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Comparison of Resolvent Energies of Laplacian Matrices

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Abstract

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of a simple graph G of order n. A graph-spectrum-based invariant, resolvent energy, put forward by Gutman et al. [Resolvent energy of graphs, *MATCH Commun. Math. Comput. Chem.* **75** (2016) 279–290], is defined as $ER(G) = \sum_{i=1}^{n} (n-\lambda_i)^{-1}$. After that two more resolvent energies defined in the literature, first one is Laplacian resolvent energy (RL) and the second one is signless Laplacian resolvent energy (RQ). In this paper we define normalized Laplacian resolvent energy (ERN), and give some lower and upper bounds on ER, RL and ERN of graphs, and characterize the extremal graphs. In particular, we obtain some relations between Laplacian resolvent energy (RL) with popular graph invariants, like Kirchhoff index and the number of spanning trees of graphs. Moreover we compare between resolvent energies of different graph matrices.

1 Introduction

Throughout this paper we assume that graphs are finite, undirected and unweighted. Let $G = (V, E)$ be a simple graph of order $|V(G)| (= n)$ with $|E(G)| (= m)$ edges. The maximum degree and the minimum degree of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. If the vertices v_i and v_j are adjacent, we write $v_iv_j \in E(G)$. The adjacency

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matrix $A(G)$ of a graph G is defined by $A(G) = (a_{ij})_{n\times n}$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_iv_j \in E(G), \\ 0 & \text{Otherwise.} \end{cases}$$

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $A(G)$. In what follows, the adjacency spectrum of the graph G, i.e., $\{\lambda_1,\, \lambda_2, \ldots,\, \lambda_n\}$ will be denoted by $S_A(G)$. Very recently, Gutman et al. introduced the resolvent energy [21], and it is defined by

$$ER(G) = \sum_{i=1}^{n} \frac{1}{n-\lambda_i}\ . \tag{1}$$

For its basic mathematical properties, including various lower and upper bounds, see [1, 18–22, 28] and the references therein.

Let $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ be, respectively, the Laplacian matrix and the signless Laplacian matrix of the graph G, where $D(G)$ is the diagonal matrix of vertex degrees. The eigenvalues of $L(G)$ and $Q(G)$ will be denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $q_1 \geq q_2 \geq \cdots \geq q_n$, respectively. The Laplacian spectrum and the signless Laplacian spectrum of graph G are denoted by $S_L(G) = \{\mu_1,\, \mu_2, \ldots,\, \mu_n\}$ and $S_Q(G) = \{q_1,\, q_2, \ldots,\, q_n\}$, respectively. The Laplacian resolvent energy and signless Laplacian resolvent energy were recently put forward in [7] and are defined as

$$RL(G) = \sum_{i=1}^{n} \frac{1}{n+1-\mu_i} \tag{2}$$

and

$$RQ(G) = \sum_{i=1}^{n} \frac{1}{2n-1-q_i}, \tag{3}$$

respectively. Since the interlacing property holds for $L(G)$ and $Q(G)$, it is easy to see that $RL(G) \geq RL(H)$ and $RQ(G) \geq RQ(H)$, where H is a subgraph of G. For more results on RL and RQ, we refer readers to the references [7, 25].

The normalized Laplacian matrix $\mathcal{L}(G)$ of graph G is defined as $D^{-1/2}(G)L(G)D^{-1/2}(G)$ (with the convention that if the degree of v_i is 0, then $d_i^{-1/2} = 0$), with eigenvalues $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_{n-1}(G) \geq \rho_n(G) = 0$. The normalized Laplacian spectrum of graph G is denoted by $S_N(G) = \{\rho_1,\, \rho_2, \ldots,\, \rho_n\}$. From the motivation of ER, RL and RQ, taking into account that the condition $\rho_i \leq 2$ $(1 \leq i \leq n)$ is satisfied by all eigenvalues of all n-vertex graphs, we also define the normalized Laplacian resolvent energy of a graph

G as follows:

$$ERN(G) = \sum_{i=1}^{n} \frac{1}{3-\rho_i}. \tag{4}$$

Since the interlacing property does not satisfy for normalized Laplacian matrix of a graph, ERN is quite different from RL and RQ. Moreover, ERN is not related with the number of closed walks, so it is also different from ER.

As usual, P_n, K_n, C_n and $K_{p,q}\,(p+q=n,\, p \geq q)$ denote, respectively, the path, the complete, the cycle, and the complete bipartite graph on n vertices. For a subset W of $E(G)$, let $G - W$ be the subgraph of G obtained by deleting edges of W. When more than one graph is under consideration, then we write λ_i (μ_i or q_i or ρ_i) instead of $\lambda_i(G)$ ($\mu_i(G)$ or $q_i(G)$ or $\rho_i(G)$). Farrugia discussed about the increase in the resolvent energy of a graph due to the addition of a new edge in [19]. In [1], Allem et al. presented some results on the extremal resolvent energy of unicyclic graphs, bicyclic graphs and tricyclic graphs. Ghebleh et al. [20] characterized the extremal graphs on the k-th smallest resolvent energy. In [9], Das proved that $ER(T) < ER(S_n^*) < ER(K_{1,n-1})$ for any tree T ($\ncong K_{1,n-1}$, S_n^*, where S_n^* is a tree with maximum degree $n-2$ of order n), and $ER(C_n) > ER(K_{1,n-1})$ for even n.

One of the referees mentioned the following: "resolvent energy" can be defined for any matrix. By this, the entire theory of graph energies would be duplicated, which would be a most unfortunate direction of future research in this area. His/her opinion is that one should stop at the Laplacian matrices (adjacency, Laplacian, signless Laplacian, normalized Laplacian). The comparison between any two resolvent energies will be reported in Section 4. Results of this kind deserve to be published (on a single occasion) as a kind of curiosity. The results in this paper are very far from being chemically relevant, which additionally makes them of borderline suitability for MATCH.

The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we present some lower and upper bounds on ER, RL and ERN of graph G and characterize graphs for which these bounds are best possible. In particular, we obtain some relations between Laplacian resolvent energy (RL) with popular graph invariants, like Kirchhoff index and the number of spanning trees of graphs. In Section 4, we compare between resolvent energies of different graph matrices. In the last section, some concluding remarks are presented.

2 Preliminaries

In this section, we shall list some previously known results that will be needed in the next two sections. First we introduce some well-known results on majorization theory. For this, suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are two non-increasing sequences of real numbers, we say that $\mathbf{x}$ majorizes $\mathbf{y}$ and write $\mathbf{x} \succ \mathbf{y}$ if

$$\sum_{i=1}^{j} x_i \geq \sum_{i=1}^{j} y_i \text{ for } 1 \leq j \leq n-1, \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

Lemma 2.1. [23] *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a non-increasing sequence of real numbers and $\sum\limits_{i=1}^{n} x_i = s$. Then $\mathbf{x} \succ \mathbf{y}$, where $\mathbf{y} = (\frac{s}{n}, \ldots, \frac{s}{n})$.*

Lemma 2.2. [23] *For any convex (resp. concave) function g, if $\mathbf{x} \succ \mathbf{y}$, we have*

$$\sum_{i=1}^{n} g(x_i) \geq \sum_{i=1}^{n} g(y_i) \quad \left(\textit{resp.} \sum_{i=1}^{n} g(x_i) \leq \sum_{i=1}^{n} g(y_i) \right).$$

Moreover, g is strictly convex (resp. strictly concave), then the above equality holds if and only if $\mathbf{x} = \mathbf{y}$.

