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Results on Two Kinds of Steiner Distance–Based Indices for Some Classes of Graphs

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Abstract

The kth Steiner Szeged index and the Steiner k-Wiener index are defined from Steiner distance, in order to generalize the Szeged index and the Wiener index, respectively. These two indices are aimed to analyze the connectedness of each k-subset of the vertex set. In this paper we first give a counterexample for a conjecture on the kth Steiner Szeged index of trees. Then, we calculate the kth Steiner Szeged index and the Steiner k-Wiener index for cycles, by establishing a correspondence to a special integer partition problem. In the end, we calculate the kth Steiner Szeged index and the Steiner k-Wiener index for wheels.

1 Introduction

All graphs in this paper are assumed to be finite, simple and undirected. We refer the reader to [1] for terminology and notation not explained here. The *degree* of a vertex is the number of edges incident with it. A *walk* $v_0e_1v_1e_2\cdots v_{l-1}e_lv_l$ is a vertex-edge alternative sequence beginning and ending with vertices and $e_i = v_{i-1}v_i$ for each $i \in \{1, \cdots, l\}$. A graph is *connected* if and only if for each pair of vertices in $V(G)$, there is a walk connecting them. For a subset $S \subseteq V(G)$, the graph $G[S]$ is $(S, E(G[S]))$, where $E(G[S])$ consists of all the edges in $E(G)$ with both ends lying in S. And the graph $G\backslash S$ is $(V(G) - S,\ E(G\backslash S))$, where $E(G\backslash S)$ consists of all the edges with no end lying in S. A *tree* is a connected graph with n vertices and $n - 1$ edges. A *cycle* C_n is a connected graph with n vertices, each of whose degree is 2. A *path* P_n is a tree with n vertices, formed from deleting an arbitrary edge from C_n. A *star* S_n is a tree with n vertices, among which one vertex has degree $n-1$ and the others have degree 1. A *wheel* W_n is a graph with n vertices, formed by connecting a single universal vertex to all the vertices of C_{n-1}. For $u, v \in V(G)$, the *distance* $d_G(u,v)$ counts the number of edges of a shortest path connecting u and v in G. Given an edge $e = uv \in E(G)$, consider a partition of $V(G)$ as follows:

$$N_u(e) = \{w \in V(G) : d_G(u,w) < d_G(v,w)\},$$
$$N_0(e) = \{w \in V(G) : d_G(u,w) = d_G(v,w)\},$$
$$N_v(e) = \{w \in V(G) : d_G(u,w) > d_G(v,w)\}.$$

And we denote the cardinality of $N_u(e)$, $N_0(e)$ and $N_v(e)$ by $n_u(e)$, $n_0(e)$ and $n_v(e)$, respectively.

In 1947, Wiener [16] introduced the *Wiener index* $W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)$ as a graph invariant. And in [7] an expansion form of the Wiener index for partial cubes was deduced: $W(G) = \sum_{e=uv \in E(G)} n_u(e)n_v(e)$. Motivated by this symmetric form, Gutman [6] introduced another graph invariant, the *Szeged index* as $Sz(G) = \sum_{e=uv \in E(G)} n_u(e)n_v(e)$. Shortly afterwards, Randić [13] raised a modified version of the Szeged index, i.e., the *revised Szeged index*, $Sz^*(G) = \sum_{e=uv \in E(G)}(n_u(e) + \frac{1}{2}n_0(e))(n_v(e) + \frac{1}{2}n_0(e))$. There are lots of results about the indices mentioned above so far; see [8, 11, 15, 17] for example.

On the other hand, Chartrand et al. generalized the distance to the *Steiner distance* $d_G(S)$ in [2]: For each subset $S \subseteq V(G)$, $d_G(S)$ is the size of a minimum subtree connecting S and we denote the subtree by T_S. For an arbitrary edge $e = uv \in E(G)$ and an integer k $(2 \leq k \leq |V(G)| - 1)$, we can similarly construct three distinct kinds of $(k-1)$-subsets of $V(G)$ as follows:

$$N_u(e,k) = \{S \subseteq V(G), |S| = k-1 : d_G(S \cup \{u\}) < d_G(S \cup \{v\}), u, v \notin S\},$$
$$N_0(e,k) = \{S \subseteq V(G), |S| = k-1 : d_G(S \cup \{u\}) = d_G(S \cup \{v\}), u, v \notin S\},$$
$$N_v(e,k) = \{S \subseteq V(G), |S| = k-1 : d_G(S \cup \{u\}) > d_G(S \cup \{v\}), u, v \notin S\}.$$

And we denote the cardinality of $N_u(e,k)$, $N_0(e,k)$ and $N_v(e,k)$ by $n_u(e,k)$, $n_0(e,k)$ and $n_v(e,k)$, respectively. Note that $n_u(e,2) + 1 = n_u(e)$ and $n_0(e,2) = n_0(e)$.

