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On the Lanzhou Index

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Abstract

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The Lanzhou index of a graph G is defined by $\mathrm{Lz}(G) = \sum_{u\in V(G)} d_{\overline{G}}(v)d_G^2(v)$, where $d_G(v)$ denotes the degree of the vertex v in G. In this paper, we study the Lanzhou index of several classes of hexagonal systems. Moreover, Lanzhou index of trees with some given diameters are obtained. Finally, we get this index for Cartesian product graphs and Nordhaus-Gaddum-type results.

1 Introduction

In a chemical graph, vertices represent atoms and edges represent bonds between atoms. The topological index of a chemical graph G is a number that is fixed under the automorphism of G. When analyzing the total electron energy, Balaban et al. [1] introduced the Zagreb group index (now call it the first Zagreb index). After that, Furtula and

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Gutman [5] introduced the forgotten index, and Kazemi & Behtoei [6] showed that the forgotten index can be used to test the chemical and pharmacological properties of drug molecular structures. In 2018, Vukičević et al. [8] introduced a new topological index, and showed that it behaves better than the existing ones in predicting a chemically relevant property. The index is named by Lanzhou index, and the *Lanzhou index* [8] of a graph G is defined by

$$\mathrm{Lz}(G) = \sum_{v \in V(G)} d_{\overline{G}}(v) d_G^2(v),$$

where $d_G(v)$ denotes the degree of the vertex v in G.

Recently, Bera and Das [2] obtained some bounds for the Lanzhou index of the first, second and third Zagreb indices, radius, eccentric connectivity index, Schultz index, inverse sum indeg index and symmetric division deg index, and the Lanzhou index of corona and joined graphs.

In Section 2, we study the Lanzhou index of several known classes of hexagonal systems. Moreover, Lanzhou index of trees with some given diameters are obtained in Section 3. In Section 4, we get this index for Cartesian product graphs and Nordhaus-Gaddum-type results.

2 Results for hexagonal systems

In 1997, Klavžar et al. [7] determined the Wiener index of several classes of hexagonal systems (called benzenoid systems in [7]). A general hexagonal system [7] denoted by $GH(m, k_1, k_2, k_3, k_4)$ is shown as in Figure 1, where $m \geq 1$ is the number of benzenoids in the lowest layer, $0 \leq k_1 \leq k_3 \leq m$, $0 \leq k_4 \leq k_2 \leq m$ and $k_1 + k_2 = k_3 + k_4$.

Theorem 1. *Let $m \geq 1$, $0 \leq k_1 \leq k_3 \leq m$, $0 \leq k_4 \leq k_2 \leq m$ and $k_1 + k_2 = k_3 + k_4$. Then the Lanzhou index of a general hexagonal system $GH(m, k_1, k_2, k_3, k_4)$ is*

$$\mathrm{Lz}(GH(m, k_1, k_2, k_3, k_4)) = 9n_3(n_2 + n_3 - 4) + 4n_2(n_2 + n_3 - 3),$$

where $n_3 = 2m(k_1+k_2+1)+k_1(k_1+2k_2+1)-k_4(k_4+1)-2$ and $n_2 = 2m+3k_1+2k_2-k_4+4$.

Proof. We consider the number of vertices of degree 3 firstly. Except $m-1$ vertices of degree 3 on the bottom and $m+k_1-k_4-1$ on the top, the remaining vertices of degree 3 can be divided to three parts, say T, P, T'. Note that T is a trapezium, and the all vertices in T are on k_1 layers, and the number of vertices of degree 3 is $2m+1$ on the lowest layer

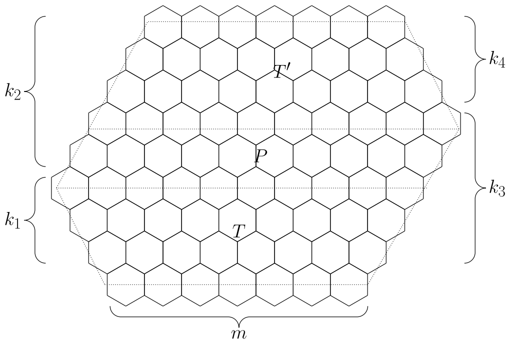


Figure 1. A general hexagonal system $GH(m, k_1, k_2, k_3, k_4)$.