Lemma 2.3. (Courant-Weyl inequalities) [8] *For a real symmetric matrix M of order n, let $\theta_1(M) \geq \theta_2(M) \geq \cdots \geq \theta_n(M)$ denote its eigenvalues. If M_1 and M_2 are two real symmetric matrices of order n and if $M = M_1 + M_2$, then for every $i = 1, 2, \ldots, n$, we have*

$$\theta_i(M_1) + \theta_1(M_2) \geq \theta_i(M) \geq \theta_i(M_1) + \theta_n(M_2)\,.$$

Next we give results on the Laplacian spectrum of a graph.

Lemma 2.4. [24] *Let G be a graph of order n with its Laplacian spectrum $S_L(G) = \{\mu_1, \mu_2, \ldots, \mu_{n-1}, 0\}$. Then $S_L(\overline{G}) = \{n-\mu_{n-1}, n-\mu_{n-2}, \ldots, n-\mu_1, 0\}$, where $\overline{G}$ is the complement of the graph G.*

The join $G_1 \vee G_2$ of graphs G_1 and G_2 is the graph obtained from the disjoint union of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$.

Lemma 2.5. [10] *Let G be a graph of order n with its Laplacian spectrum $S_L(G) = \{\mu_1, \mu_2, \ldots, \mu_{n-1}, 0\}$. Then $S_L(G \vee K_1) = \{n+1, \mu_1+1, \mu_2, \ldots, \mu_{n-1}+1, 0\}$.*

A Relation between Laplacian eigenvalues and normalized Laplacian eigenvalues of a graph is obtained as follows:

Lemma 2.6. [5] *Let G be a graph of order n with no isolated vertices. For each $1 \le i \le n$, we have*

$$\frac{\mu_i}{\Delta} \le \rho_i \le \frac{\mu_i}{\delta}.$$

Lemma 2.7. [6] *Let G be a graph of order $n \ge 2$ without isolated vertices. Then $\frac{n}{n-1} \le \rho_1 \le 2$ with left equality holding if and only if $G \cong K_n$, and right equality holding if and only if one of connected components of G is bipartite.*

Let H_1 and H_2 be two graphs obtained from K_n $(n > 3)$ by deleting two independent edges and two adjacent edges, respectively. The spectrums of them are given in the following results:

Lemma 2.8. *Let H_1 be a graph defined above. Then*

$$S_A(H_1) = \Big\{s,\, 0,\, 0,\, \underbrace{-1, \ldots, -1}_{n-5},\, t,\, -2\Big\},$$

where s and t are the roots of $x^2 - (n-3)x + 6 - 2n = 0$.

Proof. By simple calculation, one can easily obtain that the characteristic polynomial of $A(H_1)$ is

$$f(x) = x^2(x+1)^{n-5}\,(x+2)\Big(x^2 - (n-3)x + 6 - 2n\Big).$$

Hence we get the required result. ∎

Lemma 2.9. *Let H_2 be a graph defined above. Then*

$$S_A(H_2) = \Big\{a,\, b,\, \underbrace{-1, \ldots, -1}_{n-3},\, c\Big\},$$

where a, b and c are the roots of $x^3 - (n-3)x^2 - (2n-5)x + n - 3 = 0$.

Proof. The characteristic polynomial of $A(H_2)$ is

$$f(x) = (x+1)^{n-3}\Big(x^3 - (n-3)x^2 - (2n-5)x + n - 3\Big).$$

Hence we get the required result. ∎

For any non-trivial graph G, by Lemma 2.3, one can easily see that $q_1 \ge \lambda_1 + \delta_1 \ge \lambda_1 + 1$, where δ_1 is the minimum non-isolated vertex degree. Here we characterize the extremal graphs for bipartite graph G.

Lemma 2.10. *Let G be a bipartite graph of order n. Then $q_1 = 1 + \lambda_1$ if and only if $G \cong t\, K_2 \cup (n-2t)\, K_1$ $(1 \le t \le \lfloor \frac{n}{2} \rfloor)$.*

Proof. Let $q_1 = 1+\lambda_1$. Then we have to prove that $G \cong t\, K_2 \cup (n-2t)\, K_1$ $(1 \le t \le \lfloor \frac{n}{2} \rfloor)$. We consider the following two cases:

Case 1. G is a connected graph. It is well known that $q_1 \ge \mu_1 \ge \Delta + 1 \ge 1 + \lambda_1$. Thus we have $q_1 = \Delta + 1 = 1 + \lambda_1$. Since G is connected, $\mu_1 = \Delta + 1$ gives $\Delta = n - 1$. Again $\lambda_1 = \Delta$ implies that G is isomorphic to a regular graph as G is connected. Since G is bipartite, we conclude that $G \cong K_2$.

Case 2. G is a disconnected graph. Let G_i be the i-th connected component in G, $1 \le i \le k$. Since $q_1(G) = 1 + \lambda_1(G)$, one can easily see that $\lambda_1(G) = q_1(G) - 1 \ge \Delta(G)$. Moreover, we have $\lambda_1(G) \le \Delta(G)$. Thus $\lambda_1(G) = \Delta(G)$ and $q_1(G) = 1 + \Delta(G)$. Also we have

$$\begin{aligned} q_1(G) &= \max\Big\{q_1(G_1),\, q_1(G_2), \ldots,\, q_1(G_k)\Big\},\\ \lambda_1(G) &= \max\Big\{\lambda_1(G_1),\, \lambda_1(G_2), \ldots,\, \lambda_1(G_k)\Big\},\\ \Delta(G) &= \max\Big\{\Delta(G_1),\, \Delta(G_2), \ldots,\, \Delta(G_k)\Big\}. \end{aligned}$$

Claim 1. $q_1(G)$ and $\Delta(G)$ are belong to the same connected component.

Proof of Claim 1. By contradiction we prove this result. For this we assume that there exits two connected components G_p and G_q $(p \ne q)$ such that $q_1(G_p) = q_1(G) \ne q_1(G_q)$ and $\Delta(G_q) = \Delta(G) \ne \Delta(G_p)$. Now, $q_1(G_q) \ge \Delta(G_q) + 1 = \Delta(G) + 1 = q_1(G) = q_1(G_p)$. Moreover, $q_1(G_p) = q_1(G) \ge q_1(G_q)$. Therefore $q_1(G) = q_1(G_p) = q_1(G_q)$, a contradiction. ■

Claim 2. $\lambda_1(G)$ and $\Delta(G)$ are belong to the same connected component.