Let $\mathbb{N}$ be the set of nonnegative integers. By using of the Steiner distance, Li et al. generalized the Wiener index and the Szeged index naturally:

Definition 1.1. *[10] The Steiner k-Wiener index: $SW_k(G) = \sum_{\substack{S\subseteq V(G)\\ |S|=k}} d_G(S)$, where $1 \leq k \leq |V(G)|$ and $k \in \mathbb{N}$.*

Definition 1.2. *[4] The kth Steiner Szeged index: $Sz_k(G) = \sum_{e=uv\in E(G)} (n_u(e,k)+1)(n_v(e,k)+1)$, where $2 \leq k \leq |V(G)|-1$ and $k \in \mathbb{N}$.*

Definition 1.3. *[4] The kth Steiner revised Szeged index: $rSz_k(G) = \sum_{e=uv\in E(G)} (n_u(e,k)+\frac{1}{2}n_0(e,k)+1)(n_v(e,k)+\frac{1}{2}n_0(e,k)+1)$, where $2 \leq k \leq |V(G)|-1$ and $k \in \mathbb{N}$.*

Because of the difficulty of analyzing the k-subsets, the previous work was very limited, but some basic results were obtained in [3, 4, 9, 10, 12].

At first, we will disprove a conjecture of [4] in Section 2, and then calculate the two indices for cycles in Sections 3 and 4 and the indices of wheels in Section 5.

2 Disproof of a conjecture on trees

In a recent paper [4], the authors proposed the following conjecture for trees.

Conjecture 2.1. *[4] For any two trees T and T', $Sz_k(T) < Sz_k(T')$ if and only if $Sz(T) < Sz(T')$.*

We claim that this is not true. To disprove it, we first calculate $Sz(P_n)$, $Sz(S_n)$, $Sz_{n-1}(P_n)$ and $Sz_{n-1}(S_n)$; see the following:

$Sz(P_n) = \sum_{e=uv\in E(G)} n_u(e)n_v(e) = \sum_{i=1}^{n-1}(i)(n-i) = \frac{n^3-n}{6}$.

$Sz(S_n) = \sum_{e=uv\in E(G)} n_u(e)n_v(e) = (n-1)(1)(n-1) = (n-1)^2$.

$Sz_{n-1}(P_n) = \sum_{e=uv\in E(G)} (n_u(e,n-1)+1)(n_v(e,n-1)+1) = 2+n-3+2 = n+1$.

$Sz_{n-1}(S_n) = \sum_{e=uv\in E(G)} (n_u(e,n-1)+1)(n_v(e,n-1)+1) = 2(n-1) = 2n-2$.

It is obvious that when n is getting relatively large, $Sz(P_n) > Sz(S_n)$ and $Sz_{n-1}(P_n) < Sz_{n-1}(S_n)$. Moreover, explicit formulas of $Sz_k(P_n)$ and $Sz_k(S_n)$ were obtained in [4]:

Proposition 2.2. *[4]*

(1) $Sz_k(P_n) = \sum\limits_{i=1}^{n-1} \left(\binom{i-1}{k-1} + 1\right)\left(\binom{n-i-1}{k-1} + 1\right).$

(2) $Sz_k(S_n) = (n-1)\binom{n-2}{k-1} + (n-1).$

It is easy to see that when n is getting relatively large and k approaches to n (assume $k = n - t$), $Sz_k(P_n) = \Omega(n^{t-1})$ and $Sz_k(S_n) = \Omega(n^t)$. Thus, we actually have $Sz_k(P_n) < Sz_k(S_n)$ in many cases. ■

These disprove the above conjecture.

3 The kth Steiner Szeged indices of cycles

For an integer k, let $\mathbb{N}^k$ be the set of the k-tuples of nonnegative integers. The main result of this section is stated as follows.