and $2(m+k_1-1)+1$ on the highest layer. Then the number of vertices of degree 3 in T is $\sum_{j=1}^{k_1}\big(2(m+j-1)+1\big) = k_1(2m+k_1)$. Similarly, the number of vertices of degree 3 in T' is $k_4(2m+2k_1-k_4)$. Since P is a parallelogram and the number of vertices on every layer is $2(m+k_1)$, it follows that the number of all vertices in P is $2(m+k_1)(k_3-k_1)$, and hence the number of all vertices of degree 3 in $GH(m, k_1, k_2, k_3, k_4)$ is

$$2m(k_1+k_2+1)+k_1(k_1+2k_2+1)-k_4(k_4+1)-2.$$

Furthermore, the number of vertices of degree 2 on the bound of $GH(m, k_1, k_2, k_3, k_4)$ is $m+2+k_1+1+k_2+1+k_3+1+k_4+1+m+k_1-k_4-2 = 2m+3k_1+2k_2-k_4+4$. Then the result follows. ■

Due to parallelogram, trapezium, bitrapezium and corona hexagonal systems [7] denoted by $P(m,n)$, $T(m,n)$, $BT(m,k_1,k_2)$ and H_m are the special cases of general hexagonal system, that is, $P(m,n) = GH(m,0,n-1,n-1,0)$, $T(m,n) = GH(m,0,n-1,0,n-1)$, $BT(m,k_1,k_2) = GH(n-k_1,k_1,k_2,k_1,k_2)$ and $H_m = GH(m,m-1,m-1,m-1,m-1) = BT(2m-1,m-1,m-1)$, and so Corollaries 2, 3 and 4 are immediate.

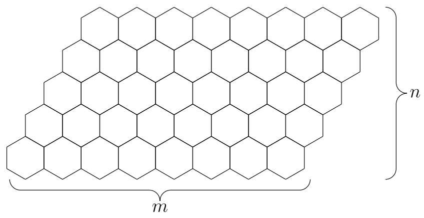


Figure 2. A parallelogram hexagonal system $P(m,n)$.

Corollary 2. *Let m,n be two positive integers. Then the Lanzhou index of a parallelogram hexagonal system $P(m,n)$ as shown in Figure 2 is*

$$Lz(P(m,n)) = 8(m+n+1)(2mn+2m+2n-3) + 36(mn-1)(mn+m+n-2).$$

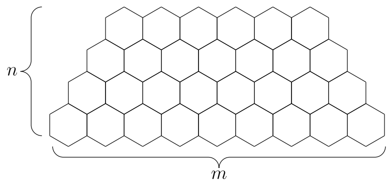


Figure 3. A trapezium hexagonal system $T(m,n)$.

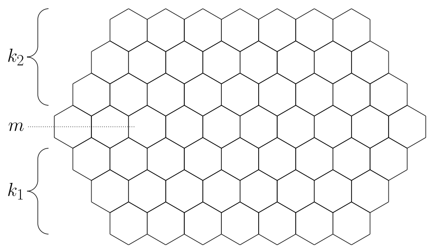


Figure 4. A bitrapezium hexagonal system $BT(m,k_1,k_2)$.

Corollary 3. *Let m and n be positive integers. The Lanzhou index of a trapezium hexagonal system $T(m,n)$ as shown in Figure 3 is*

$$\begin{aligned} \mathrm{Lz}(T(m,n)) &= 9(2mn-m^2+m-2)(2m+2n+2mn-m^2-3) \\ &\quad +4(m+2n+3)(2m+2n+2mn-m^2-2). \end{aligned}$$

For two non-negative integers k_1 and k_2. The Lanzhou index of bitrapezium hexagonal system $BT(m, k_1, k_2)$ as shown in Figure 4 is

$$\begin{aligned}\mathrm{Lz}(BT(m,k_1,k_2)) &= 9(2mn - n - k_1^2 - k_2^2 - 1)(2m(n+1) - k_1^2 - k_2^2 - 2)\\ &\quad 4(2m + n + 3)(2m(n+1) - k_1^2 - k_2^2 - 1),\end{aligned}$$

where $n = k_1 + k_2 + 1$.

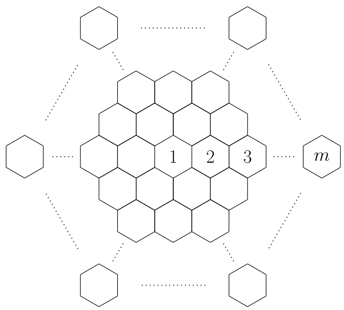


Figure 5. A corona H_m.

