1 + \frac{d_j + d_k - 1}{4(d_j - 1)}\right) \frac{1}{2\sqrt{d_j - 1}} \\
&\quad \leq \frac{\sqrt{d_k}}{2(d_j - 1)\sqrt{d_j - \frac{1}{2} + 6d_k}} \cdot \frac{(d_j - d_k - \frac{1}{2})\sqrt{d_j - 1}\sqrt{d_j - \frac{1}{2} + 6d_k}}{(d_j + d_k)(d_j + d_k - 1)}
\end{aligned}$$

$$
\begin{aligned}
&< \frac{\sqrt{d_k}}{2(d_j-1)\sqrt{d_j-\frac{1}{2}+6d_k}} \cdot \frac{(d_j-d_k-\frac{1}{2})(d_j+3d_k-\frac{1}{2})}{(d_j+d_k)(d_j+d_k-1)} \\
&< \frac{\sqrt{d_k}}{2(d_j-1)\sqrt{d_j-\frac{1}{2}+6d_k}} \\
&\leq \frac{\sqrt{d_{\max}^{(j)}}}{2(d_j-1)\sqrt{d_j-\frac{1}{2}+6d_{\max}^{(j)}}}, \qquad (24)
\end{aligned}
$$

where the last inequality holds as $d_{\max}^{(j)} \geq d_k$ and $f(x) = \frac{\sqrt{x}}{\sqrt{a+6x}}$ is an increasing function in $x > 0$ whenever $a > 0$. Now, using (23) and (24), one may get

$$
\sum_{v_k \in N(v_j)\setminus\{v_i\}} \left(\frac{\sqrt{(d_j-1)d_k}}{d_j+d_k-1} - \frac{\sqrt{d_jd_k}}{d_j+d_k} \right) < \frac{\sqrt{d_{\max}^{(j)}}}{2\sqrt{d_j-\frac{1}{2}+6d_{\max}^{(j)}}}. \qquad (25)
$$

Therefore, if the condition (ii) is satisfied, then $GA(G) > GA(G-e)$.

The proof is complete. ■

Remark that, in Example 3, it is easy to verify that G_1 satisfies the condition (ii) of Theorem 12. So, $GA(G_1) > GA(G_1-e)$. However, G_2 does not satisfy each one of conditions of Theorem 11. Hence, Theorem 12 is an improvement on Theorem 11 when e is a pendent edge of a graph.

Theorem 13. Assume that G is a graph with non–pendent edge $e = v_iv_j$. If one of the following conditions is satisfied, then $GA(G) > GA(G-e)$:

(i) $\max\left\{\frac{d_i}{d_{\min}^{(i)}}, \frac{d_j}{d_{\min}^{(j)}}\right\} \leq 1$, or

(ii) $\max\left\{\frac{d_i}{d_j}, \frac{d_j}{d_i}\right\} \leq \min\left\{\frac{d_i-\frac{1}{2}}{d_{\max}^{(i)}}, \frac{d_j-\frac{1}{2}}{d_{\max}^{(j)}}\right\}$.

Proof. Since $e = v_iv_j$ is a non–pendent edge of G. Then, from the definition of GA index, one gets

$$
\begin{aligned}
GA(G) - GA(G-e) = 2 &\sum_{v_l \in N(v_i)\setminus\{v_j\}} \left(\frac{\sqrt{d_id_l}}{d_i+d_l} - \frac{\sqrt{(d_i-1)d_l}}{d_i+d_l-1} \right) \\
&+ 2 \sum_{v_k \in N(v_j)\setminus\{v_i\}} \left(\frac{\sqrt{d_jd_k}}{d_j+d_k} - \frac{\sqrt{(d_j-1)d_k}}{d_j+d_k-1} \right) + 2\frac{\sqrt{d_id_j}}{d_i+d_j}. \qquad (26)
\end{aligned}
$$

If G satisfies the condition (i), then $d_l \geq d_i$ for any $v_l \in N(v_i)\setminus\{v_j\}$ and $d_k \geq d_j$ for any $v_k \in N(v_j)\setminus\{v_i\}$. Now, from (23), one can easily see that

$$
\frac{\sqrt{(d_i-1)d_l}}{d_i+d_l-1} - \frac{\sqrt{d_id_l}}{d_i+d_l} < 0,
$$

and

$$\frac{\sqrt{(d_j-1)d_k}}{d_j+d_k-1}-\frac{\sqrt{d_jd_k}}{d_j+d_k}<0.$$

So, it follows from (26) that $GA(G) > GA(G-e)$. Otherwise, again from (23) and (24), one has

$$\sum_{v_k\in N(v_j)\setminus\{v_i\}}\left(\frac{\sqrt{(d_j-1)d_k}}{d_j+d_k-1}-\frac{\sqrt{d_jd_k}}{d_j+d_k}\right)<\frac{\sqrt{d_{\max}^{(j)}}}{2\sqrt{d_j-\frac{1}{2}+6d_{\max}^{(j)}}}.$$

