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# A Note on the Normalized Laplacian Estrada Index of Bipartite Graphs

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**Abstract**

For a connected graph $G$ with $n$ vertices and normalized Laplacian eigenvalues $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n = 0$ , the normalized Laplacian Estrada index of $G$ is defined as $NEE = NEE(G) = \sum_{i=1}^{n} e^{(\gamma_i - 1)}$. In this paper, we establish a lower bound on $NEE$ of non-complete bipartite graphs and characterize the graph achieving the lower bound. In addition, we point out that $NEE$ and Randić Estrada index ($REE$) coincide in the case of bipartite graphs. Herewith, we realize that the results on $REE$ of bipartite graphs given by Maden [*MATCH Commun Math. Comput Chem.* **74** (2015) 367–387] are actually same with the results on $NEE$ of bipartite graphs previously given by Li et al. [*Filomat* **28** (2014) 365–371].

# 1 Introduction

Let $G$ be a finite, simple and connected graph with $n$ vertices and $m$ edges. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of $G$. Denote by $d_i$ the degree of the vertex $v_i$, where $i = 1, 2, \ldots, n$.

Let $A(G)$ be the $(0,1)$-adjacency matrix of a graph $G$ and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote its eigenvalues. The eigenvalues of $A(G)$ are said to be the eigenvalues of $G$ [9]. The Laplacian matrix of $G$ is the matrix $L(G) = D(G) - A(G)$, where $D(G) = diag(d_1, d_2, \ldots, d_n)$ denote the diagonal matrix of vertex degrees of $G$ [25]. Because $G$ is connected, $D(G)$ is non-singular, then the normalized Laplacian matrix of $G$ is defined as [7]

$$\mathcal{L}(G) = D(G)^{-1/2} L(G) D(G)^{-1/2} = I_n - R(G)$$

where $I_n$ is the $n \times n$ unit matrix and $R(G)$ is the Randić matrix of $G$ [5]. Denote by $\rho_1 = 1 \geq \rho_2 \geq \cdots \geq \rho_n$ and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n = 0$ the eigenvalues $R(G)$ and $\mathcal{L}(G)$, respectively [5,7,22]. These eigenvalues are respectively called as the Randić eigenvalues and the normalized Laplacian eigenvalues of $G$ [5,7].

The Estrada index of a graph $G$ was defined as [11]

$$EE = EE(G) = \sum_{i=1}^{n} e^{\lambda_i}. \tag{1}$$

This concept has found remarkable applications in various areas such as chemistry [11,12], complex networks [13] and statistical thermodynamics [14,15]. In addition, an extensive literature exists regarding $EE$ and its mathematical properties and bounds. For survey and more details, see [16,18,23].

In full analogy with Eq. (1), the Randić Estrada index of $G$ was put forward in [3] as

$$REE = REE(G) = \sum_{i=1}^{n} e^{\rho_i}. \tag{2}$$

For several lower and upper bounds on $REE$, see [3,24].

In an analogous manner with the Estrada index defined by (1) and the Laplacian Estrada index defined in [19], Li et al. introduced the normalized Laplacian Esrada index of $G$ as [20]

$$NEE = NEE(G) = \sum_{i=1}^{n} e^{(\gamma_i - 1)}. \tag{3}$$

In [20], the authors also obtained some bounds on $NEE$ as well as some inequalities between $NEE$ and the normalized Laplacian energy [6]. Some results of [20] were improved in [8] via majorization techniques. More detailed information on $NEE$ can be found in [26,27].

Independently from [20], another definition of normalized Laplacian Estrada index was given in [17] as

$$\ell EE = \ell EE(G) = \sum_{i=1}^{n} e^{\gamma_i}.$$

Note that $NEE = \frac{1}{e} \ell EE$ and therefore, any results derived for $NEE$ can be directly re-stated for $\ell EE$ and vice versa [8].

In recent paper [1], a new graph invariant so called the sum of powers of normalized signless Laplacian eigenvalues was defined, with emphasis on it coincides with the sum of powers of normalized Laplacian eigenvalues [4] in the case of bipartite graphs. Some lower and upper bounds on this graph invariant were presented in [1,2] for (non)-bipartite graphs. In [28], Sun and Das reported a lower bound on Randić energy [5] in order to find the smallest Randić energy among all connected bipartite graphs except complete bipartite graph. In the similar spirit with the papers [1, 2, 28], in this study, we establish a lower bound on $NEE$ of non-complete bipartite graphs and characterize the graph achieving the lower bound. In addition, we point out that $NEE$ and $REE$ coincide in the case of bipartite graphs. Herewith, we realize that the results on $REE$ of bipartite graphs given in [24] are actually same with the results on $NEE$ of bipartite graphs previously given in [20].

