On the Algebraic Analysis of Chiral Amplification in Chemical Reaction Networks

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Abstract

We introduce an algebraic criterion that we call the MM-condition, and which can be used to analyze the chemical mechanisms that are used to explain spontaneous enantioselective synthesis, something that apparently occurred in prebiotic earth. We exemplify the power and soundness of our criterion using a concrete network as probe model. We choosed to work with a complex network introduced by Plasson and which is called the APED model. We can explain, using our mathematical criteria, some facts related to the qualitative behavior of this model and which were previously observed by means of computer simulations. It is important to remark that our analytical machinery can be fully automatized, and that it can be used to analyze any feasible network model of the prebiotic synthesis of enantiopure compounds.

1 Introduction

The strongly biased chirality of biomolecules is one of the most intriguing phenomena related to the chemistry of life. We would like to shed some light on the chemical mechanisms that favour *enantioselective synthesis* [3]. We have that most of the proposed models of *enantioselective synthesis* are constituted by two main steps:

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1. *Spontaneous mirror symmetry breaking*, which corresponds to the creation of a small enantiomeric excess [9].

2. *Chiral amplification*, which corresponds to the creation of a large enantiomeric excess from a negligible gap between the initial concentrations of the enantiomeric species.

Known mechanisms for the production of small enantiomeric gaps include electroweak interaction and circularly polarized light [3]. We are interested in the second step, namely chiral amplification, which corresponds to a qualitative feature of the dynamical systems that are determined by the chemical reaction networks that exhibit this kind of behavior. We focus on the following problem:

To develop a mathematical (and algorithmic) methodology that can be used to detect the chemical reaction networks exhibiting chiral amplification.

Linear stability analysis (LSA, for short) is a mathematical methodology that can be used to detect the reaction networks exhibiting chemical instabilities. However, we are not interested in general instabilities, we are interested in a specific type of instabilities, the ones that can produce chiral amplification. Thus, we would try to adapt some of the tools provided by LSA in order to deal with our problem.

Let Ω be a chemical reaction network, let $\mathbf{s}$ be a state of Ω, and let $\mathcal{J}_\Omega(\mathbf{s})$ be the *Jacobian* of $\mathbf{s}$. According to LSA: $\mathbf{s}$ encodes a chemical instability depending on the structure of the eigenvalues of $\mathcal{J}_\Omega(\mathbf{s})$. Intuition tell us that: $\mathbf{s}$ produces chiral amplification depending on the eigenvectors of $\mathcal{J}_\Omega(\mathbf{s})$ and not only in its eigenvalues. We claim that $\mathbf{s}$ can produce chiral amplification, if and only if, matrix $\mathcal{J}_\Omega(\mathbf{s})$ is non-singular and it has a *symmetry-breaking eigenvector* (see below). Thus, our problem can be formulated in the following way:

Problem 1 *Check-and-Sample*

- *Input: Ω, where Ω is a chemical network.*

- *Problem: decide if Ω admits symmetry-breaking eigenvectors and, in that case, sample the set of steady states admitting symmetry-breaking eigenvectors.*

We would like to remark that any algorithmic solution to the above problem presupposes a mathematical characterization of the states that can produce chiral amplification.

We are mainly interested in the latter mathematical problem, and not in the implementation of an algorithmic tool that can be obtained as a product of our investigations. We have discovered an algebraic criterion that allow us to effectively solve the above problem for the restricted class of *chiral networks* ([2], [7]), and for the more general class of *pseudochiral networks*. The aforementioned criterion yields an algorithm that can be used to automatize the analysis of pseudochiral networks. However, we have to remark that this algorithm is inefficient, and it could be useless when applied to networks whose size is moderately large. We use the aforementioned algebraic criterion to develop a rigorous and detailed mathematical analysis of a network model of chiral amplification: the APED Model introduced by Plasson et al [8]. Actually, this paper could be read as a rigorous mathematical analysis of the APED model, but taking into account that it is not an ad-hoc analysis, and that any other network model of chiral amplification can be analyzed along the lines of the present work.

2   Chemical Networks and Mass Action Kinetics

A *chemical reaction* over the *chemical species* $X_1, ..., X_n$ is an expression like

$$c_1 X_1 + \cdots + c_n X_n \to d_1 X_1 + \cdots + d_n X_n,$$
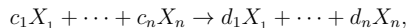
where $c_1, ..., c_n$ and $d_1, ..., d_n$ are small integers (some of which could be equal to zero). The above expression indicates that the mixture of c_1 units of $X_1, ...,$ and c_n units of X_n gives place to d_1 units of $X_1, ...,$ and d_n units of X_n.

Definition 2 *A chemical network over the species* $\{X_1, ..., X_n\}$ *is a set of chemical reactions, say the set* $\{R_1, ..., R_r\}$, *over this set of species.*

Given a chemical network $\Omega = ((X_1, ..., X_n), (R_1, ..., R_r))$ we use the expression

$$c_{1i} X_1 + \cdots + c_{ni} X_n \to d_{1i} X_1 + \cdots + d_{ni} X_n$$
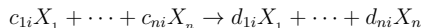
to denote the reaction R_i.

Notation 3 *Let* $\Omega = ((X_1, ..., X_n), (R_1, ..., R_r))$ *be a chemical network, we use variables* $[X_1], ..., [X_n]$ *to denote the concentrations of the n chemical species.*

Let $(k_1, ..., k_r)$ be a vector of rate constants related to the reactions $R_1, ..., R_r$, if the values of $k_1, ..., k_r$ correspond to the rate constants of those reactions then, and according

to the *law of mass-action,* the dynamics of the network is governed by the polynomial system of differential equations given by

$$\frac{dx_i}{dt} = \sum_{j=1}^{r} k_j \left(d_{ij} - c_{ij} \right) \left([X_1]^{c_{ij}} \cdot \cdots \cdot [X_n]^{c_{ij}} \right), \ i = 1, ..., n.$$

We say, by an abuse of language, that the above system is the ODE system determined by Ω.

We want to study the dynamics of the chemical reaction networks that could model enantioselective synthesis in prebiotic earth, and which we suppose governed by the law of mass-action. Notice that all those dynamics are deterministic, and hence the time-evolution of all those chemical mechanisms are completely determined by their initial states. An initial state of the network Ω can be fully described by a $(n + r)$-tuple

$$\left([X_1^0], ..., [X_n^0], k_1^0, ..., k_s^0 \right),$$

of non-negative reals.