Theorem 3.1. $Sz_k(C_n) = n\left(1 + \sum\limits_{j=1}^{n-k} \sum\limits_{\substack{r+sj=n-k-j+1 \\ (r,s)\in\mathbb{N}^2}} (-1)^s \left(\binom{k+r-2}{r}\binom{k-1}{s} - \binom{k+r-3}{r}\binom{k-2}{s}\right)\right)^2,$ *where* $2 \le k \le n-1$ *and* $k \in \mathbb{N}$.

Before proving, we need the following lemma.

Lemma 3.2. $Sz_k(C_n) = n(a+1)^2$, *where* $a = |\{(x, y, x_1, \cdots, x_{k-2}) \in \mathbb{N}^k | x + y + \sum\limits_{i=1}^{k-2} x_i = n - k + 1, 0 \le x_i < y$ *for all* $i, 0 < x < y\}|$, $2 \le k \le n-1$ *and* $k \in \mathbb{N}$.

Proof. Because of the symmetry of C_n, we consider an arbitrary edge $e = uv$. Label u with 1, v with n and all the other vertices with $\{2, 3, \cdots, n-1\}$ clockwise, as shown in Figure 1 (left). Denote the unique path $P = wz\cdots y$ (oriented clockwise) by $\overrightarrow{P}(wz\cdots y)$, as shown in Figure 1 (right). For every $(k-1)$-subset $S \subseteq V(G)$ mentioned below, we arrange its elements from small to large, i.e., $S = \{a_1, \cdots, a_{k-1}\}$, where $a_1 < \cdots < a_{k-1}$.

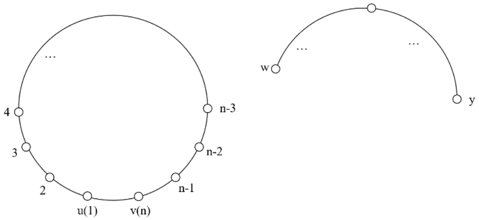

Figure 1

For each $(k-1)$-subset $S = \{a_1, \cdots, a_{k-1}\}$, we need to compare $d(S \cup \{u\})$ with $d(S \cup \{v\})$, i.e., the size of $T_{S\cup\{u\}}$ and $T_{S\cup\{v\}}$. Note that both $T_{S\cup\{u\}}$ and $T_{S\cup\{v\}}$ are paths. Moreover, all the possible $T_{S\cup\{u\}}$'s will be $\overrightarrow{P}(ua_1a_2\cdots a_{k-1})$, $\overrightarrow{P}(a_1\cdots a_{k-1}u)$, $\cdots$, $\overrightarrow{P}(a_{k-1}u\cdots a_{k-2})$, some of which are shown in Figure 2.

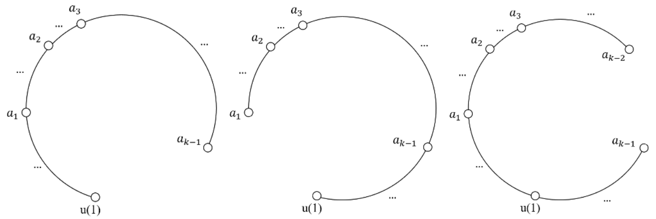

Figure 2

Now let us show how to choose the right $T_{S\cup\{u\}}$ according to the set S. Since a right $T_{S\cup\{u\}}$ is of the smallest size, it will leave out the largest number of vertices among $V(C_n) - (S \cup \{u\})$. That means that we have to find a connected component of $C_n \backslash (S \cup \{u\})$ with the largest number of vertices. In fact, this problem corresponds to an integer partition problem.

Now consider an array $A_u = a_1a_2\cdots a_{k-1}u$. Mark the minimal positive value $(a_{i+1} - a_i - 1)$ in the upper right corner of a_i, $(u - a_{k-1} - 1)(\text{mod } n)$ in the upper right corner of a_{k-1} and $(a_1 - u - 1)(\text{mod } n)$ in the upper right corner of u. Then we have $A_u = a_1^{x_1}a_2^{x_2}\cdots a_{k-1}^{x}u^{y-1}$ and $A_v = a_1^{x_1}a_2^{x_2}\cdots a_{k-1}^{x-1}v^{y}$, where the sum of all the superscripts of A_u (A_v) is $n - |S| - 1$. For example, for C_{10} and $S = \{3, 4, 7\}$, we have $A_u = 3^0 4^2 7^3 1^1$ and $A_v = 3^0 4^2 7^2 10^2$.