Corollary 4. *Let m be a positive integer. Then the Lanzhou index of corona H_m as shown in Figure 5 is*

$$Lz(H_m) = 324m^4 - 180m^3 - 216m^2 + 144m.$$

Let $P_j(m, n)$ be the parallelogram-like hexagonal system [7] of type j as shown in Figure 6 for $j = 1, 2, 3$. A simple calculation gives Proposition 5.

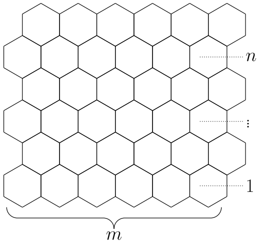
Proposition 5. *Let m and n be positive integers. Then*

$$\mathrm{Lz}(P_1(m,n)) = 36(2mn - 1)(m + 2n + 2mn - 2) + 8(m + 2n + 1)(2m + 4n + 4mn - 3);$$

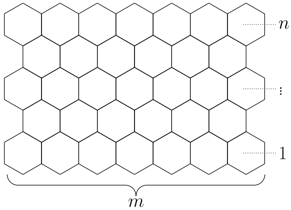
$$\mathrm{Lz}(P_2(m,n)) = 36(2mn - m - n)(n + 2mn - 2) + 8(m + 2n)(2n + 4mn - 3);$$

$$\mathrm{Lz}(P_3(m,n)) = 36(2mn - m + n - 2)(3n + 2mn - 4) + 8(m + 2n)(6n + 4mn - 7).$$

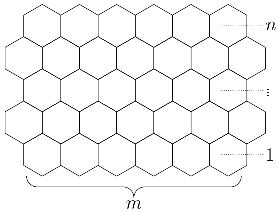
A vertex shared by three hexagons is called an *internal vertex* of the respective hexagonal system. A hexagonal system is said to be *catacondensed* if it does not possess internal vertices, and thus the inner dual graph (obtained from dual graph [3] by deleting the vertex corresponding to outer face) is a tree.



(a): $P_1(m, n)$ of Type 1.



(b): $P_2(m, n)$ of Type 2.



(c): $P_3(m, n)$ of Type 3.

Figure 6. Three parallelogram-like hexagonal systems.

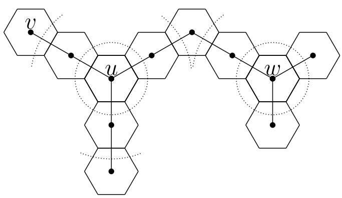


Figure 7. A catacondensed hexagonal system with inner dual graph.

Theorem 6. *Let n be a positive integer, and let HS_n denote a catacondensed hexagonal system with n hexagons. Then*

$$\mathrm{Lz}(HS_n) = 104n^2 - 52n + 20.$$

Proof. Let T_n denote the tree corresponding to the inner dual graph of a hexagonal system HS_n. Clearly $\Delta(T_n) \leq 3$ and let n_3 be the number of vertices of degree 3 in T_n. If $n_3 > 0$, then the number of vertices of degree 1 in T_n is $n_3 + 2$. As shown in Figure 7, we divide all vertices of HS_n to n parts.

If $\deg(v) = 1$, $\deg(u) = 3$ and $\deg(s_j) = 2$ for $j = 1, 2, \ldots, k$ in uv-path $us_1s_2 \cdots s_kv$, then there are two vertices of degree 2 and two vertices of degree 3 in HS_n corresponding to s_j for $j = 1, 2, \ldots, k$. If $\deg(w) = 3$ and $\deg(t_j) = 2$ for $j = 1, 2, \ldots, m$ in uw-path $ut_1t_2 \cdots t_mw$, then exists just one vertex $t' \in \{t_1, t_2, \ldots, t_m\}$ such that there are two remaining vertices of degree 2 in HS_n corresponding to t'; and there are two vertices of degree 2 and two vertices of degree 3 in HS_n corresponding to every vertex $t'' \in \{t_1, t_2, \ldots, t_m\} \setminus \{t'\}$. It is difficult to see that there are exactly $n_3 - 1$ paths like uw-path.

Note that the degree of vertices in a hexagon in HS_n corresponding to a vertex of degree 3 in T_n are 3; and there are four vertices of degree 2 in HS_n corresponding to a vertex of degree 1 in T_n.

