MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **82** (2019) 255-292

ISSN 0340 - 6253

On the Chirality of Triangular Prism Links

Luxuan Guo[a], Wuyang Sun[b], Jingcheng Hao[a], Shuya Liu[a*]

[a]*Key Laboratory of Colloid and Interface Chemistry, Shandong University, Jinan 250100, China*

[b]*School of Mathematics and Statistics, Shandong University, Weihai 264200, China*

liushuya@sdu.edu.cn

(Received October 5, 2018)

Abstract

Triangular prism links, as mathematical models of DNA triangular prisms with double-helix edges, have been constructed and classified into 366 link types in our previous work [1]. In the present paper, the chirality of such links are identified by firstly giving some invariants such as the component number, crossing number and writhe number. Then, there are 77 link types of triangle prism links whose lowest-degree terms of HOMFLY polynomials are given as some general formulas by establishing a new algorithm. Also, there are 100 pairs of triangular prism links such that these invariants of each pair have the same value respectively, due to each pair of links having the symmetrical relationship disclosed in this paper. These results altogether show that these triangular prism links are all chiral, which means that the chirality of DNA triangular prisms can be determined by their topological structures. Meanwhile, they also confirm that the synthesized DNA triangular prism of five components is chiral. Hence our work provide a necessary theoretical support for predicting and controlling the chiral structure of DNA Triangular Prism.

1 Introduction

Chirality is an essential feature of nature, and play the important role in physical and chemical properties of biological molecules [2, 3]. In recent decades, DNA have became an ideal building material for its well-defined secondary structure and excellent molecular recognition [4, 5]. A variety of DNA molecules in polyhedral shape have been assembled and commonly studied [6–17]. It is well-known that DNA is intrinsically chiral for its

*Corresponding author.

duplex structure [18]. However, the chirality of DNA nanostructures are not recognized widely due to the complexity of the geometric folding and association of component DNA molecules in the construction process. In 2005, Turberfield and coworkers synthesized and confirmed a pair of chiral DNA tetrahedrons with a DNA duplex on each edge by designing four component oligonucleotides [19]. More recently, Mao's group construct a serial of DNA octahedra and triangular prisms, each with two parallel DNA duplexes on each edge, to investigate their chiral structures by designing three or four symmetrical cross motif [20, 21]. However, these studies so far only exploit a small fraction on the chirality of DNA nanomolecules. How to predict the chirality of DNA nanostructures become a challenge work. Herein, we employ a theoretical method to determine the chirality of DNA triangle prisms with double-helical edges [22, 23].

Oriented triangle prism links (OTP links) are interlocked and interlinked architectures in triangle-prism shape such that each edge is two oriented and twisted strands with odd or even crossings number. These links are further classified into 366 link types, which are used to describe topological structures of DNA triangle prisms with a double-helix on each edge [1]. To date, there are many researches devoted to calculating the chirality of polyhedral links [25–37], particularly with even crossing number on each edge. These works are mainly dependent on establishing the corresponding relationship between polyhedral links and polyhedra to obtain the polynomial invariants such as Homfly polynomial and Jone polynomial. However, this relationship will be not existing for the oriented polyhedral links with edges of odd crossings number. In fact, there are very little work involved in calculating the chirality of these oriented polyhedral links [24, 27, 37] owing to the complexity of their structures. OTP links, as oriented polyhedral links, allow the edges of odd crossing number as well as the edges of even crossing number, which make it more difficult to determine their chirality. Thus, we must appeal to a new approach to giving the chirality of OTP links.

Link invariants, as important tools in knot theory to determine whether two links are equivalent, play the significant roles in identifying the chirality of links [38]. However, there is no invariant which always work for all links. In the present paper, three link invariants including component number, crossing number and HOMFLY polynomial are calculated for OTP links. The component number of OTP links are completely dependent on the building blocks on each edge, which are used to identify the chirality of links

with even component number. The crossing number of OTP links, together with their writhe number as a regular isotopy invariant of oriented link diagrams, are used to determine the chirality of 111 link types of OTP links with odd component number. For the remaining 77 link types of triangle prism links, we must resort to a more powerful invariant of oriented links, Homfly polynomial [39, 40], which can distinguish many links from their mirror images. Note that these OTP links each contain the crossing number as a parameter. Hence there is no software package which can be used for computing HOMFLY polynomials of such links as well as the lowest-degree terms of their HOMFLY polynomials with respect to the variable z. Also, these links exhibit less regularity on the change of their HOMFLY polynomials with the appearance of the edges of odd crossing number. Hence a new algorithm is to supposed for establishing a general formula for the lowest-degree terms of HOMFLY polynomial for each link type. In particular, the second lowest-degree term of HOMFLY polynomial of the link $D(2b^{\alpha}_{2n}, 4a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n})_{64}$ is given separately. These polynomials are shown to be asymmetrical over the variant v, and hence the remaining 77 OTP links are all chiral. Thus, our works show that the chirality of DNA triangular prism with double-helix edges are determined by their topological structures, which open a new door to predict and control the chiral structures of DNA Triangular Prism from the theoretical viewpoint.

2 OTP link diagrams

The symmetry of link diagrams as a new definition and OTP link diagrams constructed in reference [1] are introduced in this section.

Let L be a link, and D be a link diagram of L. L can be oriented by giving one of the two directions along each component. L with the opposite orientation, denoted by $-L$, is called *the reverse of* L. An oriented link L is called achiral if it is equivalent to its mirror image L^*. Otherwise, it is called chiral. Similarly, D also can be oriented, and its reverse and mirror image are denoted by $-D$ and D^* respectively.

The diagram D^s is called *a symmetrical link diagram* of D if it is obtained from $-D^*$ by switching the over and under-line position at each crossing. The link L^s corresponding to D^s is called as *a symmetrical link* of L. According to this definition, we have the following lemma.