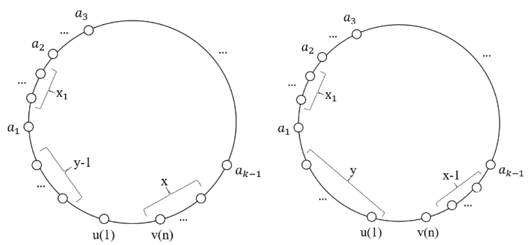

Figure 3

Each superscript x_i represents how many vertices will be left out when choosing $\overrightarrow{P}(a_{i+1}$ $a_{i+2}\cdots a_i)$ as $T_{S\cup\{u\}}$. The same occurs for x, $x-1$, $y-1$ and y. So, if we find the largest superscript, we can determine $d(S \cup \{u\})$.

Let M be the value $max\{x, y, x_1, \cdots, x_{k-2}\}$. Then we have

(1) $d(S \cup \{u\}) < d(S \cup \{v\})$ if only x achieves M;

(2) $d(S \cup \{u\}) > d(S \cup \{v\})$ if only y achieves M;

(3) otherwise, $d(S \cup \{u\}) = d(S \cup \{v\})$.

Because we have the fixed u and v as reference points, we have a bijection:

$\{(x, y, x_1, \cdots, x_{k-2}) \in \mathbb{N}^k | x + y + \sum_{i=1}^{k-2} x_i = n - k + 1, 0 \leq x_i < y \text{ for all } i, 0 < x < y\}$ $\leftrightarrow \{S \subseteq V(G), |S| = k - 1 : d_G(S \cup \{u\}) > d_G(S \cup \{v\}), u, v \notin S\}$.

As a result, $|\{(x, y, x_1, \cdots, x_{k-2}) \in \mathbb{N}^k | x+y+\sum_{i=1}^{k-2} x_i = n-k+1, 0 \leq x_i < y \text{ for all } i, 0 < x < y\}| = n_v(e, k) = n_u(e, k)$. ■

Now we are ready to prove Theorem 3.1 by using the following enumerative result, which can be found in [14].

Lemma 3.3. *[14]*

Let $\kappa(n, j, k)$ be the number of compositions of n into k nonnegative parts, each part less than j. Then

$$\kappa(n, j, k) = \sum_{r+sj=n} (-1)^s \binom{k+r-1}{r} \binom{k}{s},$$

where the sum is over all pairs $(r, s) \in \mathbb{N}^2$ satisfying $r + sj = n$.

Proof of Theorem 3.1. From Lemma 3.2, it is reduced to get the value of a.

$$\begin{aligned}
a &= |\{(x, y, x_1, \cdots, x_{k-2}) \in \mathbb{N}^k | x + y + \sum_{i=1}^{k-2} x_i = n - k + 1, 0 \leq x_i < y \text{ for all } i, 0 < x < y\}| \\
&= |\{(x, y, x_1, \cdots, x_{k-2}) \in \mathbb{N}^k | x + y + \sum_{i=1}^{k-2} x_i = n - k + 1, 0 \leq x_i < y \text{ for all } i, 0 \leq x < y\}| - \\
&\quad |\{(y, x_1, \cdots, x_{k-2}) \in \mathbb{N}^{k-1} | y + \sum_{i=1}^{k-2} x_i = n - k + 1, 0 \leq x_i < y \text{ for all } i\}| \\
&= \sum_{j=1}^{n-k+1} (\kappa(n - k - j + 1, j, k - 1) - \kappa(n - k - j + 1, j, k - 2)) \\
&= \sum_{j=1}^{n-k+1} \sum_{\substack{r+sj=n-k-j+1 \\ (r,s)\in\mathbb{N}^2}} (-1)^s \left(\binom{k+r-2}{r} \binom{k-1}{s} - \binom{k+r-3}{r} \binom{k-2}{s} \right).
\end{aligned}$$

Then, we can obtain the formula for the kth Steiner revised Szeged indices of cycles as well. ■

4 The Steiner k-Wiener indices of cycles

Let k, s be positive integers and $X_k = \{x = (x_0, x_1, \cdots, x_{k-1}) \in \mathbb{N}^k | \sum_{i=0}^{k-1} x_i = n-k, x_i \geq 0\}$. Our main result of this section is stated below.