Proof of Claim 2. There exits a connected component G_p $(1 \le p \le k)$ such that $\lambda_1(G) = \lambda_1(G_p)$. Then we have $\Delta(G) = \lambda_1(G) = \lambda_1(G_p) \le \Delta(G_p)$ and $\Delta(G) \ge \Delta(G_p)$. This implies that $\Delta(G) = \Delta(G_p)$. This completes the proof of the **Claim 2**. ■

By **Claims 1** and **2**, we suppose that $q_1(G)$ & $\Delta(G)$ are in the same connected component G_p $(1 \le p \le k)$, and $\lambda_1(G)$ & $\Delta(G)$ are in the same connected component G_q $(1 \le q \le k)$, where $p \ne q$. Then $q_1(G) = q_1(G_p)$, $\Delta(G_p) = \Delta(G) = \Delta(G_q)$ and

$\lambda_1(G) = \lambda_1(G_q)$. Now,

$$q_1(G_q) \geq \Delta(G_q) + 1 = \Delta(G) + 1 = q_1(G) \ \text{ and } \ q_1(G) \geq q_1(G_q).$$

Thus we have $q_1(G_q) = q_1(G) = q_1(G_p)$. Hence $q_1(G)$, $\lambda_1(G)$ and $\Delta(G)$ are in the same connected component G_q. Therefore $q_1(G_q) = \lambda_1(G_q) + 1$ and $\lambda_1(G_q) = \Delta(G_q)$. By **Case 1**, we have $G_q \cong K_2$ and hence $\Delta(G) = 1$. Therefore $G \cong t\,K_2 \cup (n-2t)\,K_1$ $(1 \leq t \leq \lfloor \frac{n}{2} \rfloor)$.

Conversely, if $G \cong t\,K_2 \cup (n-2t)\,K_1$ $(1 \leq t \leq \lfloor \frac{n}{2} \rfloor)$, then $q_1 = 2 = 1 + \lambda_1$ holds. ■

We recall the definition of the k-th spectral moment of the graph G, which is defined as

$$M_k(G) = \sum_{i=1}^{n} \lambda_i^k.$$

In [21], the relation between resolvent energy and spectral moments is obtained as follows:

$$ER(G) = \frac{1}{n} \sum_{k=0}^{\infty} \frac{M_k(G)}{n_k}. \tag{5}$$

Now we introduce the following transformations:

Let u be a non-isolated vertex of a simple graph G. If G_1 and G_2 are the graphs obtained from G by identifying an end vertex and the internal vertex of path P_r to u $(r \geq 3)$, respectively.

Transformation A: $G_2 \longrightarrow G_1$.

The following result is obtained in [17].

Lemma 2.11. [17] *Let G_1 and G_2 be the graphs defined above. Then $M_{2k}(G_1) < M_{2k}(G_2)$ and $M_{2k-1}(G_1) \leq M_{2k-1}(G_2)$ for $k \geq 2$.*

Let uv be an edge of a complete graph K_{n-s-t} with $n \geq s+t+3$, $s, t \geq 1$. If G_3 is the graph obtained from K_{n-s-t} by adding an edge between u and an end vertex of path P_{s+t}, and G_4 is the graph obtained from K_{n-s-t} by adding an edge between u and an end vertex of path P_s, and another edge between v and an end vertex of another path P_t.

Transformation B: $G_4 \longrightarrow G_3$.

According to Lemma 2.11, we obtain a similar result on **Transformation B**. As the proof of the following result is very similar to the proof of Lemma 2.11, we omit the proof.

Lemma 2.12. *Let G_3 and G_4 be the graphs defined above. Then $M_{2k}(G_3) < M_{2k}(G_4)$ and $M_{2k-1}(G_3) \leq M_{2k-1}(G_4)$ for $k \geq 2$.*

We now give results on resolvent energy which are obtained in [21].

Lemma 2.13. [21] *Let G be a graph of order n. Then $ER(G) \leq \frac{2n}{n+1}$ with equality holding if and only if $G \cong K_n$.*

Lemma 2.14. [21] *If e is an edge in graph G, denote by $G - e$ the subgraph obtained by deleting e from G. For any edge e of G, we have $ER(G - e) < ER(G)$.*

3 Bounds on different resolvent energies of graphs

In this section we give some lower and upper bounds on ER, RL and ERN of graphs. Recall that a kite $Ki_{n,\omega}$ is the graph obtained from a clique K_ω and a path $P_{n-\omega}$ by adding an edge between a vertex from the clique and an endpoint from the path. First we give a lower bound on ER in terms of n and clique number ω.

Theorem 3.1. *Let G be a connected graph of order n and clique number ω. Then $ER(G) \geq ER(Ki_{n,\omega})$ with equality holding if and only if $G \cong Ki_{n,\omega}$.*

Proof. If $G \cong Ki_{n,\omega}$, then the equality holds. Otherwise, $G \ncong Ki_{n,\omega}$. Then we have to prove that $ER(G) > ER(Ki_{n,\omega})$. First we assume that G is a tree. Then we have $G \ncong P_n$. Now we apply **Transformation A** on G several times (at least one time), we obtain a path P_n. Hence by Lemma 2.11 in (5), we have $ER(G) > ER(P_n)$. Next we assume that G is not a tree. Let G' be a subgraph of G with order n and clique number ω such that $G' - E(K_\omega)$ is a forest. Then we have

$$G' - E(K_\omega) = \bigcup_{i=1}^{\omega} T_i,$$

where T_i is a tree of order $|V(T_i)| \geq 1$, $1 \leq i \leq \omega$. Therefore by Lemma 2.14, we have $ER(G) \geq ER(G')$ with equality holding if and only if $G \cong G'$. For $G' \cong Ki_{n,\omega}$, we have $ER(G) > ER(G') = ER(Ki_{n,\omega})$ as $G \ncong Ki_{n,\omega}$. Otherwise, $G' \ncong Ki_{n,\omega}$. We apply **Transformation A** on G' several times, we obtain a new graph G'' such that

$$G'' - E(K_\omega) = \bigcup_{i=1}^{\omega} P_{n_i},$$

where P_{n_i} is a path of order n_i and $n_1 \geq n_2 \geq \cdots \geq n_\omega$. If $n_2 = 1$, then $G'' \cong Ki_{n,\omega}$ and hence $ER(G) \geq ER(G') > ER(G'') = ER(Ki_{n,\omega})$. Otherwise, $n_2 \geq 2$. We apply **Transformation B** on G'' for several times until the resultant graph is isomorphic to $Ki_{n,\omega}$. Then by Lemmas 2.11 and 2.12 in (5), we obtain $ER(G) \geq ER(G') \geq \cdots \geq ER(G'') > \cdots > ER(Ki_{n,\omega})$. This completes the proof of the theorem. ■

Denote by $K_{n_1, n_2, \ldots, n_k}$ a complete k-partite graph of order n whose partition sets are of size $n_1, n_2, \ldots, n_k$, respectively. Hereafter we always assume that $n_1 \geq n_2 \geq \cdots \geq n_k$. The Turán graph $T(n, k)$ is a complete multipartite graph formed by partitioning a set of n vertices into k subsets, with sizes as equal as possible, and connecting two vertices by an edge if and only if they belong to different subsets. We now present an upper bound on Laplacian resolvent energy of graphs in terms of order n and chromatic number k.

Theorem 3.2. *Let G be a graph of order n with the chromatic number k. Then*

$$RL(G) \leq 2k - 1 + \frac{1}{n+1} - \frac{2(2k + 2tk - n)}{(t+2)(t+1)},$$

where $t = \left\lfloor \frac{n}{k} \right\rfloor$. Moreover, the equality holds if and only if $G \cong T(n, k)$.