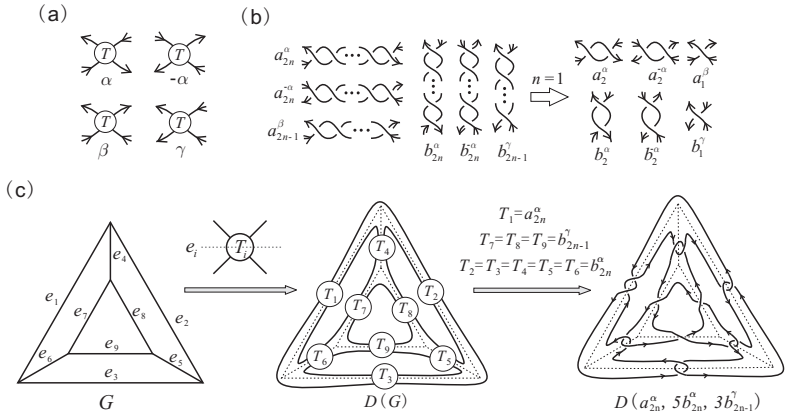

Figure 1. (a) Four orientations of T: α, $-\alpha$, β, γ ; (b) Six oriented twist tangles of T: a^{α}_{2n}, $a^{-\alpha}_{2n}$, a^{β}_{2n-1}, b^{α}_{2n}, $b^{-\alpha}_{2n}$ and b^{γ}_{2n-1}; (c) The construction of $D(G)$ and $D(a^{\alpha}_{2n}, 5b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})$.

A twist tangle T is two parallel strands twisted round each other, where T allows four orientations α, β, γ and $-\alpha$ (Fig. 1(a)). Also, T has six oriented twist tangles denoted by a^{α}_{2n}, $a^{-\alpha}_{2n}$, a^{β}_{2n-1}, b^{α}_{2n}, $b^{-\alpha}_{2n}$ and b^{γ}_{2n-1} for $n \in \mathbb{Z}^+$ (Fig. 1(b)) . Let G be a plane graph of a triangular prism, and let $D(G)$ be an oriented triangular prism link diagram (OTP link diagram) obtained from G by replacing each edge e_i with an oriented twist tangle T_i for $1 \leq i \leq 9$ (Fig. 1(c)). $D(G)$ can be denoted as $D(T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9)_n$ for $1 \leq n \leq 366$ by recording the twist tangle on each edge in a sequence from outside to inside and in the clockwise direction, where the subscript n is the number labeling the OTP link diagram [1]. And the orientation of $D(G)$ is denoted accordingly by $o(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8, \tau_9)$, where $\tau_i \in \{\alpha, -\alpha, \beta, \gamma\}$ is the orientation of T_i for $1 \leq i \leq 9$. The oriented triangular prism link (OTP link) corresponding to $D(G)$ is denoted by $L(G)$ [1].

Lemma 2.1. *There are 100 pairs of symmetrical links among 366 OTP links, as shown in table 1.*

This lemma can be directly obtained from the definition of symmetrical links and OTP links, and the proof is left as an exercise for the reader. With some abuse of terminology in this paper, the word ‘ link ’ is applied to mean a whole equivalence class (a knot type) or a particular representative member.

3 Chirality of OTP links

3.1 Component number of OTP link diagrams

In knot theory, *component number* $\mu(L)$ of the link L is defined as the number of components of L. Similarly, *a component* of a link diagram D is plane projection of a component of L. Evidently, $\mu(L) = \mu(D)$. Component number is an invariant of links, and also can be used to identify the chirality of some links.

Lemma 3.1. *[41] All oriented alternating links with an even number of components are chiral.*

Each OTP link diagram is oriented and alternating, then we obtain the following theorem.

Theorem 3.2. *The component number of all OTP link diagrams are given in table 1 (Appendix A). There are 178 OTP link diagrams with an even number of components, which are all chiral.*

3.2 Crossing number and writhe number of OTP link diagrams

In knot theory, *crossing number* $c(D)$ of a link L is the minimal number of crossings in any link diagram for L. It is well-known that the number of crossings in a reduced alternating link diagram of L is a topological invariant of L [42].

For any link diagram D, each crossing is given a sign of plus or minus according to the conventions shown in Fig. 2. *The writhe number* $\omega(D)$ of a link diagram D is the sum of the signs of all the crossings. As we known, writhe number is the simplest invariant of regular isotopy for oriented diagrams. Since all OTP link diagrams we constructed are all alternating, we obtain the following lemma (proof is omitted).

Lemma 3.3. *Let D be an OTP link diagram. let x_α, $x_{-\alpha}$, x_β and y_γ be the number of the twist tangles a_{2n}^{α}, $a_{2n}^{-\alpha}$, a_{2n-1}^{β} and b_{2n-1}^{γ} in D respectively. Then*

$$c(D(G)) = 18n - x_\beta - y_\gamma \tag{1}$$

$$\text{and} \quad \omega(D(G)) = 18n - 4n(x_\alpha + x_{-\alpha}) - (4n-1)x_\beta - y_\gamma. \tag{2}$$

In terms of the crossing number and writhe number, the following inequality is introduced to identify chiral links by Kauffman.

Lemma 3.4. *[42] Let D be a simple alternating diagram which is not the unknotted circle diagram, and $T(D) = |\omega(D)|$. If $T(D) > \frac{c(D)}{3}$, then D is chiral.*

Using the above two lemmas, we obtain the following theorem.

Theorem 3.5. *Crossing number and writhe number of all OTP link diagrams are given in table 1 (Appendix A). Then there are 111 OTP link diagrams with an odd number of components (without the label $*$ in table 1), which are all chiral.*

3.3 HOMFLY polynomials of OTP link diagrams

Let $min_z H$ denote the lowest-order term of z in the multi-variable polynomial f taken over terms with non-zero coefficients. Our result begins with the definition of HOMFLY polynomial [38–40].