## 2 Preliminaries

We now present some previously known results that will be needed in the subsequent section.

**Lemma 2.1.** *[7] Let $G$ be a connected graph of order $n \geq 2$. Then, the following properties regarding the normalized Laplacian eigenvalues hold:*

*(1)* $\sum\limits_{i=1}^{n} \gamma_i = tr\left(\mathcal{L}\left(G\right)\right) = n$.

*(2)* $\gamma_1 \leq 2$ *with equality holding if and only if $G$ is a bipartite graph.*

*(3)* $\gamma_n = 0$ *and* $\gamma_{n-1} \neq 0$.

**Lemma 2.2.** *[7] Let $G$ be a bipartite graph of order $n$. Then, $\gamma_i + \gamma_{n-i+1} = 2$, for $i = 1, 2, \ldots, n$.*

Let $K_{p,q} - e$ denote the graph obtained by deleting any edge $e$ from the complete bipartite graph $K_{p,q}$. In [28], Sun and Das considered the first two smallest values on $\gamma_2$ among all connected bipartite graphs with fixed size of bipartition as follows:

**Lemma 2.3.** *[21, 28] Let $G$ ($\ncong K_{p,q}$) be a connected bipartite graph with bipartition $V(G) = X \cup Y$ and $p = |X| > 1$, $q = |Y| > 1$. Then*

$$\gamma_2\left(G\right) \geq 1 + \frac{1}{\sqrt{pq}} > \gamma_2\left(K_{p,q}\right) = 1.$$

*The first equality holds if and only if $G \cong K_{p,q} - e$ ($e$ is any edge in $K_{p,q}$).*

Let $RS(G) = \{\rho_1, \rho_2, \ldots, \rho_n\}$ denote the Randić spectrum of a graph $G$. The following result was given in [10].

**Lemma 2.4.** *[10] The Randić spectrum of the graph $K_{p,q} - e$ is*

$$RS(K_{p,q} - e) = \left(1, \frac{1}{\sqrt{pq}}, \underbrace{0, \ldots, 0}_{n-4}, -\frac{1}{\sqrt{pq}}, -1\right).$$

The following lemma is well known from [7,9] and has also been utilized in [1,2,10,17].

**Lemma 2.5.** *[7, 9] Let $G$ be a bipartite graph of order $n$. Then, $\gamma_i = 1 + \rho_i$, for $i = 1, 2, \ldots, n$.*

# 3 Main Results

Now, we are ready to give the main results of this paper. At first, we will consider the problem in Theorem 4.2 of [28] for $NEE$.

**Theorem 3.1.** *Let $G$ be a connected bipartite graph with bipartition $V(G) = X \cup Y$ and $p = |X| > 1$, $q = |Y| > 1$. If $G \cong K_{p,q}$, then $NEE(G) = e + e^{-1} + (n-2)$ [20]. Otherwise,*

$$NEE(G) \geq e + e^{\frac{1}{\sqrt{pq}}} + e^{-\frac{1}{\sqrt{pq}}} + e^{-1} + (n-4). \tag{4}$$

*Moreover, the equality holds in (4) if and only if $G \cong K_{p,q} - e$ ($e$ is any edge in $K_{p,q}$).*

*Proof.* If $G \cong K_{p,q}$, then by Theorem 3.2 of [20], $NEE(G) = e + e^{-1} + (n-2)$. Otherwise, $G$ is not complete bipartite graph. Note that by Lemmas 2.1 and 2.2, $\sum_{i=1}^{n} \gamma_i = n$, $\gamma_1 = 2$, $\gamma_n = 0$ and $\gamma_2 + \gamma_{n-1} = 2$. Then, considering these with Eq. (3), we obtain that

$$\begin{aligned} NEE &= e + e^{(\gamma_2 - 1)} + e^{(1-\gamma_2)} + e^{-1} + \sum_{i=3}^{n-2} e^{(\gamma_i - 1)} \\ &\geq e + e^{(\gamma_2 - 1)} + e^{(1-\gamma_2)} + e^{-1} + (n-4)\left(\prod_{i=3}^{n-2} e^{(\gamma_i - 1)}\right)^{1/(n-4)} \\ &= e + e^{(\gamma_2 - 1)} + e^{(1-\gamma_2)} + e^{-1} + (n-4), \text{ as } \sum_{i=3}^{n-2} (\gamma_i - 1) = 0. \end{aligned}$$

Let us consider the function $f(x) = e^x + e^{-x}$. It is easy to see that $f'(x) = e^x - e^{-x} > 0$, for $x > 0$. By Lemma 2.3, we have that $\gamma_2 - 1 \geq \frac{1}{\sqrt{pq}} > 0$. Then, we get

$$NEE \geq e + e^{\frac{1}{\sqrt{pq}}} + e^{-\frac{1}{\sqrt{pq}}} + e^{-1} + (n-4).$$

Hence the inequality (4) follows. The equality holds in (4) if and only if $\gamma_2 = 1 + \frac{1}{\sqrt{pq}}$ and $\gamma_3 = \gamma_4 = \cdots = \gamma_{n-2}$. Note that the similar idea in [2] will be followed for the equality condition.

