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The Maximal Geometric–Arithmetic Energy of Trees*

Xuan Zhao[a], **Yanling Shao**[b,†], **Yubin Gao**[b]

[a]*School of Data Science and Technology, North University of China, Taiyuan, Shanxi 030051, P.R. China*

[b]*School of Sciences, North University of China, Taiyuan, Shanxi 030051, P.R. China*

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Abstract

Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, and $d(v_i)$ be the degree of the vertex v_i. The geometric-arithmetic matrix of G is the matrix of order n whose (i, j)-entry is equal to $\frac{2\sqrt{d(v_i)d(v_j)}}{d(v_i)+d(v_j)}$ if $v_iv_j \in E(G)$, and 0 otherwise. The geometric-arithmetic energy of G is the sum of the absolute values of the eigenvalues of its geometric-arithmetic matrix. In 2019, Y. Shao et al. [1] conjectured that the path P_n has the maximal geometric-arithmetic energy among all trees of order $n \geq 4$. In this paper, we prove that the conjecture is true, and in fact, the result holds also for $n = 1, 2, 3$.

1 Introduction

Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For $i = 1, 2, \ldots, n$, denote by $d_G(v_i)$ (or $d(v_i)$ for short) the degree of the vertex v_i in G. A edge $v_iv_j \in E(G)$ is called a pendent edge of G if $d_G(v_i) = 1$ or $d_G(v_j) = 1$. Let T be a tree. A vertex v is called a branched vertex of T if $d_T(v) \geq 3$.

The geometric-arithmetic (or GA, for short) index of a graph G, introduced by Vukičević and Furtula as a topological index ([2]), is defined as $GA(G) = \frac{2\sqrt{d(v_i)d(v_j)}}{d(v_i)+d(v_j)}$

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†Corresponding author. E-mail address: ylshao@nuc.edu.cn (Y. Shao).

if $v_iv_j \in E(G)$. In [3, 4], Rodríguez and Sigarreta introduced the GA matrix and GA energy of a graph. Let G be a graph of order n. The GA matrix of G, denoted by $M(G)$, is the matrix of order n whose (i, j)-entry is equal to $\frac{2\sqrt{d(v_i)d(v_j)}}{d(v_i)+d(v_j)}$ if $v_iv_j \in E(G)$, and 0 otherwise. The characteristic polynomial of $M(G)$, denoted by $\phi_{GA}(G, x) = |xI - M(G)|$, is called the GA characteristic polynomial of G. The n roots of the equation $\phi_{GA}(G, x) = 0$, denoted by $\mu_1(G), \mu_2(G), \ldots, \mu_n(G)$, are called the GA eigenvalues of G. Since $M(G)$ is real and symmetric, all GA eigenvalues of G are real. The GA energy of G is defined as $\mathcal{E}_{GA}(G) = \sum_{i=1}^{n} |\mu_i(G)|$.

In [1], Shao and Gao studied the maximal GA energy of trees of order n. They proved that the path of order n has the maximal geometric-arithmetic energy among all trees of order n with at most two branched vertices, and conjectured the result also hold for all trees of order n.

Theorem 1.1 ([1]) *Let T be a tree of order $n \geq 4$ with at most two branched vertices. Then*

$$\mathcal{E}_{GA}(T) \leq \mathcal{E}_{GA}(P_n).$$

Equality holds if and only if T is isomorphic to the path P_n.

It is clear that the result holds also for $n = 1, 2, 3$.

Conjecture 1.2 ([1]) *Let T be a tree of order $n \geq 4$. Then*

$$\mathcal{E}_{GA}(T) \leq \mathcal{E}_{GA}(P_n).$$

Equality holds if and only if T is isomorphic to the path P_n.

In this paper, we prove that this conjecture is true, and in fact, the result holds also for $n = 1, 2, 3$.