Proof. Since G has order n and the chromatic number k, we can assume that $V(G) = S_1 \cup S_2 \cup \cdots \cup S_k$ with $S_i \cap S_j = \emptyset$ $(1 \leq i \neq j \leq k)$ such that for any edge $v_i v_j \in E(G)$, $v_i \in S_p$, $v_j \in S_q$ $(1 \leq p \neq q \leq k)$ and $|S_i| = n_i$, $1 \leq i \leq k$, $n_1 \geq n_2 \geq \cdots \geq n_k$, where $\sum\limits_{i=1}^{k} n_i = n$. Therefore G is a subgraph of $K_{n_1, n_2, \ldots, n_k}$ and hence

$$RL(G) \leq RL(K_{n_1, n_2, \ldots, n_k}) \tag{6}$$

with equality holding if and only if $G \cong K_{n_1, n_2, \ldots, n_k}$. Since

$$S_L(K_{n_1, n_2, \ldots, n_k}) = \Big\{ \underbrace{n, \ldots, n}_{k-1}, \underbrace{n - n_1, \ldots, n - n_1}_{n_1 - 1}, \underbrace{n - n_2, \ldots, n - n_2}_{n_2 - 1}, \underbrace{n - n_k, \ldots, n - n_k}_{n_k - 1}, 0 \Big\}.$$

Now,

$$RL(K_{n_1, n_2, \ldots, n_k}) = k - 1 + \frac{1}{n+1} + \sum_{i=1}^{k} \frac{n_i - 1}{n_i + 1} = 2k - 1 + \frac{1}{n+1} - 2\sum_{i=1}^{k} \frac{1}{n_i + 1}. \tag{7}$$

Since $n_1 \geq n_2 \geq \cdots \geq n_k$ with $\sum\limits_{i=1}^{k} n_i = n = (n - tk)(t+1) + (k + tk - n)t$ $\left(t = \left\lfloor \frac{n}{k} \right\rfloor\right)$, then by Lemma 2.1, we obtain

$$(n_1, n_2, \ldots, n_k) \succ \Big(\underbrace{t+1, \ldots, t+1}_{n-tk}, \underbrace{t, \ldots, t}_{k+tk-n} \Big).$$

By Lemma 2.2 with the fact that $f(x) = \frac{1}{x+1}$ for $x > 0$ is strictly convex, from (6) and (7), we have

$$RL(G) \leq 2k-1+\frac{1}{n+1}-\frac{2(n-tk)}{t+2}-\frac{2(k+tk-n)}{t+1}=2k-1+\frac{1}{n+1}-\frac{2(2k+2tk-n)}{(t+2)(t+1)},$$

with equality holding if and only if G is the Turán Graph. This completes the proof of the theorem. ■

A complete split graph $CS_{n,\alpha}$ is a graph on n vertices consisting of a clique on $n-\alpha$ vertices and a stable set on the remaining α vertices in which each vertex of the clique is adjacent to each vertex of the stable set. Note that $CS_{n,\alpha} \cong K_{\alpha,\underbrace{1,\,1,\,\ldots,\,1}_{n-\alpha}}$. We now give an upper bound on RL in terms of order n and independence number α.

Theorem 3.3. *Let G be a graph of order n with independence number α. Then*

$$RL(G) \leq n-\alpha+1-\frac{2}{\alpha+1}+\frac{1}{n+1} \tag{8}$$

with equality holding if and only if $G \cong CS_{n,\alpha}$.

Proof. Since the independence number of graph G is α, therefore G is a subgraph of $CS_{n,\alpha}$. For $G \cong CS_{n,\alpha}$, since

$$S_L(CS_{n,\alpha}) = \Big\{\underbrace{n,\ldots,n}_{n-\alpha},\underbrace{n-\alpha,\ldots,n-\alpha}_{\alpha-1},0\Big\},$$

then the equality in (8) holds. Otherwise, $G \ncong CS_{n,\alpha}$. Then G is a proper subgraph of $CS_{n,\alpha}$. By interlacing property, one can easily see that $\mu_i(G) \leq \mu_i(CS_{n,\alpha})$ for $1 \leq i \leq n-1$ and there exists at least one value (say j) such that $\mu_j(G) < \mu_j(CS_{n,\alpha})$. Hence we get

$$RL(G) = \sum_{i=1}^{n}\frac{1}{n+1-\mu_i(G)} < \sum_{i=1}^{n}\frac{1}{n+1-\mu_i(CS_{n,\alpha})} = n-\alpha+\frac{\alpha-1}{\alpha+1}+\frac{1}{n+1}.$$

This completes the proof of the theorem. ■

The Kirchhoff index of a connected graph G with order n is denoted by $Kf(G)$ and is defined as

$$Kf(G) = n\sum_{i=1}^{n-1}\frac{1}{\mu_i}.$$

The Kirchhoff index found noteworthy applications in chemistry, as a molecular structure descriptor [2,27], and many of its mathematical properties have been established in [16,26] and their references. Here we present a relation between Laplacian resolvent energy and Kirchhoff index of graphs.

Proposition 3.4. *Let G be a graph of order n with its complement graph $\overline{G}$. Then $Kf(\overline{G} \vee K_1) = (n+1)\, RL(G)$.*

Proof. By Lemmas 2.4 and 2.5, we have

$$S_L(\overline{G} \vee K_1) = \Big\{n+1,\, n-\mu_{n-1}+1,\, n-\mu_{n-2}+1, \ldots,\, n-\mu_1+1,\, 0\Big\}. \tag{9}$$

Hence

$$Kf(\overline{G} \vee K_1) = \frac{n+1}{n+1} + \sum_{i=1}^{n-1} \frac{n+1}{n-\mu_i+1} = (n+1)\, RL(G).$$

■

Corollary 3.5. *Let G be a graph of order n with m edges and maximum degree Δ. Then*

$$RL(G) \geq \frac{1}{n+1} + \frac{1}{n-\Delta} + \frac{(n-2)^2}{n^2-2m-2}$$

with equality holding if and only if $G \cong K_n$ or $G \cong K_{1,\,n-1}$.

Proof. Using $\Delta + 1 \leq \mu_1 \leq n$ and the arithmetic-harmonic-mean inequality, from the proof of Proposition 3.4, we obtain

$$\begin{aligned} RL(G) &= \frac{1}{n+1} + \frac{1}{n-\mu_1+1} + \sum_{i=2}^{n-1} \frac{1}{n-\mu_i+1} \\ &\geq \frac{1}{n+1} + \frac{1}{n-\Delta} + \frac{(n-2)^2}{\sum\limits_{i=2}^{n-1}(n-\mu_i+1)} \\ &= \frac{1}{n+1} + \frac{1}{n-\Delta} + \frac{(n-2)^2}{n^2-2m-2}. \end{aligned}$$

Moreover, the equality holds if and only if $\Delta + 1 = \mu_1 = n$ and $\mu_2 = \mu_3 = \cdots = \mu_{n-1}$, that is, if and only if $G \cong K_n$ or $G \cong K_{1,\,n-1}$, by Theorem 2.8 in [11]. ■

Let $t(G)$ be the number of spanning trees of a graph G. It is well-known fact that

$$t(G) = \frac{1}{n} \prod_{i=1}^{n-1} \mu_i,$$

where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ are the Laplacian eigenvalues of graph G. Here we present a lower bound on $RL(G)$ of graphs.

Theorem 3.6. *Let G be a graph of order n with its complement graph $\overline{G}$. Then*

$$RL(G) \geq (n-1) \Big(t(\overline{G} \vee K_1)\Big)^{-1/(n-1)} + \frac{1}{n+1}$$

with equality holding if and only if $G \cong \overline{K}_n$ or $G \cong K_n$.