Theorem 4.1. *Let C_n be a cycle with n vertices, $1 \leq k \leq n$ ($k \in \mathbb{N}$). Then*

$$SW_k(C_n) = \frac{n}{k} \sum_{x \in X_k} (n - 1 - max\{x_0, \cdots, x_{k-1}\}).$$

To prove this, we need more definitions and notations. Define π_k^s to be a map from X_k to X_k by $\pi_k^s(x_0, x_1, \cdots, x_{k-1}) = (x_{(0+s)(\mathrm{mod}\ k)}, x_{(1+s)(\mathrm{mod}\ k)}, \cdots, x_{(k-1+s)(\mathrm{mod}\ k)})$, where $1 \leq s \leq k$. For $(x_0, x_1, \cdots, x_{k-1})$, $(y_0, y_1, \cdots, y_{k-1}) \in \mathbb{N}^k$, if there exists an s such that $\pi_k^s(x_0, x_1, \cdots, x_{k-1}) = (y_0, y_1, \cdots, y_{k-1})$, then we say that $(x_0, x_1, \cdots, x_{k-1})$ and $(y_0, y_1, \cdots, y_{k-1})$ are in a relation R. It is easy to see that the relation R is an equivalent relation. Moreover, define the smallest s such that $\pi_k^s(x_0, x_1, \cdots, x_{k-1}) = (x_0, x_1, \cdots, x_{k-1})$ to be the order σ of $(x_0, x_1, \cdots, x_{k-1})$, where $1 \leq \sigma \leq k$ and $\sigma | k$. It is obvious that all the elements in a same equivalent class have the same order.

For the set $X_k = \{(x_0, x_1, \cdots, x_{k-1}) \in \mathbb{N}^k | \sum_{i=0}^{k-1} x_i = n - k, x_i \geq 0\}$, partition it into l equivalent classes and choose one (not necessarily unique) element with the largest initial term from each equivalent class as the representative element. Form a subset X_k^0 consisting of all the representative elements. Because there are exactly $\sigma(x)$ elements in the equivalent class of x, we have an equivalent formula of Theorem 4.1.

Lemma 4.2. *Let C_n be a cycle with n vertices, $1 \leq k \leq n$ ($k \in \mathbb{N}$). Then*

$$SW_k(C_n) = \frac{n}{k} \sum_{x \in X_k^0} \sigma(x)(n - 1 - x_0).$$

We prove Lemma 4.2 here and Theorem 4.1 holds naturally.

Proof of Lemma 4.2. The main idea is pretty much the same as the former one. For each k-subset $S \subseteq V(G)$, let the corresponding vector be $(x_0, x_1, \cdots, x_{k-1})$, where $\sum_{i=0}^{k-1} x_i = n - k$ and $x_0 = max\{x_0, x_1, \cdots, x_{k-1}\}$. Then $d_G(S) = n - 1 - x_0$. Note that the corresponding vector is not necessarily unique but all the possible corresponding vectors lie in the same equivalent class. Also, the corresponding is not injective because of the symmetry of cycles and the lack of fixed reference points. So, the tricky part is to count how many sets will correspond to the same vector $(x_0, x_1, \cdots, x_{k-1})$.

A particular vector $(x_0, x_1, \cdots, x_{k-1})$ can determine at most n k-sets $\{1(\text{mod } n), (1+x_0+1)(\text{mod } n), (1+x_0+x_1+2)(\text{mod } n), \cdots, (1+\sum_{i=0}^{k-2} x_i+k-1)(\text{mod } n)\}$, $\{2(\text{mod } n), (2+x_0+1)(\text{mod } n), (2+x_0+x_1+2)(\text{mod } n), \cdots, (2+\sum_{i=0}^{k-2} x_i+k-1)(\text{mod } n)\}$, $\cdots$, $\{n(\text{mod } n), (n+x_0+1)(\text{mod } n), (n+x_0+x_1+2)(\text{mod } n), \cdots, (n+\sum_{i=0}^{k-2} x_i+k-1)(\text{mod } n)\}$. Provided $\sum_{i=0}^{k-1} x_i = n-k$ and the consecutive cyclic structure, if $\{a(\text{mod } n), (a+x_0+1)(\text{mod } n), (a+x_0+x_1+2)(\text{mod } n), \cdots, (a+\sum_{i=0}^{k-2} x_i+k-1)(\text{mod } n)\}$ and $\{b(\text{mod } n), (b+x_0+1)(\text{mod } n), (b+x_0+x_1+2)(\text{mod } n), \cdots, (b+\sum_{i=0}^{k-2} x_i+k-1)(\text{mod } n)\}$ are identical, there will be an $1 \le i \le k$, such that