Definition 4 *We say that the state $([X_1], ..., [X_n], k_1, ..., k_s)$ is a steady state, if and only if, it satisfies the steady state conditions given by*

$$0 = \sum_{i=1}^{r} (d_{1i} - c_{1i}) k_i [X_1]^{c_{1i}} \cdots [X_n]^{c_{ni}}.$$

$$\vdots$$

$$0 = \sum_{i=1}^{r} (d_{ni} - c_{ni}) k_i [X_1]^{c_{1i}} \cdots [X_n]^{c_{ni}}.$$

Notation 5 *Let Ω be a chemical reaction network, we use the symbol $\mathcal{J}_\Omega$ to denote the symbolic Jacobian of the ODE system determined by Ω. Notice that all the entries of $\mathcal{J}_\Omega$ are polynomials over the variables*

$$[X_1], ..., [X_n], k_1, ..., k_r.$$

Given a steady state $\mathbf{s}$, we use the symbol $\mathcal{J}_\Omega(\mathbf{s})$ to denote the Jacobian of $\mathbf{s}$, which is the numerical matrix that is obtained after evaluating $\mathcal{J}_\Omega$ at $\mathbf{s}$ (after evaluating all the polynomial entries of $\mathcal{J}_\Omega$ at $\mathbf{s}$)
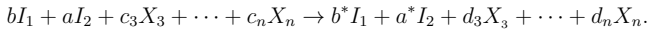
3 Analysis of Chiral Networks

Enantiomers are, to some extent and in the absence of chiral agents, indiscernible from the point of view of chemical kinetics. The latter implies that any realistic network model of

chiral amplification must satisfy some symmetries that are related to the *indiscernibility* of enantiomers. The first symmetry constraint that comes to mind is the following one

Definition 6 *Suppose that* $\Omega = ((I_1, I_2, X_3, ..., X_n), (R_1, ..., R_r))$ *is a chemical network and suppose that* I_1 *and* I_2 *represent a pair of enantiomers. We say that* $\Omega = ((I_1, I_2, X_3, ..., \quad X_n), (R_1, ..., R_r))$ *is a chiral network, if and only if, given* R_i *equal to*

$$aI_1 + bI_2 + c_3X_3 + \cdots + c_nX_n \to a^*I_1 + b^*I_2 + d_3X_3 + \cdots + d_nX_n,$$
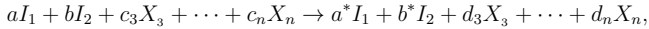
there exists $j \leq r$ *such that* R_j *is equal to*

$$bI_1 + aI_2 + c_3X_3 + \cdots + c_nX_n \to b^*I_1 + a^*I_2 + d_3X_3 + \cdots + d_nX_n.$$

We say in the latter case that reactions R_i *and* R_j *are dual reactions. Moreover, if* R_i *and* R_j *are dual reactions their reactions rates must be the same. The latter fact allows us to talk about the reaction rate of the dual pair* (R_i, R_j).

If reaction R_j *has the form*

$$aI_1 + aI_2 + c_3X_3 + \cdots + c_nX_n \to bI_1 + bI_2 + d_3X_3 + \cdots + d_nX_n,$$
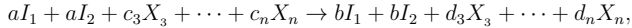
we say that it is a self-dual reaction since it is equal to its dual reaction.

It happens that many different models of biological homochirality are chiral networks, as it is the case with the classical Frank model [4], the network of Kondepudi-Nelson [6] and the network of Iwamoto [5]. It also happens that those networks are fairly easy to analyze.

Definition 7 *We say that* **s** *is a symmetry-breaking state for the pair* $(\{1\}, \{2\})$, *if and only if, there exists an eigenvalue of* $\mathcal{J}_\Phi(\mathbf{s})$, *say the eigenvalue* λ, *such that:*

1. $Re(\lambda) > 0$.

2. There exists a vector **v** *such that* $\mathcal{J}_\Phi(\mathbf{s}) \cdot \mathbf{v} = \lambda \mathbf{v}$ *and* $\mathbf{v}[1] \neq \mathbf{v}[2]$.

Let $\Omega = ((I_1, I_2, X_3, ..., X_n), (R_1, ..., R_r))$ be a chiral network. It can be argued that the symmetry-breaking states of Ω are the states that can produce chiral amplification [1], [2]. Thus, we are interested in testing and sampling the set

$$\mathcal{SB}(\Omega) = \left\{ \mathbf{s} \in \mathbb{R}^{n+r} : \mathbf{s} \text{ is a symmetry-breaking state of } \Omega \text{ for the pair } (\{1\}, \{2\}) \right\}.$$

We have that (see reference [2]).

Theorem 8 *Frank Inequality*

Let Ω be a chiral network, let $\mathbf{s}$ be a steady state of Ω and let $\mathcal{J}_\Omega(\mathbf{s})$ be the Jacobian of $\mathbf{s}$, we have that state $\mathbf{s}$ is a symmetry-breaking state, if and only if, the inequality

$$\mathcal{J}_\Omega(\mathbf{s})[1,1] - \mathcal{J}_\Omega(\mathbf{s})[1,2] > 0$$

holds.

The above theorem (algebraic criterion) allows us to compute a good *semialgebraic* definition of the set $\mathcal{SB}(\Omega)$, a definition that can be effectively used for checking non-emptiness and for sampling the set of symmetry-breaking states. We have illustrated the latter claim with the analysis of Frank and Kondepudi-Nelson models (see [1] and [2]).

4   Analysis of Pseudochiral Networks

If a chiral network has symmetry-breaking states, then it contains a dual pair of autocatalytic reactions within its set of chemical reactions. The latter fact explains a common feature of chiral models: all those models are based on (contain) autocatalytic steps. It is important to remark, at this point, that there are examples of autocatalytic reactions in nature, and that it has been proved that those reactions give place to chiral amplification [11]. However, autocatalytic reactions seem to be very scarce. Therefore, different authors have introduced (pseudochiral) network models of biological homochirality that avoid the presence of the latter type of reactions. Plasson et al introduced one of those models, the APED model [8], which is equivalent to the pseudochiral network Ω_{APED} defined below.

The constituent species are the species $L, L^*, LL, DL, D, D^*, DD$ and LD. The set of reactions is given by

1. A dual pair of **A**ctivation reactions

$$L \rightarrow L^* \text{ and } D \rightarrow D^*,$$

   whose reaction rate constant is denoted with the symbol a. The network also includes a dual pair of reverse reactions

$$L^* \rightarrow L \text{ and } D^* \rightarrow D,$$

   whose reaction rate constant is denoted with the letter b.

2. A dual pair of homochiral **P**olymerization reactions

$$L + L^* \to LL \text{ and } D + D^* \to DD,$$
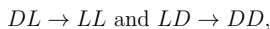
whose reaction rate constant is denoted with the letter p. The network also includes a dual pair of heterochiral polymerization reactions

$$L + D^* \to DL \text{ and } D + L^* \to LD,$$

whose reaction rate constant is denoted with the symbol αp.

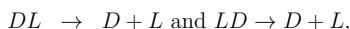3. A dual pair of **E**pimerization reactions

$$DL \to LL \text{ and } LD \to DD,$$

whose reaction rate constant is denoted with the letter e. The network also includes a dual pair of reverse reactions

$$LL \to DL \text{ and } DD \to LD,$$

whose reaction rate constant is denoted with the symbol γe.

4. **D**epolymerization of the formed dimers, given by the dual pairs of reactions

$$LL \quad \to \quad L + L \text{ and } DD \to D + D;$$
$$DL \quad \to \quad D + L \text{ and } LD \to D + L,$$

whose reaction rate constants are denoted with the symbols h and βh.