2 Preliminaries

For a graph G, we use $\mathcal{M}_k(G)$ to denote the set of all k-matchings of G. If $e = v_iv_j \in E(G)$, then we denote

$$GA_G(e) = GA_G(v_iv_j) = \left(\frac{2\sqrt{d(v_i)d(v_j)}}{d(v_i) + d(v_j)}\right)^2,$$

and we say that $GA_G(e)$ is the GA value of the edge e. If $\alpha_k = \{e_1, e_2, \dots, e_k\} \in \mathcal{M}_k(G)$, we call that $\prod_{i=1}^{k} GA_G(e_i)$ is the GA value of matching α_k, and write $GA_G(\alpha_k) = \prod_{i=1}^{k} GA_G(e_i)$.

Let T be a tree T of order n with GA matrix $M(T)$. Then the GA characteristic polynomial of T can be written as ([1])

$$\phi_{GA}(T, x) = |xI - M(T)| = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b(M(T), k) x^{n-2k}$$

where $b(M(T), 0) = 1$, and $b(M(T), k) = \sum_{\alpha_k \in \mathcal{M}_k(T)} GA_G(\alpha_k)$ for $1 \le k \le \lfloor \frac{n}{2} \rfloor$.

Lemma 2.1 **([1])** *Let T_1 and T_2 be two trees of order n, and their GA characteristic polynomials be*

$$\phi_{GA}(T_1, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b(M(T_1), k) x^{n-2k}, \quad \phi_{GA}(T_2, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b(M(T_2), k) x^{n-2k},$$

respectively. If $b(M(T_1), k) \ge b(M(T_2), k)$ for all $k \ge 0$, and there is a positive integer k such that $b(M(T_1), k) > b(M(T_2), k)$, then

$$\mathcal{E}_{GA}(T_1) > \mathcal{E}_{GA}(T_2).$$

Lemma 2.2 **([5])** *Let T be a tree of order n. Then $|\mathcal{M}_k(T)| \le |\mathcal{M}_k(P_n)|$ for $1 \le k \le \lfloor \frac{n}{2} \rfloor$.*

Lemma 2.3 *Let $f(x, y) = \frac{2\sqrt{xy}}{x+y}$.*

(1) *For $x = 1$ and $y \ge 1$, $f(1, y)$ is a monotonically decreasing function on y.*

(2) *For $x \ge 1$ and $y \ge 1$, $f(x, y) \le 1$, and equality holding if and only if $x = y$.*

3 Main result

In this section, we will prove the following main theorem, it implies that Conjecture 1.2 holds.

Theorem 3.1 *Let T be a tree of order $n \ge 1$. Then*

$$\mathcal{E}_{GA}(T) \le \mathcal{E}_{GA}(P_n).$$

Equality holds if and only if T is isomorphic to the path P_n.

Proof. For $n = 1, 2, 3$, it is clear that the result holds. For $n \geq 4$, by Theorem 1.1, we only need to prove that $\mathcal{E}_{GA}(T) < \mathcal{E}_{GA}(P_n)$ for any tree T of order n with at least three branched vertices.

Let T be a tree of order n with at least three branched vertices. Then there are at least two pendent edges which have no common vertex. This fact implies that T is a tree as depicted in Figure 3.1, where T_1 is a tree of order $n-2$, and $v_2, v_{n-1} \in V(T_1)$.

Consider the tree T' which is obtained from T by replacing T_1 with the path of order $n-2$. Clearly, T' is a path of order of n as depicted in Figure 3.2. For $1 \leq i < j \leq n$, use P_{v_i,v_j} to denote the path of T' from v_i to v_j. We will prove that $\mathcal{E}_{GA}(T) < \mathcal{E}_{GA}(T')$.

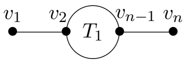

Figure 3.1 Tree T

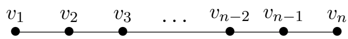

Figure 3.2 Tree T'

Let the GA characteristic polynomials of T and T' be

$$\phi_{GA}(T,x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b(M(T),k) x^{n-2k}, \quad \phi_{GA}(T',x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b(M(T'),k) x^{n-2k},$$

respectively.