Similarly,

$$\sum_{v_l\in N(v_i)\setminus\{v_j\}}\left(\frac{\sqrt{(d_i-1)d_l}}{d_i+d_l-1}-\frac{\sqrt{d_id_l}}{d_i+d_l}\right)<\frac{\sqrt{d_{\max}^{(i)}}}{2\sqrt{d_i-\frac{1}{2}+6d_{\max}^{(i)}}}.$$

The following proof is similar to that of Theorem 3.4 in [3]. For the convenience of readers, here we give the detailed proof. Let $t_1=\max\left\{\frac{d_i}{d_j},\frac{d_j}{d_i}\right\}$, $t_2=\frac{d_i-\frac{1}{2}}{d_{\max}^{(i)}}$ and $t_3=\frac{d_j-\frac{1}{2}}{d_{\max}^{(j)}}$. Without loss of generality, assume that $t_2\le t_3$. The condition (ii) implies that $1\le t_1\le t_2\le t_3$. After some rearrangements, one has

$$\frac{\sqrt{d_{\max}^{(i)}}}{2\sqrt{d_i-\frac{1}{2}+6d_{\max}^{(i)}}}+\frac{\sqrt{d_{\max}^{(j)}}}{2\sqrt{d_j-\frac{1}{2}+6d_{\max}^{(j)}}}=\frac{1}{2\sqrt{t_2+6}}+\frac{1}{2\sqrt{t_3+6}}$$

$$\le\frac{1}{\sqrt{t_2+6}}\le\frac{\sqrt{t_1}}{t_1+1}=\frac{\sqrt{d_id_j}}{d_i+d_j}.$$

Hence, it follows from (26) that $GA(G) > GA(G-e)$. The proof is complete. ■

Remark that, in Example 2, $d_i = 100$, $d_j = 500$, $d_{\max}^{(i)} = d_{\max}^{(j)} = 2$. Clearly, G_1 satisfies the condition (ii) of Theorem 13. So, $GA(G) > GA(G-e)$. Hence, Theorem 13 is an improvement on Theorem 11. In addition, if G has an edge $e = v_iv_j$ with the property (i) in Theorem 13, we say e is an *ascending edge* of G.

Corollary 6. If $e = v_iv_j$ is an ascending edge of G, then $GA(G) > GA(G-e)$.

4.2. Effect on *AG* index of deleting an edge

Theorem 14. Let $e = v_iv_j$ be an edge of a graph G with non–pendent vertex v_j. If one of the following conditions is satisfied, then $AG(G) > AG(G-e)$:

(i) $\min\left\{\frac{d_i}{d_{\max}^{(i)}},\frac{d_j}{d_{\max}^{(j)}}\right\}>1$, or

(ii) $\frac{d_{\max}^{(i)}-d_i+1}{2\sqrt{d_{\max}^{(i)}}\sqrt{d_i}}+\frac{d_{\max}^{(j)}-d_j+1}{2\sqrt{d_{\max}^{(j)}}\sqrt{d_j}}\le\frac{d_i+d_j}{\sqrt{d_id_j}}$,

where $d_i/d_{\max}^{(i)}$ is stipulated as $\propto$ and $\frac{d_{\max}^{(i)}-d_i+1}{2\sqrt{d_{\max}^{(i)}}\sqrt{d_i}}=\frac{d_{\max}^{(i)}}{2}=0$ when v_i is a pendent vertex.

Proof. First suppose that $e = v_iv_j$ is a non–pendent edge of G. Then, in the light of the definition of GA index,

$$AG(G) - AG(G-e) = \frac{1}{2}\sum_{v_k\in N(v_i)\backslash\{v_j\}}\left(\frac{d_i+d_k}{\sqrt{d_id_k}} - \frac{d_i+d_k-1}{\sqrt{(d_i-1)d_k}}\right)$$

$$+\frac{1}{2}\sum_{v_l\in N(v_j)\backslash\{v_i\}}\left(\frac{d_j+d_l}{\sqrt{d_jd_l}} - \frac{d_j+d_l-1}{\sqrt{(d_j-1)d_l}}\right) + \frac{d_i+d_j}{2\sqrt{d_id_j}}.$$

If the graph G satisfies the condition (i), then $d_i > d_k$ for any $v_k \in N(v_i)\backslash\{v_j\}$ and $d_j > d_l$ for any $v_l \in N(v_j)\backslash\{v_i\}$. Since $f(x) = x + \frac{1}{x}$ is an increasing function for $x \geq 1$. Thus, one has

$$\frac{d_i+d_k}{\sqrt{d_id_k}} > \frac{d_i+d_k-1}{\sqrt{(d_i-1)d_k}} \tag{27}$$

and

$$\frac{d_j+d_l}{\sqrt{d_jd_l}} > \frac{d_j+d_l-1}{\sqrt{(d_j-1)d_l}}.$$

Hence, $AG(G) > AG(G-e)$.