We have to observe that the enantiomers L and D can be discerned in the network: notice that L reacts with L^* to produce the homochiral dimer LL, while D reacts with L^* to produce the heterochiral dimer LD. However, we have to observe that the network is constituted by two *enantiomeric clusters*, the sets $\{L, L^*, LL, DL\}$ and $\{D, D^*, DD, LD\}$, which cannot be distinguished in the network.

Definition 9 *A pseudochiral network is a chemical reaction network that contains a pair of enantiomeric clusters within its set of species, and which are indiscernible in the network.*

Let Ω be pseudochiral network, and let $\{L_1, ..., L_k\}; \{D_1, ..., D_k\}$ be its pair of enantiomeric clusters, we say that Ω is a *pseudochiral network of order k*. Notice that chiral networks are exactly the pseudochiral networks of order 1. As far as we know all the network models of *symmetric enantioselective synthesis* proposed in the literature are pseudochiral networks. We claim that it has to be that way since the symmetry constraint that defines the class of pseudochiral networks is the weakest symmetry constraint that must be satisfied by any feasible network model of biological homochirality. If network Ω is not pseudochiral, there is an asymmetry in the structure of Ω that favours the synthesis of the species in one of the two clusters, and it implies that the synthesis mechanism modeled by Ω is an asymmetric one (the kinetical structure of the network encodes the action of a chiral agent like e.g. circular polarized light).

4.1 The MM-Condition

Frank inequality is specific of chiral networks, and we cannot use it to analyze pseudochiral networks of order higher than 1. Thus, we have to look for a mathematical criterion that can be applied to the more general class of pseudochiral networks.

Let

$$\Omega = \{(L_1, ..., L_k, D_1, ..., D_k, X_{2k+1}, ..., X_n); R_1, ..., R_r\}$$

be a pseudochiral network of order k. We want to track the evolution of the concentrations

$$[L_1], ..., [L_k], [D_1], ..., [D_{k-1}] \text{ and } [D_k].$$

Remark 10 *From now on we use the above ordering of the species in Ω.*

Let $\mathbf{s}$ be a steady state (with respect to those $2k$ chiral species), and let $\mathcal{J}_\Omega(\mathbf{s})$ be the $2k \times 2k$ Jacobian matrix at state $\mathbf{s}$. Let us suppose that $\mathbf{s}$ is a racemic state, and let us remark that the racemic condition for Ω is given by the equalities

$$[L_1] = [D_1], ..., [L_k] = [D_k].$$

The indiscernibility of the enantiomeric clusters, together with the racemic condition, imply that there exist two $k \times k$ matrices $A_{\mathbf{s}}$ and $B_{\mathbf{s}}$ such that

$$\mathcal{J}_\Omega(\mathbf{s}) = \begin{bmatrix} A_{\mathbf{s}} & B_{\mathbf{s}} \\ B_{\mathbf{s}} & A_{\mathbf{s}} \end{bmatrix}.$$

We say that $(A_{\mathbf{s}}, B_{\mathbf{s}})$ is the pair of *k-blocks* of matrix $\mathcal{J}_\Omega(\mathbf{s})$.

Definition 11 *Let* **v** *be a n-dimensional vector and let* $I = \{i_1, ..., i_k\}$, $J = \{j_1, ..., j_k\}$ *be two disjoint subsets of* $\{1, ..., n\}$ *of the same size. We say that* **v** *is racemic with respect to the pair* (I, J), *if and only if, for all* $l \leq k$ *the equality* $\mathbf{v}[i_l] = \mathbf{v}[j_l]$ *holds. On the other hand, we say that* **v** *is a symmetry-breaking vector with respect to the pair* (I, J), *if and only if, vector* **v** *is not racemic with respect to the latter pair.*

Let **s** be a steady state of the pseudochiral network Ω, let $\mathcal{J}_\Omega(\mathbf{s})$ be its Jacobian matrix and let $(A_\mathbf{s}, B_\mathbf{s})$ be its pair of k-blocks. We say that **s** is a *symmetry-breaking* state, if and only if, the following two conditions are satisfied:

1. Matrix $\mathcal{J}_\Omega(\mathbf{s})$ has an eigenvalue with a positive real part, and this eigenvalue has an eigenvector which is symmetry-breaking with respect to the pair $(\{1, ..., k\}, \{k+1, ..., 2k\})$.

2. $\det(A_\mathbf{s} - B_\mathbf{s}) \neq 0$.

We claim that the symmetry-breaking states are the states that can produce chiral amplification.

Remark 12 *If we suppose that* $\mathcal{J}_\Omega(\mathbf{s})$ *is hyperbolic, we get that* **s** *is a symmetry-breaking state, if and only if, it can locally produce chiral amplification. We conjecture that the same is true if we replace the hyperbolic condition by the weaker (non-singularity) condition given by* $\det(A_\mathbf{s} - B_\mathbf{s}) \neq 0$.

Definition 13 *Let* $n \geq 2$ *and let* $m \leq \frac{n}{2}$, *we say that a* $n \times n$ *matrix is a pseudochiral matrix of order* m, *if and only if, there exist two* $m \times m$ *matrices* A *and* B, *and there exist two matrices* H *and* R, *such that* M *is equal to*

$$\begin{bmatrix} A & B & R \\ B & A & R \\ H & H & * \end{bmatrix}.$$

We say in the latter case that (A, B) *is the pair of* m-*blocks of matrix* M.

It is important to remark that chiral matrices are the pseudochiral matrices of order 1

Definition 14 *We say that a matrix* N *is Hurwitz-unstable, if and only if, there exists an eigenvalue of* N *whose real part is positive.*

Next theorem is a generalization of Frank inequality to pseudochiral networks of higher orders.

Theorem 15 *The MM-Condition*

Let M be a pseudochiral matrix of order m, and let (A, B) be its pair of m-blocks, matrix M has a symmetry-breaking eigenvector for the pair

$$(\{1, ..., m\}, \{m + 1, ..., 2m\}),$$

if and only if, the $m \times m$ matrix $A - B$ is non-singular and Hurwitz-unstable.

Proof. Suppose that $A - B$ has an eigenvalue λ whose real part is positive, and let $\mathbf{v}$ be a non-null eigenvector of M related to the eigenvalue λ. Set $\mathbf{w} = (\mathbf{v}, -\mathbf{v}, 0, ..., 0)$. Notice that $\mathbf{w}$ is symmetry-breaking for the pair $(\{1, ..., m\}, \{m + 1, ..., 2m\})$. Moreover, it is easy to check that $M \cdot \mathbf{w} = \lambda \cdot \mathbf{w}$. We get that the Hurwtiz-instability of $A - B$ entails the existence of an eigenvector of M that is symmetry-breaking for the pair

$$(\{1, ..., m\}, \{m + 1, ..., 2m\}).$$

Now suppose that λ is not an eigenvalue of $A - B$ and let $\mathbf{v}$ be a non-null vector such that $M \cdot \mathbf{v} = \lambda \cdot \mathbf{v}$. Let us write $\mathbf{v}$ as the triple $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{s})$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are m-dimensional vectors. Let us use the symbol F_A^i to denote the ith row of A, and let us use the symbol F_B^i to denote the ith row of B. If we fix $i \leq m$ we get that the equalities

$$\langle F_A^i : \mathbf{v}_1 \rangle + \langle F_B^i : \mathbf{v}_2 \rangle + \sum_{j=2m+1}^{n} M[i, j] \cdot \mathbf{s}[j] = \lambda \cdot \mathbf{v}_1[i],$$

$$\langle F_B^i : \mathbf{v}_1 \rangle + \langle F_A^i : \mathbf{v}_2 \rangle + \sum_{j=2m+1}^{n} M[m + i, j] \cdot \mathbf{s}[j] = \lambda \cdot \mathbf{v}_2[i].$$

hold. If we sum up those two equalities we get that

$$\left(F_A^i - F_B^i\right) \cdot (\mathbf{v}_1 - \mathbf{v}_2) = \lambda \cdot ((\mathbf{v}_1 - \mathbf{v}_2)[i]).$$