Definition 3.6. *The HOMFLY polynomial $H(L) = H(L; v, z) \in \mathbb{Z}[v, z]$ for an oriented link L is defined by the following relationships:*
(1) $H(L; v, z)$ is invariant under ambient isotopy of L.
(2) If L is a trivial knot, then $H(L; v, z) = 1$.
(3) Suppose that three link diagrams L_+, L_- and L_0 are different only on a local region, as shown in Fig. 2, then $v^{-1}H(L^+; v, z) - vH(L_-; v, z) = zH(L_0; v, z)$.

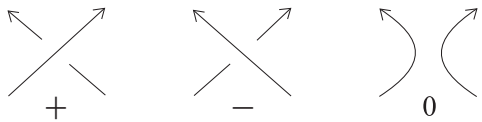

Figure 2. Three link diagrams are different from a local region, and each diagram denotes its corresponding HOMFLY polynomial.

Let D be a link diagram of L. According to the above definition, to obtain the HOMFLY polynomial of L, repeatedly applying the skein relation to the crossings of D until each diagram D_i obtained from D is trivial for $1 \leq i \leq n$. In this process, we can require that each crossing of D is switched or smoothed no more than once. Also, let $P_i(v, z) \in \mathbb{Z}[v, z]$ be the product of all polynomials produced by switching or smoothing the crossings of D for obtaining D_i. Then

$$H(D) = \sum_{i=1}^{n} P_i(v, z) H(D_i). \tag{3}$$

On the other hand, the HOMFLY polynomial also has the following properties:
(1) If L is the connected sum of L_1 and L_2, denoted by $L_1 \sharp L_2$, then

$$H(L) = H(L_1)H(L_2).$$

(2) If L is the disjoint union of L_1 and L_2, denoted by $L_1 \cup L_2$, then

$$H(L) = \left(\frac{v^{-1} - v}{z}\right) H(L_1)H(L_2).$$

(3) If L is a link, then

$$H(L; v, z) = H(L^s; v, z) = H(L^*; -v^{-1}, z).$$

The following four lemmas are introduced for the HOMFLY polynomials of the link diagram having a twist tangle a^{α}_{2n}, b^{α}_{2n}, b^{γ}_{2n-1} or a^{β}_{2n-1}. The first three lemmas have proved in our previous works [24]. The proof of four lemma is similar to the lemma 3.9, and we omitted it here.

Lemma 3.7. *Let $D_{b^{\alpha}_{2n}}$ be a link diagram with an edge b^{α}_{2n} for $n \geq 1$. Let $D_{b^{\alpha}_0}$ and $D_{b^{\alpha}_{\infty}}$ be two link diagrams obtained from $D_{b^{\alpha}_2}$ by switching and smoothing a crossing of b^{α}_2 respectively. Then*

$$H(D_{b^{\alpha}_{2n}}) = v^{2n} H(D_{b^{\alpha}_0}) + vz\frac{v^{2n} - 1}{v^2 - 1} H(D_{b^{\alpha}_{\infty}}). \tag{4}$$

Lemma 3.8. *Let $D_{a^{\alpha}_{2n}}$ be a link diagram with an edge a^{α}_{2n} for $n \geq 1$. Let $D_{a^{\alpha}_0}$ and $D_{a^{\alpha}_{\infty}}$ be two link diagrams obtained from $D_{a^{\alpha}_2}$ by switching and smoothing a crossing of a^{α}_2 respectively. Then*

$$H(D_{a^{\alpha}_{2n}}) = v^{-2n} H(D_{a^{\alpha}_0}) - v^{-1} z\frac{v^{-2n} - 1}{v^{-2} - 1} H(D_{a^{\alpha}_{\infty}}). \tag{5}$$

Lemma 3.9. *Let $D_{b^{\gamma}_{2n-1}}$ be a link diagram with an edge b^{γ}_{2n-1} for $n \geq 1$. Let $D_{b^{\gamma}_{-1}}$ and $D_{b^{\gamma}_{\infty}}$ be two link diagrams obtained from $D_{b^{\gamma}_1}$ by switching and smoothing a crossing of b^{γ}_1 respectively. Then*

$$H(D_{b^{\gamma}_{2n-1}}) = v^{2n-2} H(D_{b^{\gamma}_1}) + vz\frac{v^{2n-2} - 1}{v^2 - 1} H(D_{b^{\gamma}_{\infty}}); \tag{6}$$

$$\text{and } H(D_{b^{\gamma}_{2n-1}}) = v^{2n} H(D_{b^{\gamma}_{-1}}) + vz\frac{v^{2n} - 1}{v^2 - 1} H(D_{b^{\gamma}_{\infty}}). \tag{7}$$

Lemma 3.10. *Let $D_{a^{\beta}_{2n-1}}$ be a link diagram with a a^{β}_{2n-1} for $n \geq 1$. Let $D_{a^{\beta}_{-1}}$ and $D_{a^{\beta}_{\infty}}$ be two link diagrams obtained from $D_{a^{\beta}_{1}}$ by switching and smoothing a crossing of a^{β}_{1} respectively. Then*

$$H(D_{a^{\beta}_{2n-1}}) = v^{-(2n-2)}H(D_{a^{\beta}_{1}}) - v^{-1}z\frac{v^{-(2n-2)}-1}{v^{-2}-1}H(D_{a^{\beta}_{\infty}}); \tag{8}$$

$$\text{and } H(D_{a^{\beta}_{2n-1}}) = v^{-2n}H(D_{a^{\beta}_{-1}}) - v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}H(D_{a^{\beta}_{\infty}}). \tag{9}$$

According to the skein relation of HOMFLY polynomial and the theorem 3 in reference [24], each lowest-order term of HOMFLY polynomial can be obtained by using the following algorithm.