$$\begin{cases} b + \sum_{t=0}^{i-1} x_{t(\text{mod } k)} + i = a(\text{mod } n) \\ b + \sum_{t=0}^{i} x_{t(\text{mod } k)} + i + 1 = (a + x_0 + 1)(\text{mod } n) \\ b + \sum_{t=0}^{i+1} x_{t(\text{mod } k)} + i + 2 = (a + x_0 + x_1 + 2)(\text{mod } n) \\ \vdots \\ b + \sum_{t=0}^{i+k-2} x_{t(\text{mod } k)} + i + k - 1 = (a + \sum_{t=0}^{k-2} x_i + k - 1)(\text{mod } n) \end{cases} \tag{4.1}$$

Thus, we have

$$\begin{cases} x_{(i)(\text{mod } k)} = x_0(\text{mod } n) \\ x_{(i+1)(\text{mod } k)} = x_1(\text{mod } n) \\ \vdots \\ x_{(i+k-1)(\text{mod } k)} = x_{k-1}(\text{mod } n) \end{cases} \tag{4.2}$$

and i is a multiple of $\sigma(x)$.

Moreover, we can observe that there will be exactly $\frac{k}{\sigma(x)}$ sets the same as $\{1(\text{mod } n), (1+x_0+1)(\text{mod } n), (1+x_0+x_1+2)(\text{mod } n), \cdots, (1+\sum_{i=0}^{k-2} x_i+k-1)(\text{mod } n)\}$. That is because for $t \in \{1, 2, \cdots, \frac{k}{\sigma(x)}\}$, $\{1 (\mathrm{mod}\ n), (1+x_0+1)(\mathrm{mod}\ n), (1+x_0+x_1+2)(\mathrm{mod}\ n), \cdots, (1+\sum_{i=0}^{k-2} x_i + k - 1)(\mathrm{mod}\ n)\}$=$\{(1 - t\sum_{i=0}^{\sigma(x)-1} x_i - t\sigma)(\mathrm{mod}\ n), [(1 - t\sum_{i=0}^{\sigma(x)-1} x_i - t\sigma) + x_0 + 1](\mathrm{mod}\ n), [(1 - t\sum_{i=0}^{\sigma(x)-1} x_i - t\sigma) + x_0 + x_1 + 2](\mathrm{mod}\ n), \cdots, 1, (1 + x_0 + 1)(\mathrm{mod}\ n), (1 + x_0 + x_1 + 2)(\mathrm{mod}\ n), \cdots, [(1 - t\sum_{i=0}^{\sigma(x)-1} x_i - t\sigma) + \sum_{i=0}^{k-2} x_i + k - 1](\mathrm{mod}\ n)\}$. Therefore, a particular vector $(x_0, x_1, \cdots, x_{k-1})$ can determine exactly $\frac{n\sigma(x)}{k}$ different sets.

Also, we can see that two vectors share the same corresponding sets if and only if they lie in the same equivalent class.

Here comes our result:

$$SW_k(C_n) = \sum_{S\subseteq V(G), |S|=k} d_G(S) = \sum_{\{a_1,a_2,\cdots,a_k\}\subseteq V(G)} d_G(S) = \sum_{\substack{\{a_1,a_2,\cdots,a_k\}\subseteq V(G) \\ \{a_1,a_2,\cdots,a_k\} \text{ corresponds to } x}} n - 1 - x_0$$

$$= \sum_{x\in X_k^0} \sum_{\substack{\{a_1,a_2,\cdots,a_k\}\subseteq V(G) \\ \{a_1,a_2,\cdots,a_k\} \text{ corresponds to } x}} n - 1 - x_0 = \sum_{x\in X_k^0} (n - 1 - x_0)\frac{n\sigma(x)}{k}.$$

■

By calculation, we can obtain $SW_2(C_n)$ (first obtained in [5]) and $SW_3(C_n)$ as corollaries:

Corollary 4.3. *(1) [5]*

$$SW_2(C_n) = \begin{cases} \frac{n^3-n}{8} & \text{if } n \text{ is odd;} \\ \frac{n^3}{8} & \text{if } n \text{ is even.} \end{cases}$$