Altogether we get that

$$(A - B) \cdot (\mathbf{v}_1 - \mathbf{v}_2) = \lambda \cdot (\mathbf{v}_1 - \mathbf{v}_2).$$

Recall that λ does not belong to the spectrum of $A - B$. Then, we have that $\mathbf{v}_1 - \mathbf{v}_2 = 0$, and the vector $\mathbf{v}$ is racemic for the pair $(\{1, ..., m\}, \{m + 1, ..., 2m\})$. ∎

Remark 16 *It is interesting to observe that Frank inequality corresponds to the MM-Condition for pseudochiral matrices of order* 1. *Notice that if* $m = 1$, *the pair of* m-*blocks of* M *is equal to* $([M[1,1]], [M[1,2]])$. *Moreover, we have that the* 1×1 *real matrix* $[M[1,1]] - [M[1,1]]$ *has an eigenvalue whose real part is positive, if and only if, the inequality* $M[1,1] - M[1,2] > 0$ *holds. If* (A, B) *is the pair of* 1-*blocks of* M, *the inequality* $M[1,1] - M[1,2] > 0$ *also implies that* $A - B$ *is non-singular.*

Let Ω be a pseudochiral network, we can use the MM-Condition to determine the set of symmetry-breaking states of Ω. The latter set is defined by the semialgebraic conditions listed below (see [2] and the references therein):

1. The steady state conditions which, as remarked before, constitute a system of polynomial equations.

2. The positivity conditions given by

$$L_1, ..., L_k, D_1, ..., D_k, X_{2k+1}, ..., X_n, R_1, ..., R_r \geq 0.$$

3. The polynomial inequality
$$\det (A_{\mathbf{s}} - B_{\mathbf{s}})^2 > 0.$$

4. The polynomial inequalities (*Hurwitz-Routh inequalities,* see [10]) asserting that $A_{\mathbf{s}} - B_{\mathbf{s}}$ is Hurwtiz-unstable.

The above theorem yields an algorithm that can be used to analyze any pseudochiral network. This algorithm can be used to check if the input network has symmetry-breaking states, and, in that case, it can also be used to sample the set constituted by all those states. The aforementioned algorithm is based on the above semialgebraic definition of the set of symmetry-breaking states.

We can also obtain, besides of the latter algorithm, an important reduction in the dimensionality of the problem. Suppose that Ω is a pseudochiral network of order k. We are interested in tracking the evolution of the $2k$ chiral species. Then, we can focus the analysis on the Jacobian matrix for those $2k$ species. Let $\mathbf{s}$ be a racemic state of Ω, and let $J_\Omega(\mathbf{s})$ be the corresponding $2k \times 2k$ Jacobian matrix. We have that

$$J_\Omega(\mathbf{s}) = \begin{bmatrix} A & B \\ B & A \end{bmatrix}.$$

We use the symbol $\mathcal{SP}(J_\Omega(\mathbf{s}))$ to denote the spectrum of $J_\Omega(\mathbf{s})$. It is important to observe that

$$\mathcal{SP}(J_\Omega(\mathbf{s})) = \mathcal{SP}(A - B) \cup \mathcal{SP}(A + B).$$

The above equation allows us to reduce the spectral analysis of $J_\Omega(\mathbf{s})$ to the spectral analysis of the matrices $A - B$ and $A + B$. We know that the eigenvalues in $\mathcal{SP}(A - B)$ are the eigenvalues that contribute with symmetry-breaking eigenvectors. Therefore, we say that these are the *symmetry-breaking eigenvalues* of M. It is easy to check that the eigenvalues in $\mathcal{SP}(A + B)$ contribute with racemic eigenvectors.

5   Analysis of the APED Model

We discovered, in the previous section, an algebraic criterion that can be used to detect and analyze the pseudochiral networks that exhibit chiral amplification. We would like to use our criterion in the analysis of a real material system exhibiting mirror symmetry breaking and chiral amplification. We only know of some few real material systems that behaves approximately in this way, as it is the case with Soai reaction [11] and Viedma process [12], and we have that none of those chemical mechanisms count with a suitable network description. However, there are some network models that approximately describe those chemical mechanisms, and we decided to choose one of those networks to test our methodology. We have chosen to work with the APED model of Plasson et al [8], as introduced above, and which can be considered as a rough network description of Soai reaction.

Let us begin with the analysis. The steady state conditions for the species L, L^*, LL and DL are equal to

$$
\begin{aligned}
\frac{d[L]}{dt} &= 0 = -a[L] + b[L^*] - p[L][L^*] - \alpha p[L][D^*] \\
&\quad + 2h[LL] + \beta h([DL] + [LD]), \\
\frac{d[L^*]}{dt} &= 0 = a[L] - b[L^*] - p[L][L^*] - \alpha p[D][L^*], \\
\frac{d[LL]}{dt} &= 0 = p[L][L^*] + e[DL] - \gamma e[LL] - h[LL], \\
\frac{d[DL]}{dt} &= 0 = \alpha p[L][D^*] - e[DL] - \gamma e[LL] - \beta h[DL].
\end{aligned}
$$

The pair of symbolic 4-blocks, denoted with the symbol (A, B), is given by the equal-

ities

$$A = \begin{bmatrix} -a - p\,[L^*] - \alpha p\,[D^*] & b - p\,[L] & 2h & \beta h \\ a - p\,[L^*] & -b - p\,[L] - \alpha p\,[D] & 0 & 0 \\ p\,[L^*] & p\,[L] & -\gamma e - h & e \\ \alpha p\,[D^*] & 0 & -\gamma e & -e - \beta h \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & -\alpha p\,[L] & 0 & \beta h \\ -\alpha p\,[L^*] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha p\,[L] & 0 & 0 \end{bmatrix}.$$