Algorithm 3.11.

Step 1. Switching some crossings enable D into a trivial link diagram D^T_n $(n \geq 1)$. Output $P_{D^T_n}H(D^T_n)$, where $P_{D^T_n}$ be the polynomial produced from D by switching and smoothing crossings to obtain the trivial link diagram D^T_n. Go to step 2.

Step 2. Among these crossings, if there exist m crossings numbered from 1 to m such that each crossing belong to a component of D $(m \geq 1)$, go to step 3. Otherwise, this loop is terminated.

Step 3. Smooth i^{th} crossing and keep switching $1^{st}, 2^{nd}, ..., (i-1)^{th}$ crossings for $i = 1, 2, ..., m$. The resulting link diagrams are denoted by $D'_1, D'_2, ..., D'_m$. Replacing D'_i with D, go to step 1.

The following lemma can be obtained directly from the above equation (3) and Algorithm 3.11.

Theorem 3.12. *Each $P_{D^T_n}H(D^T_n)$ is one lowest-order term of $H(D)$. Then*

$$min_z H(D) = \sum_n P_{D^T_n}(v,z)H(D^T_n). \tag{10}$$

By using the above lemmas, we obtain the following theorem.

Theorem 3.13. *The lowest-order terms of HOMFLY polynomials of 77 OTP links are given in table 2.*