Proof. By the arithmetic-geometric-mean inequality with (9), we have

$$\begin{aligned} RL(G) &= \frac{1}{n+1}+\sum_{i=1}^{n-1}\frac{1}{n-\mu_i(G)+1}\\ &= \frac{1}{n+1}+\sum_{i=2}^{n}\frac{1}{\mu_i(\overline{G}\vee K_1)}\\ &\geq \frac{1}{n+1}+(n-1)\left(\prod_{i=2}^{n}\mu_i(\overline{G}\vee K_1)\right)^{-1/(n-1)}\\ &= \frac{1}{n+1}+(n-1)\left(t(\overline{G}\vee K_1)\right)^{-1/(n-1)}, \end{aligned}$$

which gives the required result. Moreover, one can easily see that the equality holds if and only if $\mu_1=\mu_2=\cdots=\mu_{n-1}$. By [11], for connected graph G, we have $\mu_1=\mu_2=\cdots=\mu_{n-1}$ if and only if $G\cong K_n$. From the above results, we conclude that the equality holds if and only if $G\cong\overline{K}_n$ or $G\cong K_n$. This completes the proof of the theorem. ■

We now give lower and upper bounds on the normalized Laplacian resolvent energy of graphs in terms of order n.

Theorem 3.7. *Let G be a graph of order n without isolated vertices. Then*

$$\frac{(n-1)^2}{2n-3}+\frac{1}{3}\leq ERN(G)\leq\frac{2n}{3}-\frac{1-(-1)^n}{12}$$

with left equality holding if and only if $G\cong K_n$, and right equality holding if and only if $G\cong\lfloor\frac{n-2}{2}\rfloor K_2\cup P_a$, where $a=n-2\lfloor\frac{n-2}{2}\rfloor$.

Proof. Since G does not contain any isolated vertex, then $\sum\limits_{i=1}^{n-1}\rho_i=n$. Then by Lemma 2.1, we have

$$(\rho_1,\,\rho_2,\ldots,\,\rho_{n-1},\,0)\succ\Big(\underbrace{\frac{n}{n-1},\ldots,\,\frac{n}{n-1}}_{n-1},\,0\Big).$$

Since the function $f(x)=\frac{1}{3-x}$ is strictly convex function for $0\leq x\leq 2$, by Lemma 2.2, we obtain

$$ERN(G)=\sum_{i=1}^{n}\frac{1}{3-\rho_i}\geq\frac{(n-1)^2}{2n-3}+\frac{1}{3}$$

with equality holding if and only if $\rho_i=\frac{n}{n-1}$, $1\leq i\leq n-1$. Then by Lemma 2.7, the above equality holds if and only $G\cong K_n$. Moreover, from $0\leq\rho_i\leq 2$, we have

$$(\rho_1,\,\rho_2,\ldots,\,\rho_{n-1},\,0)\prec\begin{cases}(\underbrace{2,\ldots,\,2}_{n/2},\underbrace{0,\ldots,\,0}_{n/2}) & \text{if } n \text{ is even}\\ (\underbrace{2,\ldots,\,2}_{(n-1)/2},\,1,\underbrace{0,\ldots,\,0}_{(n-1)/2}) & \text{if } n \text{ is odd.}\end{cases}$$

Similarly, by Lemma 2.2, we obtain

$$ERN(G)=\sum_{i=1}^{n}\frac{1}{3-\rho_i}\leq\frac{2n}{3}-\frac{1-(-1)^n}{12}$$

with equality holding if and only if $(\rho_1,\,\rho_2,\ldots,\,\rho_{n-1},\,0)=(\underbrace{2,\ldots,\,2}_{n/2},\underbrace{0,\ldots,\,0}_{n/2})$ for even n and $(\rho_1,\,\rho_2,\ldots,\,\rho_{n-1},\,0)=(\underbrace{2,\ldots,\,2}_{(n-1)/2},\,1,\underbrace{0,\ldots,\,0}_{(n-1)/2})$ for odd n, which follows that G has $n/2$ (resp. $(n-1)/2$) connected components for even (resp. odd) n. From the given condition that G does not contain any isolated vertex, we have $G\cong\frac{n}{2}K_2$ for even n, and $G\cong\frac{n-3}{2}K_2\cup P_3$ for odd n. This completes the proof of the theorem. ■

Theorem 3.8. *Let G be a bipartite graph of order n without isolated vertices. Then*

$$ERN(G)\geq\frac{n}{2}+\frac{1}{3}\tag{10}$$

with equality holding if and only if $G\cong K_{p,\,q}\,(p+q=n,\,p\geq q)$.

Proof. Since G is bipartite, then $\rho_1=2$ and hence $\sum\limits_{i=2}^{n-1}\rho_i=n-2$. By Lemma 2.1, we have

$$(\rho_2,\,\rho_3,\ldots,\,\rho_{n-1})\succ(1,1,\ldots,\,1).$$

Since the function $f(x)=\frac{1}{3-x}$ is strictly convex function for $0\leq x\leq 2$, by Lemma 2.2, we obtain $ERN(G)=\sum\limits_{i=1}^{n}\frac{1}{3-\rho_i}\geq\frac{n}{2}+\frac{1}{3}$. By [12], for any graph G, we have $\rho_2=\rho_3=\cdots=\rho_{n-1}$ if and only if $G\cong K_n$ or $G\cong K_{p,\,q}\,(p+q=n,\,p\geq q)$. The equality in (10) holds if and only if $(\rho_2,\,\rho_3,\ldots,\,\rho_{n-1})=(1,1,\ldots,\,1)$, that is, $G\cong K_{p,\,q}\,(p+q=n,\,p\geq q)$. ■

The Randić energy [3,4] of graph G without isolated vertices is defined as

$$RE=RE(G)=\sum_{i=1}^{n}|\rho_i-1|.$$

For several lower and upper bounds on Randić energy, see [3,4,13–15]. We now obtain a relation between normalized Laplacian resolvent energy ERN and Randić energy RE.

Theorem 3.9. *Let G be a graph of order n without isolated vertices. Then*

$$RE(G)<\sqrt{(n-1)\Big(27\,ERN(G)-13n-1\Big)}.$$

Proof. From the definition we have

$$\begin{aligned} ERN(G) &= \frac{1}{3}\sum_{i=1}^{n}\left(1-\frac{\rho_i}{3}\right)^{-1} \\ &> \frac{1}{3}\sum_{i=1}^{n}\left(1+\frac{\rho_i}{3}+\frac{\rho_i^2}{9}\right) \\ &= \frac{1}{3}\left(n+\frac{n}{3}+\frac{1}{9}\sum_{i=1}^{n}\rho_i^2\right). \end{aligned}$$

From the above, we obtain

$$27\left(ERN(G)-\frac{4n}{9}\right) > \sum_{i=1}^{n}\rho_i^2.$$

Using this result with

$$RE(G) = \sum_{i=1}^{n-1}|\rho_i - 1|,$$

we get

$$RE(G) \le \sqrt{(n-1)\sum_{i=1}^{n-1}(\rho_i^2 - 2\rho_i + 1)} < \sqrt{(n-1)\left(27\,ERN(G) - 13n - 1\right)}.$$

This completes the proof of the theorem. ■

4 Comparison between resolvent energies of different graph matrices

In this section we compare between resolvent energies of different graph matrices. First we find the difference between RL and RQ of graphs.