Therefore, the number of vertices of degree 3 in HS_n is $6n_3 + 2(n - 2n_3 - 2) - 2(n_3 - 1) = 2(n-1)$, and the number of vertices of degree 2 in HS_n is $4(n_3 + 2) + 2(n - 2n_3 - 2) = 2(n+2)$. Hence $|V(HS_n)| = 2(2n+1)$.

It is not difficult to verify that the above results hold on $n_3 = 0$, and hence

$$\mathrm{Lz}(HS_n) = 2(n-1)(2(2n+1)-1-3)3^2 + 2(n+2)(2(2n+1)-1-2)2^2 = 104n^2 - 52n + 20.$$

■

3 Trees with given diameter

The *double star* $S(a, b)$ for integers $a \geq b \geq 2$ is the graph obtained from the union of two stars $K_{1,a-1}$ and $K_{1,b-1}$ by adding the edge between their centers.

Proposition 7. *Let a and b are positive integers, and let $S(a, b)$ be a double star of order $a + b$. Then*

$$\mathrm{Lz}(S(a, b)) = (a + b)(ab - 4) + 2ab + 4.$$

Proof. Since the numbers of vertices of degree a and b are 1 in $S(a,b)$ respectively, it follows that the Lanzhou index of $S(a,b)$ is

$\mathrm{Lz}(S_{a,b}) = (b-1)a^2 + (a-1)b^2 + (a+b-2)(a+b-1-1)1^2 = (a+b)(ab-4)+2ab+4.$

■

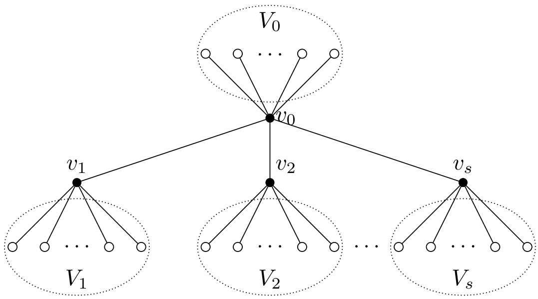


Figure 8. A tree of diameter 4.

Proposition 8. *Let $S_{r,s,t}$ $(r+s+t=n-1)$ be a tree of diameter 4 as shown in Figure 8, where $|V_0|=r$, $|V_i|=t_i$ for $i=1,2,\ldots,s$ and $t=\sum_{i=1}^{s} t_i$. Then*

$$\mathrm{Lz}(S_{r,s,t}) = (r+t)(n-2) + t(r+s)^2 + \sum_{i=1}^{s}(n-2-t_i)(t_i+1)^2.$$

Proof. As shown in Figure 8, $d_{v_0} = r+s$, $d_{v_i} = t_i+1$ for $i=1,2,\ldots,s$. Note that the number of vertices of degree 1 is $r+t$, and so

$$\begin{aligned}\mathrm{Lz}(S_{r,s,t}) &= (r+t)(n-1-1)1^2 + (n-1-r-s)(r+s)^2 + \sum_{i=1}^{s}(n-1-t_i-1)(t_i+1)^2\\ &= (r+t)(n-2) + t(r+s)^2 + \sum_{i=1}^{s}(n-2-t_i)(t_i+1)^2.\end{aligned}$$

■

Theorem 9. *Let $P_r(m_1,m_2,\ldots,m_r)$ denote the graph obtained from P_r by adding m_i pendent edges on the i-th vertex of P_r for $i=1,2,\ldots,r$. Then*

$$\mathrm{Lz}(P_r(m_1,m_2,\ldots,m_r)) = m(n-2) + \sum_{i=1,r}(n-m_i-2)(m_i+1)^2 + \sum_{i=2}^{r-1}(n-m_1-3)(m_1+2)^2,$$

where $m=\sum_{j=1}^{r} m_j$ and $n=m+r$.

Proof. Without loss of generality, assume that $V(P_r) = \{v_1, v_2, \ldots, v_r\}$, $d_{v_1} = m_1 + 1$, $d_{v_r} = m_r + 1$ and $d_{v_i} = m_i + 2$ for $i = 2, 3, \ldots, r-1$. Hence,

$$\begin{aligned}\mathrm{Lz}(P_r(m_1, m_2, \ldots, m_r)) &= m(m + r - 1 - 1)1^2 + (m + r - 1 - m_1 - 1)(m_1 + 1)^2 \\ &+ (m + r - 1 - m_r - 1)(m_r + 1)^2 \\ &+ \sum_{i=2}^{r-1}(m + r - 1 - m_1 - 2)(m_1 + 2)^2 \\ &= m(n-2) + \sum_{i=1,r}(n - m_i - 2)(m_i + 1)^2 \\ &+ \sum_{i=2}^{r-1}(n - m_1 - 3)(m_1 + 2)^2.\end{aligned}$$

■

4 Results for Cartesian product graphs and Nordhaus–Gaddum results

In this section, we derive some other results on the Lanzhou index. The Cartesian product [3] is an operation studied widely in graph theory. The Cartesian product of P_5, P_4 and P_3 is shown in Figure 9.