D	$min_z H(D)$
D_{23}	$(6v^{6n}+v^{-4+6n}-4v^{-2+6n}-4v^{2+6n}+v^{4+6n})z^{-4}$
$D_{25}, D_{29}, D_{31}, D_{39}$	$(-2v^{6n}+2v^{8n}-2v^{12n}+v^{-2+6n}+v^{2+6n}-v^{-2+8n}-v^{2+8n}+v^{-2+12n}+v^{2+12n})z^{-2}$
$D_{27}, D_{33}, D_{37}, D_{41}, D_{47}$	$(-2v^{12n}+v^{-2+12n}+v^{2+12n})z^{-2}$
D_{49}, D_{51}	$(-2v^{4n}+2v^{6n}-2v^{12n}+v^{-2+4n}+v^{2+4n}-v^{-2+6n}-v^{2+6n}+v^{-2+12n}+v^{2+12n})z^{-2}$
D_{58}, D_{66}, D_{82}	$(v^{-2-4n}+v^{2-4n}-v^{-2-2n}-v^{2-2n}-2v^{-4n}+2v^{-2n}-2v^{2n}+v^{-2+2n}+v^{2+2n})z^{-2}$
D_{60}	$2+v^{-10n}-2v^{-6n}+2v^{-4n}-v^{-2n}-2v^{2n}+v^{8n}$
D_{62}, D_{78}	$4+v^{-10n}-3v^{-6n}+5v^{-4n}-5v^{-2n}-v^{2n}-v^{4n}+v^{8n}$
D_{64}	$(-2+v^{-2}+v^{2})z^{-2}$
D_{68}	$2+v^{-10n}-2v^{-6n}+3v^{-4n}-3v^{-2n}-v^{4n}+v^{8n}$
D_{72}	$(-2v^{2n}+v^{-2+2n}+v^{2+2n})z^{-2}$
D_{74}	$2+v^{-10n}-3v^{-6n}+5v^{-4n}-4v^{-2n}-v^{4n}+v^{8n}$
D_{76}, D_{346}, D_{364}	$(v^{-2-4n}+v^{2-4n}-2v^{-4n})z^{-2}$
D_{80}	$(-2+v^{-2}+v^{2}+v^{-2-4n}+v^{2-4n}-v^{-2-2n}-v^{2-2n}-2v^{-4n}+2v^{-2n})z^{-2}$
D_{88}	$7+v^{-10n}-5v^{-6n}+11v^{-4n}-12v^{-2n}-2v^{4n}+v^{8n}$
D_{94}	$4+v^{-10n}-4v^{-6n}+8v^{-4n}-8v^{-2n}+v^{2n}-2v^{4n}+v^{8n}$
D_{96}	$1+v^{-10n}-3v^{-6n}+5v^{-4n}-4v^{-2n}+2v^{2n}-2v^{4n}+v^{8n}$
D_{104}	$4+v^{-10n}-3v^{-6n}+6v^{-4n}-7v^{-2n}+v^{2n}-2v^{4n}+v^{8n}$
$D_{119}, D_{125}, D_{139}$	$(v^{2n}-2v^{2+2n}+v^{4+2n})z^{-2}$
D_{121}	$3v^{2}+3v^{2-4n}-5v^{2-2n}+v^{2n}-v^{4n}+v^{8n}-v^{2+4n}$
D_{123}, D_{131}	$2v^{2}+v^{2-4n}-v^{2-2n}+v^{8n}-v^{2+2n}-v^{2+4n}$
D_{127}	$v^{2}+2v^{2-4n}-2v^{2-2n}+v^{8n}-v^{2+4n}$
D_{133}	$5v^{2}+3v^{2-4n}-6v^{2-2n}+v^{2n}-v^{4n}+v^{8n}-v^{2+2n}-v^{2+4n}$
D_{143}, D_{199}	$(1-2v^{2}+v^{4})z^{-2}$
D_{172}	$2v^{-2}-3v^{-2-6n}+3v^{-2-4n}-2v^{-2-2n}+v^{-10n}$
D_{174}	$(v^{-4-4n}-2v^{-2-4n}+v^{-4n})z^{-2}$
D_{185}	$(3v^{4n}-v^{6n}+3v^{-4+4n}-6v^{-2+4n}-v^{-6+6n}+v^{-4+6n}+v^{-2+6n})z^{-2}$
D_{186}	$3v^{-2+4n}-3v^{-2+6n}+3v^{-2+10n}-v^{-4+12n}-v^{-2+12n}$
D_{187}	$2v^{-2}-3v^{-2+2n}+5v^{-2+4n}-v^{-4+6n}-5v^{-2+6n}+v^{-4+8n}$ $+v^{-2+8n}+3v^{-2+10n}-v^{-4+12n}-v^{-2+12n}$
D_{188}	$v^{-2}-v^{-2+2n}+2v^{-2+4n}-2v^{-2+6n}+3v^{-2+10n}-v^{-4+12n}-v^{-2+12n}$
D_{189}	$2v^{-2}-4v^{-2+2n}+7v^{-2+4n}-v^{-4+6n}-6v^{-2+6n}+v^{-4+8n}$ $+v^{-2+8n}+3v^{-2+10n}-v^{-4+12n}-v^{-2+12n}$
D_{190}	$(v^{4n}+v^{-4+4n}-2v^{-2+4n})z^{-2}$
D_{197}	$4v^{2}+3v^{2-4n}-5v^{2-2n}+v^{2n}-v^{4n}+v^{8n}-2v^{2+2n}$
D_{201}	$2v^{2}+2v^{2-4n}-2v^{2-2n}+v^{8n}-2v^{2+2n}$
D_{209}	$(v^{2-4n}-2v^{4-4n}+v^{6-4n}-v^{2-2n}+2v^{4-2n}-v^{6-2n}+v^{2n}-2v^{2+2n}+v^{4+2n})z^{-2}$
D_{214}	$v^{2}+2v^{4-8n}-2v^{4-6n}+v^{8n}-v^{2+4n}$
D_{216}	$2v^{2}+v^{4-8n}-2v^{4-4n}+v^{4-2n}+v^{8n}-v^{2+2n}-v^{2+4n}$
D_{228}	$(v^{2-4n}-2v^{4-4n}+v^{6-4n})z^{-2}$
D_{242}	$-(4v^{-4})-5v^{-2-6n}+5v^{-2-4n}+2v^{-4-2n}+v^{-10n}+2v^{-4+2n}$
D_{250}	$2v^{-2}-2v^{-2+2n}-2v^{-4+4n}+3v^{-2+4n}+3v^{-4+6n}-v^{-2+6n}-3v^{-4+8n}+v^{-4+12n}$
D_{252}	$2v^{-2}-2v^{-2+2n}-v^{-4+4n}+3v^{-2+4n}+v^{-4+6n}-v^{-2+6n}-2v^{-4+8n}+v^{-4+12n}$
D_{256}	$2v^{-2}+v^{-2+2n}-3v^{-4+4n}-v^{-2+4n}+3v^{-4+6n}-2v^{-4+8n}+v^{-4+12n}$
D_{258}	$(v^{-6+6n}-2v^{-4+6n}+v^{-2+6n})z^{-2}$
D_{261}	$2v^{2}+v^{4}+v^{4-6n}-v^{4-4n}-v^{4-2n}+v^{8n}-2v^{2+4n}$
D_{264}	$v^{2}+v^{4}+v^{4-8n}-v^{4-4n}-v^{4-2n}+v^{8n}+v^{2+2n}-2v^{2+4n}$
D_{272}	$(v^{2-4n}-2v^{4-4n}+v^{6-4n}-v^{2-2n}+2v^{4-2n}-v^{6-2n}+v^{2n}-2v^{2+2n}+v^{4+2n})z^{-2}$
D_{284}, D_{287}	$(v^{-6+4n}-2v^{-4+4n}+v^{-2+4n})z^{-2}$
D_{293}	$v^{-2}-v^{-4+2n}-v^{-4+4n}+v^{-6+6n}+2v^{-4+6n}-v^{-6+8n}+2v^{-4+8n}-v^{-6+10n}-v^{-4+10n}$
D_{297}	$v^{4}+3v^{4-4n}-4v^{4-2n}+v^{8n}+2v^{2+2n}-2v^{2+4n}$
D_{320}	$(v^{-2-2n}+v^{2-2n}-2v^{-2n})z^{-2}$
D_{322}	$3-v^{2-4n}-v^{-4n}+2v^{-2n}-v^{2n}-v^{-2+2n}$
D_{327}	$-v^{6-12n}+2v^{6-10n}+v^{4-8n}-2v^{6-8n}-v^{4-6n}+v^{4-4n}+v^{2-2n}$
D_{329}	$(v^{4-10n}-2v^{6-10n}+v^{8-10n})z^{-2}$
D_{338}	$(v^{-8+8n}-2v^{-6+8n}+v^{-4+8n})z^{-2}$
D_{348}	$(v^{-8+6n}-2v^{-6+6n}+v^{-4+6n})z^{-2}$
D_{351}	$2-3v^{-2+4n}+v^{-4+6n}+v^{-2+6n}$
D_{352}	$-v^{6-12n}-2v^{8-12n}+5v^{6-10n}+v^{8-10n}-4v^{6-8n}+2v^{4-4n}$
D_{357}	$(-2v^{2-4n}+v^{4-4n}+v^{-4n})z^{-2}$
D_{358}	$v^{4-8n}-3v^{2-4n}+v^{2-2n}-v^{-4n}+3v^{-2n}$
D_{362}	$-4-4v^{-2}+v^{2-6n}-3v^{2-4n}+v^{-2-2n}+v^{2-2n}-4v^{-4n}+11v^{-2n}+2v^{-2+2n}$

Table 2: The lowest-degree terms of z of HOMFLY polynomials of 77 OTP link diagrams. Here each link diagram denote the link diagram with the same subscript in table 1.