Recall that the racemic condition corresponds to the following set of equations

$$[L] = [D]; \quad [L^*] = [D^*]; \quad [LL] = [DD]; \quad [LD] = [DL].$$

Thus, if we assume the racemic condition, the steady state equations become equal to

$$\frac{d\,[L]}{dt} = 0 = -a\,[L] + b\,[L^*] + [L]\,[L^*]\,(-p - \alpha p) + 2h\,[LL] + 2\beta h\,[DL],$$

$$\frac{d\,[L^*]}{dt} = 0 = a\,[L] - b\,[L^*] + [L]\,[L^*]\,(-p - \alpha p),$$

$$\frac{d\,[LL]}{dt} = 0 = p\,[L]\,[L^*] - [LL]\,(\gamma e + h) - e\,[DL],$$

$$\frac{d\,[DL]}{dt} = 0 = \alpha p\,[L]\,[L^*] - \gamma e\,[LL] - [DL]\,(e + \beta h).$$

Let $\mathbf{s}$ be a steady state, we use the symbol $A_{\mathbf{s}} - B_s$ to denote the numerical matrix that is obtained from $A - B$ after evaluating it at $\mathbf{s}$. If $\mathbf{s}$ is a racemic state, the matrix $A_{\mathbf{s}} - B_{\mathbf{s}}$ becomes equal to

$$\begin{bmatrix} -a - [L^*]\,(p + \alpha p) & b - [L]\,(p - \alpha p) & 2h & 0 \\ a - [L^*]\,(p - \alpha p) & -b - [L]\,(p + \alpha p) & 0 & 0 \\ p\,[L^*] & p\,[L] & -\gamma e - h & e \\ \alpha p\,[L^*] & -\alpha p\,[L] & -\gamma e & -e - \beta h \end{bmatrix}.$$

We know that the latter matrix is the symbolic matrix that we have to analyze. However, this task seems to be a hard one because of the many independent parameters occurring in this matrix. We will try to simplify our current task as much as possible and without assuming restrictive hypothesis.

Proposition 17 *Let $\mathbf{s} = (\mathbf{X}, \mathbf{K})$ be a racemic steady state, let $\alpha > 0$ and set $\mathbf{s}_\alpha = (\mathbf{X}, \alpha \cdot \mathbf{K})$. We have that $\mathbf{s}$ is a symmetry-breaking state, if and only if, the state $\mathbf{s}_\alpha$ is also symmetry-breaking.*

Proof. Let λ be an eigenvalue of $A_{\mathbf{s}} - B_s$ with a positive real part, we have that $\alpha\lambda$ is an eigenvalue of $A_{\mathbf{s}_\alpha} - B_{s_\alpha}$ with a positive real part. Moreover, we have that $A_{\mathbf{s}} - B_s$ is non-singular, if and only if, the matrix $A_{\mathbf{s}_\alpha} - B_{s_\alpha}$ is non-singular. ∎

Let Ω be a pseudochiral network, and suppose we choose one of the reaction-rate parameters, say the parameter k, and we assign to it a random positive value c. The assignment $k = c$ determines a cross-section of the set of racemic steady states of Ω. We use the symbol $S_{k=c}(\Omega)$ to denote the latter cross-section, and we use the symbol $\mathcal{SB}(\Omega)$ to denote the whole set of symmetry-breaking states. It follows from the above proposition that $\mathcal{SB}(\Omega)$ is the disjoint union of the following two sets

$$\mathcal{SB}^0(\Omega) = \mathcal{SB}(\Omega) \cap S_{k=0}(\Omega) \text{ and}$$
$$\mathcal{SB}^{\neq 0}(\Omega) = \{\mathbf{s}_\alpha : \alpha > 0 \ \& \ \mathbf{s} \in \mathcal{SB}(\Omega) \cap S_{k=c}(\Omega)\}.$$

The above fact tells us that we can choose a reaction parameter, say the reaction parameter k, and consider only two cases, the cases $k = 0$ and $k = 1$. Let us choose the reaction rate parameter p. If we suppose that $p = \alpha p = 0$, we get the singular matrix

$$\begin{bmatrix} -a & b & 2h & 0 \\ a & -b & 0 & 0 \\ 0 & 0 & -\gamma e - h & e \\ 0 & 0 & -\gamma e & -e - \beta h \end{bmatrix}.$$

Then, we have that $\mathcal{SB}^0(\Omega_{APED}) = \emptyset$, and we are forced to suppose that $p, \alpha p > 0$. Thus, we suppose, without loss of generality, that $p = 1$. We get the symbolic matrix

$$\begin{bmatrix} -a - [L^*](1+\alpha) & b - [L](1-\alpha) & 2h & 0 \\ a - [L^*](1-\alpha) & -b - [L](1+\alpha) & 0 & 0 \\ [L^*] & [L] & -\gamma e - h & e \\ \alpha[L^*] & -\alpha[L] & -\gamma e & -e - \beta h \end{bmatrix}.$$

Now, we have to determine the parameter values that make the above matrix becomes Hurwitz-unstable. Let $\mathbf{s} \in S_{p=1}(\Omega_{APED})$, and let

$$p_{A_{\mathbf{s}} - B_{\mathbf{s}}}(X) = X^4 + a_3(\mathbf{s})X^3 + ... + a_4(\mathbf{s})$$

be the characteristic polynomial of $A_{\mathbf{s}} - B_{\mathbf{s}}$. The *Hurwitz-Routh criterion* implies that $A_{\mathbf{s}} - B_{\mathbf{s}}$ is Hurwitz-unstable, if and only if, at least one of the following four conditions holds true (see reference [10]):

1. $a_3(\mathbf{s}) < 0$.

2. $a_0(\mathbf{s}) < 0$.

3. $a_3(\mathbf{s})\, a_2(\mathbf{s}) - a_1(\mathbf{s}) < 0$.

4. $(a_3(\mathbf{s})\, a_2(\mathbf{s}) - a_1(\mathbf{s}))\, a_1(\mathbf{s}) - a_0(\mathbf{s})\, (a_3(\mathbf{s}))^2 < 0$.

Recall that

$$a_3(\mathbf{s}) = -tr_1(A_\mathbf{s} - B_\mathbf{s}),\ a_2(\mathbf{s}) = tr_2(A_\mathbf{s} - B_\mathbf{s}),$$
$$a_1(\mathbf{s}) = -tr_3(A_\mathbf{s} - B_\mathbf{s})\ \text{and}\ a_0 = \det(A_\mathbf{s} - B_\mathbf{s}),$$

where given $i = 1, 2, 3$ the symbol $tr_i(A_\mathbf{s} - B_\mathbf{s})$ denotes the sum of all the i-size diagonal minors of $A_\mathbf{s} - B_\mathbf{s}$. Notice that for all $i = 0, 1, 2, 3$ the function $a_i(\mathbf{s})$ is a polynomial function over the parameters $a, b, h, e, \alpha, \gamma, \beta, [L]$ and $[L^*]$. Thus, we have computed three polynomial inequalities, which, together with the steady state (polynomial) equalities and the polynomial inequalities

$$(\det(A_\mathbf{s} - B_\mathbf{s}))^2 > 0 \text{ and } a, b, h, e, \alpha, \gamma, \beta, [L], [L^*] \geq 0,$$

determine a semialgebraic definition of the set $\mathcal{SB}(\Omega) \cap S_{p=1}(\Omega)$.