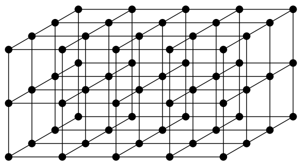


Figure 9. The Cartesian product $P_5 \Box P_4 \Box P_3$.

Theorem 10. *Let* $\prod_{j=1}^{k} P_{m_j+2} = P_{m_1+2} \Box P_{m_2+2} \Box \cdots \Box P_{m_k+2}$ *denote the Cartesian product of paths* $\{P_{m_j+2}\}_{j=1}^{k}$*, where k is a positive integer and m_j is a non-negative integer for* $j = 1, 2, \ldots, k$*. Then*

$$\mathrm{Lz}\left(\prod_{j=1}^{k} P_{m_j+2}\right) = \sum_{d=0}^{k}(n - 1 - k - d)(k + d)^2 2^{k-d} \sum_{J \in [k]^d} \prod_{j \in J} m_j,$$

where $[k]^d = \{S : S \subseteq \{1, 2, \ldots, k\} \text{ and } |S| = d\}$ *and* $n = \prod_{j=1}^{k}(m_j + 2)$*.*

Especially, if $m_j = 0$ *for* $j = 1, 2, \ldots, k$*, hence* $\Delta = k$*, then the Cartesian product is a k-dimension cube, and the Lanzhou index is* $2^k k^2(2^k - k - 1)$*; if* $m_j = m > 0$ *for*

$j = 1, 2, \ldots, k$, hence $\Delta = 2k$, then the Lanzhou index is

$$\sum_{d=0}^{k}((m+2)^k - 1 - k - d)(k+d)^2 2^{k-d}\binom{k}{d}m^d.$$

Proof. Note that there are two vertices of degree 1, and m_j vertices of degree 2 for $j = 1, 2, \ldots, k$. Then the generating function of degree sequence in $\prod_{j=1}^{k} P_{m_j+2}$ is

$$\prod_{j=1}^{k}(2x + m_j x^2) = x^k \prod_{j=1}^{k}(2 + m_j x),$$

and hence $\delta = k$ and $\Delta \le 2k$. Similarly to the Binomial Theorem, the number of vertices of degree $k + d$ is

$$2^{k-d}\sum_{J\in[k]^d}\prod_{j\in J} m_j,$$

for $d = 0, 1, \ldots, k$. Observe that the order of the Cartesian product is $n = \prod_{j=1}^{k}(m_j + 2)$. The result follows. ■

Proposition 11. *Let G be a graph of order n. Then*

$$0 \le \mathrm{Lz}(G) + \mathrm{Lz}(\overline{G}) \le \frac{1}{4}n(n-1)^3.$$

The left equality is satisfied if and only if G is either complete or empty graph. The right equality is satisfied if and only if n is odd and G is $(n-1)/2$-regular.

Proof. From the definition of Lanzhou index, the left inequality is immediate. We only consider the right inequality as follows.

$$\begin{aligned}
\mathrm{Lz}(G) + \mathrm{Lz}(\overline{G}) &= \sum_{v\in V(G)}(n-1-\deg_G(v))\deg_G(v)^2 + \sum_{v\in V(\overline{G})}\deg_G(v)(n-1-\deg_G(v))^2\\
&= (n-1)\sum_{v\in V(G)}(n-1-\deg_G(v))\deg_G(v)\\
&\le (n-1)\sum_{v\in V(G)}\left(\frac{n-1-\deg_G(v)+\deg_G(v)}{2}\right)^2\\
&= \frac{1}{4}n(n-1)^3.
\end{aligned}$$

The right equality is satisfied if and only if $n-1-\deg_G(v) = \deg_G(v)$ for all $v \in V(G)$, that is, $\deg_G(v) = (n-1)/2$ for every $v \in V(G)$. The result follows. ■

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