Proof. By using the definition (3) of HOMFLY polynomial, the 77 OTP link diagrams cover nineteen types of orientations in table 2. Then the lowest-degree terms of HOM-

FLY polynomials of these link diagrams can be divided into the following nineteen cases according to their orientations. Also, the operation that switch n or $n-1$ crossings of a twist tangle a^{α}_{2n}, a^{β}_{2n-1}, b^{α}_{2n} or b^{γ}_{2n-1} is covered in the proof below, and hence we illustrate it in Fig. 3.

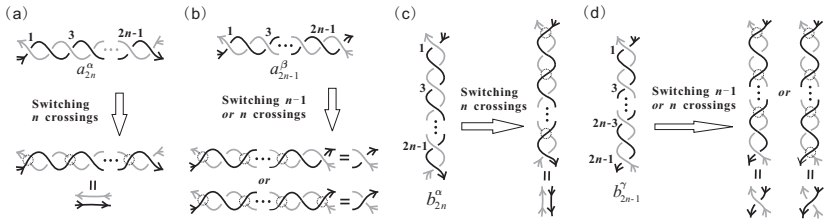

Figure 3. Each crossing switched or smoothed is bounded by a circle. (a) Switching n crossings of a^{α}_{2n}; (b) Switching $n-1$ or n crossings of a^{β}_{2n-1} respectively; (c) Switching n crossings of b^{α}_{2n}; (d) Switching $n-1$ or n crossings of b^{γ}_{2n-1} respectively.

(1) There are 28 OTP link diagrams oriented with $o(9\alpha)$ in table 2, and the subscribes of these link diagrams are numbered from 23 to 104.

For the OTP link diagram $D_{23} = D(3a^{\alpha}_{2n}, 6b^{\alpha}_{2n})_{23}$, there is only one lowest degree term of z for $H(D_{23})$. First, the link diagram D_{23} is changed into a trivial link diagram D^T_1 with five components by switching n crossings of each for three a^{α}_{2n} and six b^{α}_{2n} (Fig. 3). The resulting polynomial is denoted by $P_{D^T_1}(v,z)$. By using the lemmas 3.7 and 3.8, then

$$P_{D^T_1}(v,z) = (v^{-2n})^3 \cdot (v^{2n})^6 = v^{6n} \ .$$

Also, using the property (3) of HOMFLY polynomial, we have

$$H(D^T_1) = \left(\frac{v^{-1}-v}{z}\right)^4 \ .$$

Then

$$P_{D^T_1} \cdot H(D^T_1) = v^{6n} \cdot \left(\frac{v^{-1}-v}{z}\right)^4 = (6v^{6n} + v^{-4+6n} - 4v^{-2+6n} - 4v^{2+6n} + v^{4+6n})z^{-4} \ .$$

In addition, for the link diagram D, all twist tangles are each composed of two different components. For each such twist tangle, smoothing its any crossing don't result in a lowest-order term of z for $H(D)$ according to the algorithm 3.11. By using theorem 3.12, we have

$$min_z H(D_{23}) = (6v^{6n} + v^{-4+6n} - 4v^{-2+6n} - 4v^{2+6n} + v^{4+6n})z^{-4}.$$

Similarly, for the other link diagrams with the orientation $o(9\alpha)$, the lowest-order terms of their HOMFLY polynomials can be given in table 2.

(2) There are nine OTP link diagrams oriented with $o(4\alpha, \beta, 2\alpha, 2\beta)$ in table 2, and the subscribes of these link diagrams are numbered from 119 to 143.

For the OTP link diagram $D_{125} = D(a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})$, it is changed into a trivial link diagram D^T_1 with three components by switching some crossings. In this process, we switch n crossings of each for a a^{α}_{2n}, four b^{α}_{2n} and a a^{β}_{2n-1}, and switch $n-1$ crossings of another a^{β}_{2n-1} (Fig. 3). By using the lemmas 3.7, 3.8, 3.10 and theorem 3.12, then

$$\begin{aligned} min_z H(D_{125}) =& P_{D^T_1} \cdot H(D^T_1) \\ =& v^{-2n}(v^{2n})^4 v^{-2n} v^{-(2n-2)} \cdot \left(\frac{v^{-1}-v}{z}\right)^2 \\ =& (v^{2n} - 2v^{2+2n} + v^{4+2n}) z^{-2}. \end{aligned}$$

Similarly, for the other link diagrams with the orientation $o(4\alpha, \beta, 2\alpha, 2\beta)$, the lowest-order terms of their HOMFLY polynomials can be given in table 2.

(3) There are eight OTP link diagrams oriented with $o(6\alpha, 3\gamma)$ in table 2, and the subscribes of these link diagrams are numbered from 172 to 190.

For the OTP link diagram $D_{185} = D(3a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})$, there are three trivial link diagrams, which all together result in the lowest-degree term of z for $H(D_{185})$. A trivial link diagram D^T_1 of three components is obtained from the diagram D_{185} by switching some crossings. In this process, we switch n crossings of each for three a^{α}_{2n}, three b^{α}_{2n} and a b^{γ}_{2n-1}, and switch $n-1$ crossings of another b^{γ}_{2n-1}. By using the lemmas 3.7-3.9, then

$$P_{D^T_1} \cdot H(D^T_1) = (v^{-2n})^3 (v^{2n})^4 v^{2n-2} \cdot \left(\frac{v^{-1}-v}{z}\right)^2 = (v^{4n} + v^{-4+4n} - 2v^{-2+4n}) z^{-2}.$$

Note that two b^{γ}_{2n-1} in the above process are both on the same component of D. Respectively smoothing the crossings of these two b^{γ}_{2n-1}.