We begin analyzing the first inequality, we have that

$$a_3(\mathbf{s}) = a + [L^*](1 + \alpha) + b + [L](1 + \alpha) + \gamma e + h + e + \beta h,$$

and we get that for all $\mathbf{s}$ the coefficient $a_3(\mathbf{s})$ is non-negative. Then, we have to check the other three inequalities.

We computed the polynomial expressions related to these inequalities, but it results in three huge expressions that cannot be suitably displayed in this paper. The latter implies that we have to process those huge polynomial expressions using some of the aforementioned algorithms. It happens that all those algorithms are inefficient (*double exponential time*), hard to implement, and it was not possible to find a free-software implementation of at least one of them. Thus, we could not process our semialgebraic condition. However, it is worth to remark that our analytic methodology can be fully automatized.

5.1 Assuming the hypothesis of Plasson et al

It seems that we cannot continue with the analysis without assuming some restrictive hypothesis. Does Ω_{APED} have symmetry-breaking states? It has been claimed, before of this work, that there exist states of Ω_{APED} that produce chiral amplification [8]. The

authors of the aforementioned reference studied the set of states satisfying the conditions $b = \beta = \gamma = 0$. If we assume those equalities we get that

$$A_{\mathbf{s}} - B_{\mathbf{s}} = \begin{bmatrix} -a - [L^*](1+\alpha) & -[L](1-\alpha) & 2h & 0 \\ a - [L^*](1-\alpha) & -[L](1+\alpha) & 0 & 0 \\ [L^*] & [L] & -h & e \\ \alpha[L^*] & -\alpha[L] & 0 & -e \end{bmatrix}.$$

Notice that if one wants to analyze the above matrix, he can focus on the matrix

$$\mathcal{N}_0 = \begin{bmatrix} -a - [L^*](1+\alpha) & -(1-\alpha) & 2 & 0 \\ a - [L^*](1-\alpha) & -(1+\alpha) & 0 & 0 \\ [L^*] & 1 & -1 & 1 \\ \alpha[L^*] & -\alpha & 0 & -1 \end{bmatrix}.$$

since the equality

$$\det(A_{\mathbf{s}} - B_{\mathbf{s}}) = e \cdot h \cdot [L] \cdot \det(\mathcal{N}_0)$$

holds. Notice also that, if the conditions $h = e = [L] = 1$ are fulfilled, the equality $A_{\mathbf{s}} - B_{\mathbf{s}} = \mathcal{N}_0$ holds. Thus, we set $h = e = [L] = 1$. It is interesting to remark that Plasson et al also assumed, at some point in their analysis, that $h = e = [L] = 1$. From now on we focus on the set of racemic steady states of Ω_{APED} satisfying the conditions

$$[L] = h = e = p = 1 \text{ and } b = \beta = \gamma = 0.$$

We use the symbol $\mathcal{SB}_P(\Omega_{APED})$ to denote the above set.

5.1.1 Analysis of $\mathcal{SB}_P(\Omega_{APED})$

We use the MM-Condition to study the fine structure of $\mathcal{SB}_P(\Omega_{APED})$.

Lemma 18 *The states of $\mathcal{SB}_P(\Omega_{APED})$ can be fully described using the parameters a and α.*

Proof. Suppose that $\mathbf{s} \in \mathcal{SB}_P(\Omega_{APED})$, we know that

$$[L] = [D] = h = e = p = 1 \text{ and } b = \beta = \gamma = 0.$$

Thus, it only remains to determine the values of the parameters $[LL], [DL], a, \alpha$ and $[L^*]$. The steady state conditions, together with the racemic condition, imply that

$$[L^*] = \frac{a}{1+\alpha}; \quad [LL] = [DD] = a \text{ and } [DL] = [LD] = \frac{\alpha a}{1+\alpha}.$$

The lemma is proved. ∎

Given a pair (a, α) of non-negative real numbers, this pair determines an state in $\mathcal{S}_P(\Omega_{APED})$. Thus, from now on, we identify the set $\mathcal{S}_P(\Omega_{APED})$ with the North-East quadrant of $\mathbb{R}^2$ that we denote with the symbol $\mathbb{R}_+^2$. Plasson et al studied the set $\mathcal{S}_P(\Omega_{APED})$ by means of massive computer simulations [8]. They observed that given $(a, \alpha) \in \mathcal{S}_P(\Omega_{APED})$, if the inequality $\alpha < 1$ holds the state (a, α) produces chiral amplification [8]. We can use our analytical machinery to explain the aforementioned experimental observation.

Theorem 19 *Let $(a, \alpha) \in \mathcal{S}_P(\Omega_{APED})$, if the inequalities $\alpha < 1$ and $a > 0$ both hold, the state (a, α) is a symmetry-breaking state.*

Proof. Set

$$
\begin{aligned}
a_3(a, \alpha, [L^*]) &= a + \alpha + [L^*](\alpha + 1) + 3, \\
a_2(a, \alpha, [L^*]) &= 4a + 2\alpha + 6\alpha[L^*] + 3, \\
a_1(a, \alpha, [L^*]) &= 3a + \alpha - [L^*] + 3\alpha[L^*] + 1, \\
a_0(a, \alpha, [L^*]) &= 2\alpha(a - 2[L^*]).
\end{aligned}
$$

We have that the polynomial

$$
p_{APED}(X) = X^4 + a_3(a, \alpha, [L^*]) \cdot X^3 + \cdots + a_0(a, \alpha, [L^*])
$$

is equal to the characteristic polynomial of matrix $\mathcal{N}_0$.

We observed before that the polynomial $a_3(a, \alpha, [L^*])$ cannot take negative values for all the non-negative values of the parameters a, α and $[L^*]$. On the other hand, we have that $a_0(a, \alpha, [L^*]) < 0$, if and only if, the inequality

$$
[L^*] > \frac{a}{2}
$$

holds. If we substitute $[L^*]$ by $\frac{a}{1+\alpha}$ in the above inequality, we get that $a_0(a, \alpha, [L^*])$ is negative, if and only if, $\alpha < 1$. Thus, we get that for all $\alpha < 1$ and for all $a > 0$ the matrix $A_{(a,\alpha)} - B_{(a,\alpha)}$ is not singular and Hurwitz-unstable. ∎

We get that the inequality $\alpha < 1$, which was inferred by Plasson et al [8] using experimental methods, is the inequality that forces the negativity of the coefficient $a_0(a, \alpha, [L^*])$ as well as the non-singularity of the matrix $A_{(a,\alpha)} - B_{(a,\alpha)}$.

It is easy to sample the set of symmetry-breaking states of Ω_{APED} satisfying the conditions

$$
[L] = h = e = p = 1; \quad b = \beta = \gamma = 0 \text{ and } \alpha < 1.
$$

To do the latter we can proceed as follows:

1. *Set* $[L] = [D] = h = e = p = 1; b = \beta = \gamma = 0$

2. *Pick* $\alpha \in [0, 1)$.

3. *Pick* $a \in (0, \infty)$.

4. *Set* $[L^*] = [D^*] = \frac{a}{(1+\alpha)}$ *and* $[LL] = [DD] = a$.

5. *Set* $[DL] = \frac{\alpha a}{1+\alpha}$.