Assume that the equality holds in (4). Then, from Lemma 2.3, we have that $G \cong K_{p,q} - e$. Since $G$ is bipartite graph, by Lemma 2.2, $\gamma_2 = 1 + \frac{1}{\sqrt{pq}}$ requires that $\gamma_{n-1} = 1 - \frac{1}{\sqrt{pq}}$. Furthermore, by Lemmas 2.1 and 2.2, one can get $\sum_{i=3}^{n-2} \gamma_i = n - 4$ which implies that $\gamma_3 = \gamma_4 = \cdots = \gamma_{n-2} = 1$. From Lemmas 2.4 and 2.5, these also verify that $G \cong K_{p,q} - e$.

Conversely, one can easily see that the equality holds in (4) for $G \cong K_{p,q} - e$, by Lemmas 2.4 and 2.5. ■

**Remark 3.2.** *From Theorem 3.2 of [20], Li et al. concluded that among all bipartite graphs of order $n$, the complete bipartite graphs with minimum normalized Laplacian Estrada index [20]. From Lemma 2.3 and Theorem 3.1, we also conclude that among all connected bipartite graphs except complete bipartite graph, $K_{p,q} - e$ with minimum normalized Laplacian Estrada index.*

**Remark 3.3.** *From Theorem 3.1, it is easy to see that the lower bound in (4) is better than the lower bound in Theorem 3.2 of [20] for any connected bipartite graph $G$ ($\ncong K_{p,q}$) with bipartition $V(G) = X \cup Y$ and $p = |X| > 1$, $q = |Y| > 1$.*

Using majorization techniques, in [8] Clemente and Cornaro obtained the following lower bound on $NEE$ of bipartite graphs.

**Theorem 3.4.** *[8] Let $G$ be a connected bipartite graph with $n \geq 2$ vertices and $\gamma_2 \geq \beta$, where $1 < \beta \leq 2$. Then*

$$NEE(G) \geq e + e^{-1} + e^{\beta-1} + (n-3)\, e^{\frac{1-\beta}{n-3}}. \tag{5}$$

**Example 3.5.** *Using Lemma 2.3 and Theorem 3.4, one can compute the lower bound in (5) as*

$$NEE(G) \geq e + e^{-1} + e^{\frac{1}{\sqrt{pq}}} + (n-3)\, e^{-\frac{1}{(n-3)\sqrt{pq}}} \tag{6}$$

*where $G$ ($\ncong K_{p,q}$) is a connected bipartite graph with bipartition $V(G) = X \cup Y$ and $p = |X| > 1$, $q = |Y| > 1$.*

**Remark 3.6.** *Note that the lower bound (4) is incomparable with the lower bound (6). Further note that $K_{p,q} - e$ is an extremal graph for the lower bound (4). Hence the lower bound (4) is better than the lower bound (6) for this graph.*

**Remark 3.7.** *We should note that $NEE$ and $REE$ coincide in the case of bipartite graphs. This is an immediate consequence of Lemma 2.5 and the definitions of $REE$ and $NEE$ given by Eqs. (2) and (3).*

**Remark 3.8.** *For a connected bipartite graph $G$ of order $n$, Maden obtained that [24]*

$$REE\,(G) \geq e + e^{-1} \tag{7}$$

*with equality holding if and only if $G$ is a complete bipartite graph. Unfortunately, there exists a small mistake in the derivation of (7). This is also clearly seen its equality condition. Using the procedure in Theorem 2.23 of [24], one can get that*

$$REE\,(G) \geq e + e^{-1} + (n-2) \tag{8}$$

*with equality holding if and only if $G$ is a complete bipartite graph. Hence, by Remark 3.7, it can be easily seen that the result in (8) is same with the result in Theorem 3.2 of [20].*

**Remark 3.9.** *From Lemma 2.5 and Remark 3.7, it is easy to conclude that the results in Theorem 2.26 and Corollary 2.27 of [24] are same with the results in Theorem 3.4 of [20].*

**Remark 3.10.** *Considering Theorem 2 of [5], Lemma 2.5 and Remark 3.7, one can easily see that the results in Theorem 2.31 of [24] are same with the results in Theorem 3.6 of [20].*

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