If $d_k \geq d_i > 1$ for some $v_k \in N(v_i)\backslash\{v_j\}$, then

$$\begin{aligned}\frac{d_i+d_k-1}{\sqrt{(d_i-1)d_k}} - \frac{d_i+d_k}{\sqrt{d_id_k}} &= \frac{\sqrt{d_i}\sqrt{(d_i-1)}(d_i+d_k-1)-(d_i-1)(d_i+d_k)}{(d_i-1)\sqrt{d_k}\sqrt{d_i}} \\ &< \frac{(d_i-\frac{1}{2})(d_i+d_k-1)-(d_i-1)(d_i+d_k)}{(d_i-1)\sqrt{d_k}\sqrt{d_i}} \\ &= \frac{d_k-d_i+1}{2(d_i-1)\sqrt{d_k}\sqrt{d_i}} \\ &\leq \frac{d_{\max}^{(i)}-d_i+1}{2(d_i-1)\sqrt{d_{\max}^{(i)}}\sqrt{d_i}},\end{aligned} \tag{28}$$

where the last inequality holds as $d_{\max}^{(i)} \geq d_k$ and $f(x) = \frac{x-a}{b\sqrt{x}}$ $(a, b > 0)$ is an increasing function for $x \geq 0$. Hence, from (27) and (28), one has

$$\sum_{v_k\in N(v_i)\backslash\{v_j\}}\left(\frac{d_i+d_k-1}{\sqrt{(d_i-1)d_k}} - \frac{d_i+d_k}{\sqrt{d_id_k}}\right) < \frac{d_{\max}^{(i)}-d_i+1}{2\sqrt{d_{\max}^{(i)}}\sqrt{d_i}}.$$

Similarly,

$$\sum_{v_l\in N(v_j)\backslash\{v_i\}}\left(\frac{d_j+d_l-1}{\sqrt{(d_j-1)d_l}} - \frac{d_j+d_l}{\sqrt{d_jd_l}}\right) < \frac{d_{\max}^{(j)}-d_j+1}{2\sqrt{d_{\max}^{(j)}}\sqrt{d_j}}.$$

Therefore, if G is a graph satisfying the condition (ii), then $AG(G) > AG(G-e)$.

If $e = v_iv_j$ is a pendent edge, then v_i is a pendent vertex. After a simple check, the result still holds. The proof is complete. ■

Let H be any graph of order $n-2$ with maximum degree $\Delta(H) < n-3$ and $G = K_2 \vee H$. If $e = v_iv_j$ is the edge with $d_i = d_j = n-1$, then $AG(G) > AG(G-e)$. We say $e = v_iv_j$ is a *descending edge* of G if the edge e has the property (i) in Theorem 14.

Corollary 7. If $e = v_iv_j$ is a descending edge of G, then $AG(G) > AG(G-e)$.

Example 4. For given two graphs G_1, G_2 in Figure 4, one can see that $AG(G_1) - AG(G_1 - e) = -1.0170$, whereas $AG(G_2) - AG(G_2 - e) = 0.6309$. In fact, G_2 satisfies the (ii) of Theorem 14. This example also shows that AG index may either increase or decrease when an ascending edge e is deleted from a graph. However, Corollary 6 implies that GA indices of G_1 and G_2 are all decrease when the ascending edge e is deleted. So there are considerable differences between GA and AG indices of graphs.

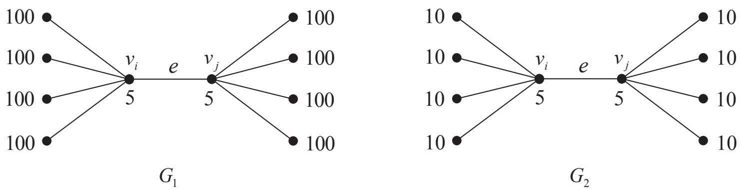

Figure 4. In the case of G_1 the AG index increases, whereas in the case of G_2 the AG index decreases.

Finally, we suggest the following problem.

Problem 1. Is there a graph G such that $GA(G) = GA(G-e)$ or $AG(G) = AG(G-e)$ for some edge $e \in E(G)$?

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