We can use the above procedure to compute infinite many symmetry-breaking states of Ω_{APED}. For instance, if we set $\alpha = 0.1$ and $a = 0.55$, we get one of the two states explicitly computed by Plasson et al [8], and exhibiting, according to the computer simulations performed by those authors, chiral amplification. If we set $\alpha = 0.1$ and $a = 0.55$ we get the second state.

The authors of [8] observed that given $a > 1$, there exists a (apparently) large value α_a such that for all $\alpha > \alpha_a$ the racemic state represented by the pair (a, α) produces chiral amplification. Does there exist such a critical value? Where does it come from?

If we substitute $[L^*]$ by $\frac{a}{1+\alpha}$, we get that the coefficients of $p_{APED}(X)$ are given by the equalities:

- $a_3(a, \alpha) = 2a + \alpha + 3$.

- $a_2(a, \alpha) = \frac{4a+5\alpha+10a\alpha+2\alpha^2+3}{1+\alpha}$.

- $a_1(a, \alpha) = \frac{2a+2\alpha+6a\alpha+\alpha^2+1}{1+\alpha}$.

- $a_0(a, \alpha) = \frac{2a\alpha^2-2a\alpha}{1+\alpha}$.

Notice that we only analyzed the inequalities

$$a_3(a, \alpha) < 0 \text{ and } a_0(a, \alpha) < 0,$$

and we did not analyze the inequalities

1. $a_3(a, \alpha) \cdot a_2(a, \alpha) - a_1(a, \alpha) < 0$.

2. $(a_3 (a, \alpha) \cdot a_2 (a, \alpha) - a_1 (a, \alpha)) \cdot a_1 (a, \alpha) - a_0 (a, \alpha) \cdot (a_3 (a, \alpha))^2 < 0.$

The inequality

$$a_3 (a, \alpha) \cdot a_2 (a, \alpha) - a_1 (a, \alpha) < 0$$

is equivalent to the polynomial inequality

$$20a^2\alpha + 8a^2 + 14a\alpha^2 + 50a\alpha + 16a + 2\alpha^3 + 12\alpha^2 + 20\alpha + 10 < 0,$$

which cannot be satisfied for non-negative values of the parameters. Thus, we have to focus on the second inequality, which is equivalent to the polynomial inequality

$$120a^3\alpha^2 + 96a^3\alpha + 16a^3 + 80a^2\alpha^3 + 384a^2\alpha^2 + 256a^2\alpha + 40a^2 + 14a\alpha^4 +$$

$$138a\alpha^3 + 286a\alpha^2 + 200a\alpha + 36a + 2\alpha^5 + 16\alpha^4 + 46\alpha^3 + 62\alpha^2 + 40\alpha + 10$$

$$< \quad 8a^3\alpha^3 + 8a^2\alpha^4 + 2a\alpha^5.$$

Theorem 20 *Let $a > 1$, there exists $\alpha_a > 1$ such that if $\alpha > \alpha_a$, the racemic steady state represented by (a, α) is a symmetry-breaking state.*

Proof. Suppose that $a > 0$ and set $\beta_a = \frac{322a^3 + 740a^2 + 674a + 176}{2a - 2}$. It is easy to check that given $\alpha \geq \beta_a$ the above inequality holds. Moreover, if $\alpha \neq 1$ the matrix $A_{(a,\alpha)} - B_{(a,\alpha)}$ is not singular. ∎

Remark 21 *Let $a > 1$, we use the symbol α_a to denote the quantity*

$$\inf \{\beta_a : \beta_a > 1 \text{ and if } \alpha > \beta_a \text{ the state } (a, \alpha) \text{ is symmetry-breaking}\}$$

Thus, we could prove, using purely analytical tools, the existence of the critical value α_a. Moreover, we could compute an upper bound for this critical value.

We can also prove that

Proposition 22 *If $a \leq 1$ the inequality*

$$(a_3 (a, \alpha) \cdot a_2 (a, \alpha) - a_1 (a, \alpha)) \cdot a_1 (a, \alpha) - a_0 (a, \alpha) \cdot (a_3 (a, \alpha))^2 < 0$$

cannot be satisfied for non-negative values of α.

Plasson et al also observed, by means of their computer simulations, that $\mathcal{SB}_P (\Omega_{APED})$ is partitioned into three regions, the regions

$$\mathcal{A} \quad = \quad \{(a, \alpha) \in \mathcal{SB}_P (\Omega_{APED}) : \alpha < 1\},$$

$$\mathcal{B} \quad = \quad \{(a, \alpha) \in \mathcal{SB}_P (\Omega_{APED}) : (a \leq 1 \text{ and } \alpha \geq 1) \text{ or } \alpha \in [1, \alpha_a]\},$$

$$\mathcal{C} \quad = \quad \{(a, \alpha) \in \mathcal{SB}_P (\Omega_{APED}) : a > 1 \text{ and } \alpha > \alpha_a\},$$

that seem to correspond to three different dynamical regimes (see [8]). According to those authors the region $\mathcal{A}$ is constituted by unstable states that trigger dynamics converging to enantiopure states; the region $\mathcal{B}$ is constituted by states that cannot produce chiral amplification; and, finally, they observed a mixture of symmetry breaking and oscillatory behavior in region $\mathcal{C}$.

We can use our analytical machinery to explain the above observations. First an easy proposition.

Proposition 23 *Let $a > 0$, if $\alpha < 1$ the inequality*

$$(a_3\,(a,\alpha) \cdot a_2\,(a,\alpha) - a_1\,(a,\alpha)) \cdot a_1\,(a,\alpha) - a_0\,(a,\alpha) \cdot (a_3\,(a,\alpha))^2 < 0$$

does not hold.

Let $(a,\alpha) \in \mathcal{A}$, and let $\left(A_{(a,\alpha)}, B_{(a,\alpha)}\right)$ be its pair of 4-blocks. We have from the previous lemma and Theorem 19 that state (a,α) satisfies only one of the Hurwitz-Routh inequalities, the inequality $a_0\,(a,\alpha) < 0$. Then, we have that there is only one change of signs in the Routh-Hurwitz array of the matrix $A_{(a,\alpha)} - B_{(a,\alpha)}$. This implies that the latter matrix has only one eigenvalue with a positive real part, and it implies, in turn, that this eigenvalue is a positive real number. Thus, the Jacobian matrix of (a,α) has only one (linear independent) symmetry-breaking eigenvector, whose action is modulated by a pure exponential function. The latter explains the convergence to enantiopure states.

On the other hand we have that region $\mathcal{B}$ is constituted by states for which the matrix $A_{(a,\alpha)} - B_{(a,\alpha)}$ is Hurwitz-stable, and as consequence they cannot produce chiral amplification.

Finally we get to region $\mathcal{C}$.