(2)

$$SW_3(C_n) = \begin{cases} \frac{n^2}{216}(14n^2 - 27n - 6) & \text{if } n = 6k, k \in \mathbb{N}; \\ \frac{n}{216}(14n^3 - 27n^2 - 6n + 19) & \text{if } n = 6k+1, k \in \mathbb{N}; \\ \frac{n}{216}(14n^3 - 27n^2 - 6n + 8) & \text{if } n = 6k+2, k \in \mathbb{N}; \\ \frac{n}{216}(14n^3 - 27n^2 - 6n + 27) & \text{if } n = 6k+3, k \in \mathbb{N}; \\ \frac{n}{216}(14n^3 - 27n^2 - 6n - 8) & \text{if } n = 6k+4, k \in \mathbb{N}; \\ \frac{n}{216}(14n^3 - 27n^2 - 6n + 35) & \text{if } n = 6k+5, k \in \mathbb{N}. \end{cases}$$

5 The *k*th Steiner Szeged indices and the Steiner *k*-Wiener indices of wheels

Label the unique vertex with degree $n-1$ by z, as shown in Figure 4. We calculate the kth Steiner Szeged index and the Steiner k-Wiener index of a wheel now. These results are easier to obtain because the diameter of a wheel is only 2.

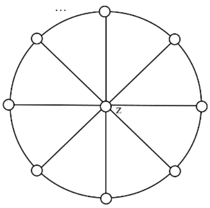

Figure 4

Proposition 5.1.

$$Sz_k(W_n) = \begin{cases} 2n-2 & \text{if } n-2 \le k \le n-1; \\ (n-1)\left(\binom{n-2}{k-1} + 5 - k\right) & \text{if } 2 \le k \le n-3. \end{cases}$$

Proof. We distinguish the following cases.

Case 1. $e = uv \in C_{n-1}$.

If S contains z, then $d_G(S \cup \{u\}) = k-1 = d_G(S \cup \{v\})$.

So, we only consider the sets without z. For such sets, $k-1 \le d_G(S \cup \{u\}) \le k$ and $k-1 \le d_G(S \cup \{v\}) \le k$ because of z. And $d_G(S \cup \{u\}) = k-1$ if and only if $W_n[S \cup \{u\}]$ is connected. Thus, the set in $N_u(e,k)$ can only be consecutive vertices from u along the cycle (it has to exclude z and v). The number of such sets would be 1 (when $k-1+3 < n$) or 0 (when $k-1+3 \ge n$).

Case 2. $e = uz \notin C_{n-1}$.

It is obvious that for any set S, $d_G(S \cup \{z\}) = k-1$ and $d_G(S \cup \{u\}) \ge k-1$. So, $n_u(e,k) = 0$. Next to count $N_z(e,k)$. It is the same to count $N_0(e,k)$ instead ($n_z(e,k) + n_0(e,k) = \binom{n-2}{k-1}$). For the sets S such that $d_G(S \cup \{u\}) = k-1$, $W_n[S \cup \{u\}]$ is connected. Thus, the set in $N_0(e,k)$ can only be consecutive vertices along the cycle containing u. The number of such sets would be k (when $k-1+2 < n$) or 1 (when $k-1+2 = n$). ■

Similarly, we can get the kth Steiner revised Szeged index of W_n, stated as follows.

Corollary 5.2.

$$Sz_k(W_n) = \begin{cases} \frac{9}{2}(n-1) & \text{if } k = n-1; \\ \frac{n^2(n-1)}{2} & \text{if } k = n-2; \\ (n-1)\left(\left(\frac{k}{2}+1\right)\left(\binom{n-2}{k-1} - \frac{k}{2} + 1\right) + \left(1 + \frac{1}{2}\binom{n-2}{k-1}\right)^2\right) & \text{if } 2 \le k \le n-3. \end{cases}$$

Proposition 5.3.

$$SW_k(W_n) = \begin{cases} n-1 & \textit{if } k = n; \\ n(n-2) & \textit{if } k = n-1; \\ k\binom{n}{k} - n + 1 - \binom{n-1}{k-1} & \textit{if } 1 \leq k \leq n-2. \end{cases}$$

Proof. If S contains z, then $d_G(S) = k-1$. Otherwise, $z \notin S$ and $d_G(S) \leq k$. For such sets, $d_G(S) = k-1$ if and only if $W_n[S]$ is connected. They can only be consecutive vertices along the cycle. The number of such sets would be $n-1$ (when $n-1 > k$), 1 (when $n-1 = k$) or 0 (when $n = k$). ■

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