Theorem 4.1. *Let G be a graph of order $n > 2$ with m edges. Then*

$$RL(G) - RQ(G) > \frac{(n-2)(2m-n)}{(2n-1)[(n+1)(n-2)-2m+n]}.$$

Proof. It is well known that $q_i \le n-2$ for $i = 2, 3, \ldots, n$. Then $n-2-q_i \ge 0$, $i = 2, 3, \ldots, n$. If $\Delta = n-1$, then $\mu_1 = n$ and hence $n-2+\mu_1-q_1 \ge 0$ as $q_1 \le 2n-2$. Otherwise, $\Delta \le n-2$. Since $\mu_1 \ge \Delta+1$ and $q_1 \le 2\Delta$, we have $n-2+\mu_1-q_1 \ge n-1-\Delta \ge 1$.

Using the above results with (2), (3) and the arithmetic-harmonic-mean inequality, we have

$$\begin{aligned}
RL(G) - RQ(G) &= \sum_{i=1}^{n} \left(\frac{1}{n+1-\mu_i} - \frac{1}{2n-1-q_i} \right) \\
&= \sum_{i=1}^{n} \frac{n-2+\mu_i-q_i}{(n+1-\mu_i)(2n-1-q_i)}
\end{aligned}$$

$$= \frac{n-2+\mu_1-q_1}{(n+1-\mu_1)(2n-1-q_1)} + \sum_{i=2}^{n} \frac{n-2+\mu_i-q_i}{(n+1-\mu_i)(2n-1-q_i)}$$

$$\geq \sum_{i=2}^{n} \frac{\mu_i}{(n+1-\mu_i)(2n-1-q_i)} \tag{11}$$

$$\geq \frac{1}{2n-1} \sum_{i=2}^{n-1} \frac{\mu_i}{n+1-\mu_i} \tag{12}$$

$$= \frac{1}{2n-1} \sum_{i=2}^{n-1} \left[-1 + \frac{n+1}{n+1-\mu_i} \right]$$

$$\geq \frac{1}{2n-1} \left[-n+2 + \frac{(n+1)\,(n-2)^2}{(n+1)(n-2)-2m+\mu_1} \right]$$

$$\geq \frac{1}{2n-1} \left[-n+2 + \frac{(n+1)\,(n-2)^2}{(n+1)(n-2)-2m+n} \right]$$

$$= \frac{(n-2)(2m-n)}{(2n-1)[(n+1)(n-2)-2m+n]}.$$

From equality in (12), we have $q_i = 0$ for $i = 2, 3, \ldots, n-1$. Then $q_n = 0$ and hence $G \cong K_2 \cup (n-2)K_1$ or $G \cong nK_1$. Using this one can easily check that the inequality in (11) is strict as $n > 2$. This completes the proof of the theorem. ■

Theorem 4.2. *Let G be a graph of order $n > 2$. Then $RL(G) > RQ(G)$.*

Proof. Let m be the number of edges in G. For $m \leq 1$, we have $G \cong \overline{K}_n$ or $G \cong \overline{K_n - e}$ (e is any edge in K_n) and hence one can easily check that $RL(G) > RQ(G)$. Otherwise, $m \geq 2$. Then $\mu_2 > 0$. From (12), we get the required result. ■

We now find the lower bound on $ER(G) - RQ(G)$ of graphs.

Theorem 4.3. *Let G be a graph of order n with m edges and maximum degree Δ. Then*

$$ER(G) - RQ(G) \geq \frac{n^2(n-\Delta-1)}{n^2(2n-\Delta-1)+2m} \tag{13}$$

with equality holding if and only if $G \cong \overline{K}_n$.

Proof. For $n = 1$, one can easily check that the result holds in (13). Otherwise, $n \geq 2$. By Lemma 2.3 on $Q(G) = D(G) + A(G)$, we have $\theta_i(Q(G)) \leq \theta_1(D(G)) + \theta_i(A(G))$ for $i = 1, 2, \ldots, n$, where $\theta_i(M)$ is the i-th largest eigenvalue of M. Thus we have $q_i \leq \Delta + \lambda_i$ for $i = 1, 2, \ldots, n$. Therefore $2n - 1 - q_i \geq 2n - \Delta - 1 - \lambda_i$ for $i = 1, 2, \ldots, n$. Using this result with arithmetic-harmonic-mean inequality, we obtain

$$\begin{aligned} ER(G) - RQ(G) &= \sum_{i=1}^{n} \left[\frac{1}{n - \lambda_i} - \frac{1}{2n - 1 - q_i} \right] \\ &\geq \sum_{i=1}^{n} \left[\frac{1}{n - \lambda_i} - \frac{1}{2n - \Delta - 1 - \lambda_i} \right] \qquad (14) \\ &= \sum_{i=1}^{n} \frac{n - \Delta - 1}{\left(n - \lambda_i\right)\left(2n - \Delta - 1 - \lambda_i\right)} \\ &\geq \frac{n^2\,(n - \Delta - 1)}{\sum\limits_{i=1}^{n} \left(n - \lambda_i\right)\left(2n - \Delta - 1 - \lambda_i\right)} \qquad (15) \\ &= \frac{n^2\,(n - \Delta - 1)}{n^2(2n - \Delta - 1) + 2m} \end{aligned}$$

as

$$\sum_{i=1}^{n} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i^2 = 2m.$$

The first part of the proof is done.

Suppose that equality holds in (13). Then the equality holds in (15). By arithmetic-harmonic-mean inequality, we have

$$\left(n - \lambda_1\right)\left(2n - \Delta - 1 - \lambda_1\right) = \cdots = \left(n - \lambda_n\right)\left(2n - \Delta - 1 - \lambda_n\right),$$

that is,

$$(\lambda_i - \lambda_j)\left(3n - \Delta - 1 - (\lambda_i + \lambda_j)\right) = 0 \quad \text{for all } 1 \leq i < j \leq n \ \text{ as } n \geq 2.$$

For $1 \leq i < j \leq n$, we have $\lambda_i + \lambda_j \leq 2\lambda_1 \leq 2(n-1) < 3n - \Delta - 1$. From these results, we conclude that $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. Since $\sum\limits_{i=1}^{n} \lambda_i = 0$, we have $\lambda_i = 0$ for all i, $1 \leq i \leq n$. Thus we have $G \cong \overline{K}_n$.

Conversely, let $G \cong \overline{K}_n$. Then $\lambda_i = q_i = 0$ for all i, $1 \leq i \leq n$ and $m = \Delta = 0$. Thus we have

$$ER(G) - RQ(G) = \frac{n-1}{2n-1} = \frac{n^2(n-\Delta-1)}{n^2(2n-\Delta-1)+2m}.$$

■

From the above result, one can easily compare $ER(G)$ and $RQ(G)$ of graph G in the following:

Theorem 4.4. *Let G be a graph of order $n > 1$. Then $ER(G) > RQ(G)$.*

From Theorems 4.2 and 4.4, it is natural to ask the question: which one is true $RL(G) > ER(G)$ or $ER(G) > RL(G)$? For $G \cong K_n$ $(n > 2)$, $RL(G) = n-1+\frac{1}{n+1} > \frac{2n}{n+1} = ER(G)$ and for $G \cong \overline{K}_n$, $RL(G) = \frac{n}{n+1} < 1 = ER(G)$. These two examples show us that $ER(G)$ and $RL(G)$ are incomparable in general case. It motivates us to find out some classes of graphs such that $RL(G) > ER(G)$ or $ER(G) > RL(G)$ holds. We now compare $ER(G)$ and $RL(G)$ of bipartite graph G as follows:

Theorem 4.5. *Let G be a bipartite graph of order n with no isolated vertices. Then $RL(G) \geq ER(G)$ with equality holding if and only if $G \cong \frac{n}{2}K_2$ (n is even).*

Proof. Using Lemma 2.3 on $Q(G) = D(G) + A(G)$, we have $q_i \geq \delta + \lambda_i$. Since G has no isolated vertices, $\delta \geq 1$ and hence $q_i \geq 1 + \lambda_i$ for $i = 1, 2, \ldots, n$. Since G is bipartite, we have $n - \lambda_i \geq n + 1 - \mu_i$ for $i = 1, 2, \ldots, n$. Hence

$$RL(G) - ER(G) = \sum_{i=1}^{n} \left[\frac{1}{n+1-\mu_i} - \frac{1}{n-\lambda_i}\right] \geq 0.$$

The first part of the proof is done.