Theorem 24 *Let $(a,\alpha) \in \mathcal{SB}_P\,(\Omega_{APED})$ and suppose that $\alpha > \alpha_a$, the matrix $A_{(a,\alpha)} - B_{(a,\alpha)}$ has exactly two eigenvalues with a positive real part.*

Proof. The first column of the Routh-Hurwitz array of $A_{(a,\alpha)} - B_{(a,\alpha)}$ is equal to

$$\begin{bmatrix} 1 \\ a_3\,(a,\alpha) \\ \frac{a_3(a,\alpha)a_2(a,\alpha)-a_1(a,\alpha)}{a_3(a,\alpha)} \\ \frac{(a_3(a,\alpha)a_2(a,\alpha)-a_1(a,\alpha))a_1(a,\alpha)-a_0(a,\alpha)(a_3(a,\alpha))^2}{\frac{a_3(a,\alpha)a_2(a,\alpha)-a_1(a,\alpha)}{a_3(a,\alpha)}} \\ a_0\,(a,\alpha) \end{bmatrix}.$$

We know that the first three entries as well as the last entry are positive. On the other hand, if $\alpha > \alpha_a$ the fourth entry is negative. Then, we have exactly two changes of signs. We conclude that $A_{(a,\alpha)} - B_{(a,\alpha)}$ is a non-singular matrix which has exactly two eigenvalues with a positive real part. ■

Notice that if the two symmetry-breaking eigenvalues of $A_{(a,\alpha)} - B_{(a,\alpha)}$ are complex numbers (conjugated to each other), then the dynamics triggered by (a, α) are oscillating. Let us consider the case $a = 1.1$. The inequality

$$(a_3(a, \alpha) \cdot a_2(a, \alpha) - a_1(a, \alpha)) \cdot a_1(a, \alpha) - a_0(a, \alpha) \cdot (a_3(a, \alpha))^2 < 0$$

is equivalent to the inequality

$$-0.2\alpha^5 + 21.74\alpha^4 + 6587.2\alpha^3 + 582.3\alpha^2 + 697.54\alpha + 119.30 < 0.$$

The polynomial

$$-0.2\alpha^5 + 21.72\alpha^4 + 6587.2\alpha^3 + 582.3\alpha^2 + 697.54\alpha + 119.30$$

has only one positive root, and this positive root is (approx) equal to 243.76. The latter means that $\alpha_{1.1} \approx 243.76$.

Remark 25 *Given $a^0 > 1$, the quantity α_{a_0} can be suitably computed as the unique positive root of the polynomial*

$$(1 + \alpha)^2 \left(\left(a_3\left(a^0, \alpha\right) \cdot a_2\left(a^0, \alpha\right) - a_1\left(a^0, \alpha\right) \right) \cdot a_1\left(a^0, \alpha\right) - a_0\left(a^0, \alpha\right) \cdot (1 + \alpha) \cdot \left(a_3\left(a^0, \alpha\right) \right)^2 \right).$$

Let us pick four states, the states $(1.1, 0.5) \in \mathcal{A}$, the state $(1.1, 2) \in \mathcal{B}$, the state $(1.1, 300) \in \mathcal{C}$ and the state $(1.1, 243.76)$. We have that

- The spectrum of the first state is equal to

  $$6.2643 \times 10^{-2}; \quad -0.87643; \quad -2.4431 + i(0.84249) \text{ and } -2.4431 - i(0.84249),$$

  and it contains exactly one eigenvalue with a positive real part, the eigenvalue 6.2643×10^{-2}.

- The spectrum of $(1.1, 2)$ is equal to

  $$-0.31697 - i(0.16418); \quad -0.31697 + i(0.16418);$$
  $$-3.283 + i(0.85546) i, \quad -3.283 - i(0.85546),$$

  and it does not contain symmetry-breaking eigenvalues.

- The spectrum of $(1.1, 300)$ is equal to

$$0.012\,1 + i\,(1.0280)\,;\ \ 0.012\,1 - i\,(1.0280)\,;\ \ -2.046, -303.18,$$

and it contains, as expected, two eigenvalues with a positive real part. Notice that those two eigenvalues are complex numbers conjugated to each other. Then, we have that the dynamics driven by the symmetry-breaking eigenvectors of this state are modulated by the product of an exponential function and a sum of trigonometric functions. The latter explains the oscillatory behavior of those dynamics.

- The spectrum of $(1.1, 243.76)$ is equal (modulus numerical error) to

$$i;\ \ -i;\ \ -2.047\,7;\ \ -246.93,$$

and we have that the latter state is a *Hopf-bifurcation*. Notice that for all $a > 1$ the state (a, α_a) must be a Hopf-bifurcation given that it represents the transition between the stable region $\mathcal{B}$ and the unstable region $\mathcal{C}$.

We would like to remark that it is easy to sample the regions $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$. For instance, if one wants to sample the region $\mathcal{C}$ he can proceed as follows:

1. *Set* $[L] = [D] = h = e = p = 1;\ b = \beta = \gamma = 0$

2. *Pick* $a \in (1, \infty)$.

3. *Compute* α_a, *and pick* $\alpha > \alpha_a$.

4. *Set* $[L^*] = [D^*] = \frac{a}{(1+\alpha)}$ *and* $[LL] = [DD] = a$.

5. *Set* $[DL] = \frac{\alpha a}{1+\alpha}$.

We claim that we have analyzed in full detail the set $\mathcal{SB}_P\left(\Omega_{APED}\right)$. Notice that we could use our analytical machinery to establish (and explain) the conditions that define the set of symmetry-breaking states included in $\mathcal{SB}_P\left(\Omega_{APED}\right)$. We could also explain the partition of $\mathcal{SB}_P\left(\Omega_{APED}\right)$ that was previously observed by means of computer experiments. It is noticeable that we can use our analytical tools to deduce some previous results that were obtained by numerical methods and massive computer simulations.

6   Concluding Remarks

We have developed an algebraic methodology for the analysis of network models of biological homochirality. The methodology is based on:

1. An algebraic criterion that seems to characterize the steady states that produce chiral amplification. We claim that our criterion, the MM-Condition, characterizes the hyperbolic steady states that (locally) produce mirror-symmetry breaking. We have decided to replace the hyperbolicity condition by the non-singularity condition, which is semialgebraic.

2. Relaxing the hyperbolicity condition makes our notion of symmetry-breaking state becomes a more comprehensive notion, and it also makes it becomes an effective notion. Given a pseudochiral network we can effectively compute a semialgebraic definition of the set of symmetry-breaking states. The latter semialgebraic definition is the conjunction of the steady state conditions, the positivity conditions, the non-singularity condition and the Hurwtiz-Routh inequalities asserting that $A - B$ is unstable.

3. We have reduced the stability analysis of pseudochiral networks of order k, to the stability analysis of two $k \times k$ symbolic matrices. The latter represents an important reduction in the dimension of the problem. Take into account that we want to track the evolution of $2k$ species, and it means that the symbolic Jacobian that we have to analyze is of order $2k \times 2k$.

We could use our mathematical machinery to analyze a complex model of biological homochirality. We think that it is the first time that this model is analyzed with this level of rigor and detail, and using an uniform methodology based on non-linear algebra and the MM-Condition. We claim that any network model can be fully analyzed using the aforementioned methodology. We think that a fine-grained understanding of the sets of parameter values that give place to symmetry-breaking states can shed some light on the origin and mechanisms of biological homochirality.

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