First, we smooth $n-1$ crossings of one b^{γ}_{2n-1} to obtain the link diagram D'_1 from D. Then D'_1 is further changed into a trivial link D^T_2 by switching n crossings of each for three a^{α}_{2n}, three b^{α}_{2n} and a b^{γ}_{2n-1}, and then switching $n-1$ crossings of the remaining b^{γ}_{2n-1}. By using the lemmas 3.7-3.9, then

$$\begin{aligned} P_{D^T_2} \cdot H(D^T_2) =& vz \frac{v^{2n-2}-1}{v^2-1} (v^{-2n})^3 (v^{2n})^4 v^{2n-2} \cdot \left(\frac{v^{-1}-v}{z}\right)^3 \\ =& (v^{4n} + v^{-4+4n} - 2v^{-2+4n} - v^{-6+6n} + 2v^{-4+6n} - v^{-2+6n}) z^{-2}. \end{aligned}$$

Second, we smooth n crossings of the other b^{γ}_{2n-1}, and keep switching $n-1$ crossings of the one b^{γ}_{2n-1} to obtain the link diagram D'_2 from D. Then D'_2 is further changed into a trivial link D^T_3 by switching n crossings of each for three a^{α}_{2n}, three b^{α}_{2n} and a b^{γ}_{2n-1}. By using the lemmas 3.7-3.9, then

$$\begin{aligned} P_{D^T_3} \cdot H(D^T_3) =& vz\frac{v^{2n}-1}{v^2-1}(v^{-2n})^3(v^{2n})^4 v^{2n-2} \cdot \left(\frac{v^{-1}-v}{z}\right)^3 \\ =& (v^{4n}-v^{6n}+v^{-4+4n}-2v^{-2+4n}-v^{-4+6n}+2v^{-2+6n})z^{-2}. \end{aligned}$$

Hence the lowest-degree term of z for $H(D_{185})$ only contains the above three cases. By using theorem 3.12, then

$$\begin{aligned} & min_z H(D_{185}) \\ & = \sum_{n=1}^{3} P_{D^T_n}(v,z)H(D^T_n) \\ & = (v^{4n}+v^{-4+4n}-2v^{-2+4n})z^{-2} + (v^{4n}+v^{-4+4n}-2v^{-2+4n}-v^{-6+6n}+2v^{-4+6n} \\ & - v^{-2+6n})z^{-2} + (v^{4n}-v^{6n}+v^{-4+4n}-2v^{-2+4n}-v^{-4+6n}+2v^{-2+6n})z^{-2} \\ & = (3v^{4n}-v^{6n}+3v^{-4+4n}-6v^{-2+4n}-v^{-6+6n}+v^{-4+6n}+v^{-2+6n})z^{-2}. \end{aligned}$$

Similarly, for the other link diagrams with the orientation $o(6\alpha, 3\gamma)$, the lowest-order terms of their HOMFLY polynomials can be given in table 2.

(4) There are three OTP link diagrams oriented with $o(3\alpha, 3\beta, 3(-\alpha))$ in table 2, and the subscribes of these link diagrams are numbered by 197, 199 and 201.

For $D_{199} = D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, 2b^{-\alpha}_{2n})$, there is only one lowest degree term of z for $H(D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, 2b^{-\alpha}_{2n}))$. First, the link diagram D_{199} is changed into a trivial link diagram D^T_1 with three components by switching some crossings. In this process, we switch n crossings of each for a a^{α}_{2n}, a $a^{-\alpha}_{2n}$, two b^{α}_{2n}, two $b^{-\alpha}_{2n}$ and a a^{β}_{2n-1}, and switch $n-1$ crossings of a a^{β}_{2n-1}. By using the lemmas 3.7, 3.8, 3.10 and theorem 3.12, then

$$min_z H(D_{199}) = (v^{-2n})^2(v^{2n})^4 v^{-2n} v^{-(2n-2)} \cdot \left(\frac{v^{-1}-v}{z}\right)^2 = (1-2v^2+v^4)z^{-2}.$$

Similarly, for the other link diagrams with the orientation $o(3\alpha, 3\beta, 3(-\alpha))$, the lowest-order terms of their HOMFLY polynomials can be given in table 2.

(5) There are four OTP link diagrams oriented with $o(4\alpha, 4\beta, -\alpha)$ in table 2, and the subscribes of these link diagrams are numbered by 209, 214, 216 and 228.

For the link diagram $D_{228} = D(2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})$, there is only one lowest degree term of z for $H(D_{228})$. First, the link diagram D_{228} is changed into a trivial link

diagram D_1^T with three components by switching some crossings. In this process, we switch n crossings of each for a a_{2n}^{α}, three b_{2n}^{α} and two a_{2n-1}^{β}, and switch $n-1$ crossings of each for the remaining two a_{2n-1}^{β}. By using the lemmas 3.7, 3.8, 3.10 and theorem 3.12, then

$$min_z H(D_{228}) = v^{-2n}(v^{2n})^3(v^{-2n})^2(v^{-(2n-2)})^2 \cdot \left(\frac{v^{-1}-v}{z}\right)^2 = (v^{2-4n} - 2v^{4-4n} + v^{6-4n})z^{-2}.$$

Similarly, for the other link diagrams with the orientation $o(4\alpha, 4\beta, -\alpha)$, the lowest-order terms of their HOMFLY polynomials can be given in table 2.

(6) There are five OTP link diagrams oriented with $o(2\alpha, \gamma, \alpha, 2\gamma, 2\alpha, \gamma)$ in table 2, and the subscribes of these link diagrams are numbered by 242, 250, 252, 256 and 258.