Suppose that $RL(G) = ER(G)$ holds. Then $\delta = 1$ and $q_i = 1 + \lambda_i$, $1 \leq i \leq n$. Since G is bipartite with no isolated vertices, by Lemma 2.10, we have $G \cong \frac{n}{2}K_2$ (n is even).

Conversely, one can easily see that $RL = ER$ holds for $\frac{n}{2}K_2$ (n is even). ■

By some examples we showed that $RL(G)$ and $ER(G)$ are incomparable, but we have the following result:

Theorem 4.6. *Let G be a graph of order n (> 3). Then $RL(G) + RL(\overline{G}) > ER(G) + ER(\overline{G})$.*

Proof. It is well known that either G or $\overline{G}$ must be connected. Without loss of generality we can assume that G is connected. For $G \cong K_n$, we have

$$S_L(G) = \Big\{\underbrace{n,\,\ldots,\,n}_{n-1},\,0\Big\},\ S_L(\overline{G}) = S_A(\overline{G}) = \{\underbrace{0,\ldots,0}_{n}\},\ S_A(G) = \Big\{n-1,\underbrace{-1,\,\ldots,\,-1}_{n-1}\Big\}$$

and hence

$$RL(G) + RL(\overline{G}) = n > \frac{3n+1}{n+1} = ER(G) + ER(\overline{G}),$$

as $n > 3$. For $G \cong K_n - e$, we have

$$S_L(G) = \Big\{\underbrace{n,\,\ldots,\,n}_{n-2},\,n-2,\,0\Big\},\quad S_L(\overline{G}) = \{2,\underbrace{0,\ldots,0}_{n-1}\},\ S_A(\overline{G}) = \{1,\underbrace{0,\ldots,0}_{n-2},\,-1\}$$

$$\text{and}\quad S_A(G) = \Big\{\frac{n-3+\sqrt{n^2+2n-7}}{2},\,0,\,\underbrace{-1,\ldots,-1}_{n-3},\,\frac{n-3-\sqrt{n^2+2n-7}}{2}\Big\}.$$

Since $n \geq 4$, we have

$$\begin{aligned} RL(G) + RL(\overline{G}) &= \frac{n^2-2}{n+1} + \frac{n+2}{3n-3} > 3 + \frac{1}{n-1} - \frac{1}{n} - \frac{3}{n+1} - \frac{1}{n+4} \\ &= ER(G) + ER(\overline{G}). \end{aligned}$$

Otherwise, G is a subgraph of H_1 or H_2 (see definition before Lemma 2.8). By Lemmas 2.8 and 2.9, we have

$$\begin{aligned} ER(H_1) &= \frac{n-5}{n+1} + \frac{2}{n} + \frac{1}{n+2} + \frac{2n-(s+t)}{(n-s)(n-t)} \\ &= \frac{n-5}{n+1} + \frac{2}{n} + \frac{1}{n+2} + \frac{2n-(n-3)}{n^2-(n-3)n+6-2n} \\ &= 2 + \frac{2}{n} + \frac{1}{n+2} - \frac{3}{n+6} - \frac{6}{n+1} \end{aligned}$$

and

$$\begin{aligned} ER(H_2) &= \frac{n-3}{n+1} + \frac{3n^2-2n(a+b+c)+ab+ac+bc}{(n-a)(n-b)(n-c)} \\ &= \frac{n-3}{n+1} + \frac{3n^2-2n(n-3)+5-2n}{n^3-(n-3)n^2-(2n-5)n+n-3} \\ &= 2 - \frac{4}{n+1} - \frac{2n-8}{n^2+6n-3}. \end{aligned} \tag{16}$$

After simple calculation, we have $ER(H_1) < ER(H_2)$. Since $\overline{G}$ is a proper subgraph of H_2, we have $ER(\overline{G}) < ER(H_2)$ by Lemma 2.14. Using these results with the fact that G is the subgraph of H_1 or H_2, again by Lemma 2.14, we obtain that $ER(G)+ER(\overline{G}) < 2ER(H_2)$. It is well known that $\mu_i(G) = n - \mu_{n-i}(\overline{G})$ for $i = 1, 2, \ldots, n-1$. Then we have

$$\begin{aligned} RL(G) + RL(\overline{G}) &= \sum_{i=1}^{n-1} \left[\frac{1}{n+1-\mu_i(G)} + \frac{1}{\mu_{n-i}(\overline{G})+1} \right] + \frac{2}{n+1} \\ &= \sum_{i=1}^{n-1} \left[\frac{1}{n+1-\mu_i(G)} + \frac{1}{\mu_i(G)+1} \right] + \frac{2}{n+1} \\ &= \sum_{i=1}^{n-1} \frac{n+2}{(n+1-\mu_i(G))(1+\mu_i(G))} + \frac{2}{n+1}. \qquad (17) \end{aligned}$$

Since $f(x) = (n+1-x)(x+1)$ is an increasing function on $x \leq n/2$ and a decreasing function on $x \geq n/2$, we have $f(x) \leq (n+2)^2/4$. Using this result in (17) with (16), we obtain

$$RL(G) + RL(\overline{G}) \geq \frac{4(n-1)}{n+2} + \frac{2}{n+1} \geq 2ER(H_2)$$

as $n \geq 4$. From the above results, we obtain $ER(G) + ER(\overline{G}) < 2ER(H_2) \leq RL(G) + RL(\overline{G})$, which completes the proof of the theorem. ∎

Now we would like to compare between $ERN(G)$ and $ER(G)$ of graph G.