By Lemmas 2.2, and 2.3, for $1 \leq i < j \leq n$,

$$GA_T(v_iv_j) \leq \begin{cases} \frac{2\sqrt{2}}{3} < 1, & \text{if } v_iv_j \text{ is a pendent edge of } T, \\ 1, & \text{otherwise,} \end{cases}$$

$$GA_{T'}(v_iv_j) = \begin{cases} \frac{2\sqrt{2}}{3} < 1, & \text{if } i = 1 \text{ or } j = n, \\ 1, & \text{otherwise,} \end{cases},$$

and for any positive integer ℓ,

$$\sum_{\alpha'_\ell \in \mathcal{M}_\ell(P_{v_2,v_{n-1}})} GA_{T'}(\alpha'_\ell) = |\mathcal{M}_\ell(P_{v_2,v_{n-1}})| \geq \sum_{\alpha_\ell \in \mathcal{M}_\ell(T_1)} GA_T(\alpha_\ell),$$

$$\sum_{\alpha'_\ell \in \mathcal{M}_\ell(P_{v_3,v_{n-1}})} GA_{T'}(\alpha'_\ell) = |\mathcal{M}_\ell(P_{v_3,v_{n-1}})| \geq \sum_{\alpha_\ell \in \mathcal{M}_\ell(T_1 - v_2)} GA_T(\alpha_\ell),$$

$$\sum_{\alpha'_\ell \in \mathcal{M}_\ell(P_{v_2,v_{n-2}})} GA_{T'}(\alpha'_\ell) = |\mathcal{M}_\ell(P_{v_2,v_{n-2}})| \geq \sum_{\alpha_\ell \in \mathcal{M}_\ell(T_1 - v_{n-1})} GA_T(\alpha_\ell),$$

$$\sum_{\alpha'_\ell \in \mathcal{M}_\ell(P_{v_3,v_{n-2}})} GA_{T'}(\alpha'_\ell) = |\mathcal{M}_\ell(P_{v_3,v_{n-2}})| \geq \sum_{\alpha_\ell \in \mathcal{M}_\ell(T_1 - v_2 - v_{n-1})} GA_T(\alpha_\ell).$$

Since there are at least three pendent edges of T, it is clear that

$$b(M(T),1) = \sum_{e \in E(T)} GA_T(e) < \sum_{e \in E(T')} GA_T(e) = b(M(T'),1).$$

For $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$,

$$
\begin{aligned}
& b(M(T'),k) \\
&= \sum_{\alpha'_k \in \mathcal{M}_k(T')} GA_{T'}(\alpha'_k) \\
&= \sum_{\alpha'_k \in \mathcal{M}_k(P_{v_2,v_{n-1}})} GA_{T'}(\alpha'_k) + GA_{T'}(v_1v_2) \sum_{\alpha'_{k-1} \in \mathcal{M}_{k-1}(P_{v_3,v_{n-1}})} GA_{T'}(\alpha'_{k-1}) \\
&\quad + GA_{T'}(v_{n-1}v_n) \sum_{\alpha'_{k-1} \in \mathcal{M}_{k-1}(P_{v_2,v_{n-2}})} GA_{T'}(\alpha'_{k-1}) \\
&\quad + GA_{T'}(v_1v_2)GA_{T'}(v_{n-1}v_n) \sum_{\alpha'_{k-2} \in \mathcal{M}_{k-2}(P_{v_3,v_{n-2}})} GA_{T'}(\alpha'_{k-2}) \\
&\geq \sum_{\alpha_k \in \mathcal{M}_k(T_1)} GA_T(\alpha_k) + GA_T(v_1v_2) \sum_{\alpha_{k-1} \in \mathcal{M}_{k-1}(T_1-v_2)} GA_T(\alpha_{k-1}) \\
&\quad + GA_T(v_{n-1}v_n) \sum_{\alpha_{k-1} \in \mathcal{M}_{k-1}(T_1-v_{n-1})} GA_T(\alpha_{k-1}) \\
&\quad + GA_T(v_1v_2)GA_T(v_{n-1}v_n) \sum_{\alpha_{k-2} \in \mathcal{M}_{k-2}(T_1-v_2-v_{n-1})} GA_T(\alpha_{k-2}) \\
&= b(M(T),k).
\end{aligned}
$$

By Lemma 2.1, $\mathcal{E}_{GA}(T) < \mathcal{E}_{GA}(T') = \mathcal{E}_{GA}(P_n)$. So the theorem now follows. ■

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