Proposition 4.7. *Let G be a graph of order $n > 2$. Then $ERN(G) > ER(G)$.*

Proof. If G has no isolated vertices, then by Lemma 2.13 and Theorem 3.7, we have

$$ERN(G) \geq \frac{(n-1)^2}{2n-3} + \frac{1}{3} > \frac{2n}{n+1} = ER(G) \text{ as } n > 2.$$

Otherwise, G has isolated vertices. Let $k \geq 1$ be the number of isolated vertices in G. We can assume that $G \cong H \cup kK_1$. Then $S_A(G) = S_A(H) \cup \{\underbrace{0, \ldots, 0}_{k}\}$ and $S_N(G) = S_N(H) \cup \{\underbrace{0, \ldots, 0}_{k}\}$. From the above, we have $ERN(H) \geq ER(H)$. Using these results with $n \geq 3$ and $k \geq 1$, we obtain

$$\begin{aligned} ERN(G) = ERN(H) + \frac{k}{3} \geq ER(H) + \frac{k}{3} &= \sum_{i=1}^{n-k} \frac{1}{n-k-\lambda_i(H)} + \frac{k}{3} \\ &> \sum_{i=1}^{n-k} \frac{1}{n-\lambda_i(H)} + \frac{k}{n} = ER(G). \end{aligned}$$

∎

Next we would like to compare between $ERN(G)$ and $RL(G)$ of graph G. For $n \geq 3$,

$$ERN(K_n) = \frac{(n-1)^2}{2n-3} + \frac{1}{3} < n - 1 + \frac{1}{n+1} = RL(K_n)$$

and

$$ERN(K_{1,n-1}) = \frac{n}{2} + \frac{1}{3} > 2 + \frac{1}{n+1} - \frac{2}{n} = RL(K_{1,n-1}).$$

Thus $ERN(G)$ and $RL(G)$ are incomparable. Hence it is interesting to find some classes of graphs such that one is always greater than the other. Now we give the following two classes of graphs in which $ERN(G) > RL(G)$.

Theorem 4.8. *Let G be a graph of order $n > 2$ with no isolated vertices. If $\Delta \leq \frac{n}{2}$, then $ERN(G) > RL(G)$.*

Proof. Since $\Delta \leq \frac{n}{2}$, we have $2 \leq \frac{n-2}{\Delta-1}$. Using this with Lemma 2.6, for any i, we obtain

$$\mu_i \leq n \leq n - 2 + \frac{n-2}{\Delta - 1}, \quad \text{i.e.,}\ \mu_i \left(1 - \frac{1}{\Delta}\right) \leq n - 2, \quad \text{i.e.,}\ 3 - \rho_i \leq 3 - \frac{\mu_i}{\Delta} \leq n + 1 - \mu_i.$$

Since $n \geq 3$, we have

$$ERN(G) = \sum_{i=1}^{n-1} \frac{1}{3 - \rho_i} + \frac{1}{3} \geq \sum_{i=1}^{n-1} \frac{1}{n + 1 - \mu_i} + \frac{1}{3} > RL(G).$$

This completes the proof of the theorem. ■

Theorem 4.9. *Let G be a bipartite graph of order $n \geq 3$. Then $ERN(G) > RL(G)$.*

Proof. Let k be the number of isolated vertices in G. We consider the following cases:

Case 1. $k = 0$. Let p and q be the sizes of bipartition of G. Then $p + q = n$. Without loss of generality, we can assume that $p \geq q$. It is clear that G must be a subgraph of $K_{p,q}$, and then $\mu_i(G) \leq \mu_i(K_{p,q})$, $i = 1, \ldots, n$. It is well known that

$$S_L(K_{p,q}) = \Big\{ n, \underbrace{p, \ldots, p}_{q-1}, \underbrace{q, \ldots, q}_{p-1}, 0 \Big\}.$$

Since

$$\frac{1}{p+1} + \frac{1}{q+1} \geq \frac{4}{n+2},$$

we obtain

$$RL(G) \leq RL(K_{p,q}) = \frac{p-1}{p+1} + \frac{q-1}{q+1} + \frac{n+2}{n+1} \leq 3 + \frac{1}{n+1} - \frac{8}{n+2}.$$

Using the above result with Theorem 3.8, we get

$$ERN(G) \geq \frac{n}{2} + \frac{1}{3} > 3 + \frac{1}{n+1} - \frac{8}{n+2} \geq RL(G)$$

as $n \geq 3$.

Case 2. $k \geq 1$. We can assume that $G \cong H \cup kK_1$. Then $S_L(G) = S_L(H) \cup \{\underbrace{0, \ldots, 0}_{k}\}$ and $S_N(G) = S_N(H) \cup \{\underbrace{0, \ldots, 0}_{k}\}$. If $k \geq n-2$, then $G \cong K_2 \cup (n-2)K_1$ or $G \cong nK_1$, and hence one can easily check that $ERN(G) > RL(G)$ as $n \geq 3$. Otherwise, $k \leq n-3$, which implies that H has order at least 3. Then by **Case 1**, we have $ERN(H) > RL(H)$. Using these results with $n \geq 3$ and $k \geq 1$, we obtain

$$\begin{aligned} ERN(G) = ERN(H) + \frac{k}{3} > RL(H) + \frac{k}{3} &= \sum_{i=1}^{n-k} \frac{1}{n-k+1-\mu_i(H)} + \frac{k}{3} \\ &> \sum_{i=1}^{n-k} \frac{1}{n+1-\mu_i(H)} + \frac{k}{n+1} = RL(G). \end{aligned}$$

This completes the proof of the theorem. ■

Theorem 4.10. *Let G be a graph of order $n > 2$. If $d_2(G) < n-1$ and $d_2(\overline{G}) < n-1$, then $ERN(G) + ERN(\overline{G}) > RL(G) + RL(\overline{G})$, where $d_2(G)$ and $d_2(\overline{G})$ are the second maximum degrees of G and its complement graph $\overline{G}$, respectively.*

Proof. It is clear that either G or $\overline{G}$ is connected. Moreover, $d_2(G) < n-1$ and $d_2(\overline{G}) < n-1$ are given. Without loss of generality, we can assume that G is connected. From the condition that $d_2(G) < n-1$, then G has at most one vertex of degree $n-1$. Therefore, we can assume that $\overline{G} \cong tK_1 \cup H$, where $t \leq 1$ and H is a graph with order $n-t$ and no isolated vertices. By Theorem 3.7, we get

$$ERN(G) + ERN(\overline{G}) \geq \frac{(n-1)^2}{2n-3} + \frac{(n-t-1)^2}{2n-2t-3} + \frac{t+2}{3} \geq \frac{(n-1)^2}{2n-3} + \frac{(n-2)^2}{2n-5} + 1. \quad (18)$$

It is easy to check that $(n+1-\mu_i(G))(1+\mu_i(G)) \geq n+1$ as $0 \leq \mu_i(G) \leq n$. Using this result in (17), we have

$$RL(G) + RL(\overline{G}) \leq \frac{(n+2)(n-1)}{n+1} + \frac{2}{n+1} = n.$$

Combining the above result and the result in (18) with the fact that $\frac{(n-1)^2}{2n-3} + \frac{(n-2)^2}{2n-5} + 1 > n$ $(n > 2)$, we can get the required result. ■

Remark 4.11. *From the above theorem, we can see that* $ERN(G)+ERN(\overline{G}) > RL(G)+RL(\overline{G})$ *is true for most cases. However, for the nearly complete graphs, the result does not hold.*

5 Conclusion

From the motivation of resolvent energy (ER), Laplacian resolvent energy (ENL) and signless Laplacian resolvent energy (RQ), in this paper, we defined the normalized Laplacian resolvent energy (ERN) and obtained several bounds on them. Therefore, we confirmed that

$$ERN > ER > RQ, \text{ and } RL > RQ,$$

for a graph with order $n \geq 3$. In particular, we proved that $ERN > RL \geq ER$ for a bipartite graph without isolated vertices. However, RL & ERN and RL & ER are incomparable in general, which is showed by some examples in last section. By computer, we checked that $RL > ER$ is true for all connected graphs with order $3 \leq n \leq 9$. Is it true that $RL > ER$ for all connected graphs with order $n \geq 3$? We leave it as an open problem for future research.

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