For the link diagram $D_{258} = D(b_{2n}^{\alpha}, a_{2n}^{\alpha}, b_{2n-1}^{\gamma}, a_{2n}^{\alpha}, 2b_{2n-1}^{\gamma}, a_{2n}^{\alpha}, b_{2n}^{\alpha}, b_{2n-1}^{\gamma})_{258}$, it is changed into a trivial link diagram D_1^T with three components by switching some crossings. In this process, we switch n crossings of each for three a_{2n}^{α}, two b_{2n}^{α} and two b_{2n-1}^{γ}, and switch $n-1$ crossings of each for the remaining two b_{2n-1}^{γ}. By using the lemmas 3.7-3.9 and theorem 3.12, then

$$min_z H(D_{258}) = (v^{-2n})^3(v^{2n})^2(v^{2n})^2(v^{2n-2})^2 \cdot \left(\frac{v^{-1}-v}{z}\right)^2 = (v^{-6+6n} - 2v^{-4+6n} + v^{-2+6n})z^{-2}.$$

Similarly, for the other link diagrams with the orientation $o(2\alpha, \gamma, \alpha, 2\gamma, 2\alpha, \gamma)$, the lowest-order terms of their HOMFLY polynomials can be given in table 2.

(7) There are three OTP link diagrams oriented with $o(\alpha, 2\beta, \alpha, -\alpha, 2\alpha, 2\beta)$ in table 2, and the subscribes of these link diagrams are numbered by 261, 264 and 272.

For the link diagram $D_{272} = D(b_{2n}^{\alpha}, 2a_{2n-1}^{\beta}, b_{2n}^{\alpha}, b_{2n}^{-\alpha}, b_{2n}^{\alpha}, a_{2n}^{\alpha}, 2a_{2n-1}^{\beta})_{272}$, there are two trivial link diagrams, which both result in the lowest-degree term of z for $H(D_{272})$. First, the link diagram D_{272} is changed into a trivial link diagram D_1^T with three components by switching some crossings. In this process, we switch n crossings of each for an a_{2n}^{α}, three b_{2n}^{α}, a $b_{2n}^{-\alpha}$ and an a_{2n-1}^{β}, and switch $n-1$ crossings of another a_{2n-1}^{β}. By using the lemmas 3.7, 3.8, 3.10, then

$$P_{D_1^T} \cdot H(D_1^T) = v^{-2n}(v^{2n})^4 v^{-2n} v^{-(2n-2)} \cdot \left(\frac{v^{-1}-v}{z}\right)^2 = (v^{2n} - 2v^{2+2n} + v^{4+2n})z^{-2}.$$

Also, there is only one b_{2n}^{α} in the above process is on the same component of D. We smooth n crossings of the b_{2n}^{α} to obtain the link diagram D_1' from D. Then D_1' is further changed into a trivial link D_2^T by switching n crossings of each for a a_{2n}^{α}, two b_{2n}^{α}, a $b_{2n}^{-\alpha}$,

two a^{β}_{2n-1}, and then switching $n-1$ crossings of the remaining two a^{β}_{2n-1}. By using the lemmas 3.7, 3.8, 3.10, then

$$\begin{aligned} P_{D_2^T}\cdot H(D_2^T) =& vz\frac{v^{2n}-1}{v^2-1}v^{-2n}(v^{2n})^3(v^{-2n})^2(v^{-(2n-2)})^2\cdot\left(\frac{v^{-1}-v}{z}\right)^3 \\ =&(v^{2-4n}-2v^{4-4n}+v^{6-4n}-v^{2-2n}+2v^{4-2n}-v^{6-2n})z^{-2}. \end{aligned}$$

By using theorem 3.12, we have

$$\begin{aligned} &min_z H(D_{272}) \\ &= P_{D_1^T}H(D_1^T)+P_{D_2^T}H(D_2^T) \\ &= (v^{2n}-2v^{2+2n}+v^{4+2n})z^{-2}+(v^{2-4n}-2v^{4-4n}+v^{6-4n}-v^{2-2n}+2v^{4-2n}-v^{6-2n})z^{-2} \\ &= (v^{2-4n}-2v^{4-4n}+v^{6-4n}-v^{2-2n}+2v^{4-2n}-v^{6-2n}+v^{2n}-2v^{2+2n}+v^{4+2n})z^{-2}. \end{aligned}$$

Similarly, for the other link diagrams with the orientation $o(\alpha,2\beta,\alpha,-\alpha,2\alpha,2\beta)$, the lowest-order terms of their HOMFLY polynomials can be given in table 2.

(8) There is three OTP link diagrams oriented with $o(2\alpha,\gamma,\alpha,4\gamma,-\alpha)$ in table 2, and the subscribes of these link diagrams are numbered by 284, 287 and 293. For the link diagram $D_{284}=D(2a^{\alpha}_{2n},b^{\gamma}_{2n-1},a^{\alpha}_{2n},4b^{\gamma}_{2n-1},b^{-\alpha}_{2n})_{284}$, there is only one lowest degree term of z for $H(D_{284})$. First, the link diagram D_{284} is changed into a trivial link diagram D_1^T with three components by switching some crossings. In this process, we switch n crossings of each for three a^{α}_{2n}, a $b^{-\alpha}_{2n}$ and two b^{γ}_{2n-1}, and switch $n-1$ crossings of each for two b^{γ}_{2n-1}. By using the lemmas 3.7-3.9 and theorem 3.12, then

$$min_z H(D_{284}) = (v^{-2n})^3v^{2n}(v^{2n})^2(v^{2n-2})^2\cdot\left(\frac{v^{-1}-v}{z}\right)^2 = (v^{-6+4n}-2v^{-4+4n}+v^{-2+4n})z^{-2}.$$

Similarly, for the other link diagrams with the orientation $o(2\alpha,\gamma,\alpha,4\gamma,-\alpha)$, the lowest-order terms of their HOMFLY polynomials can be given in table 2.

(9) There is only one OTP link diagram $D_{297}=D(b^{\alpha}_{2n},2a^{\beta}_{2n-1},b^{\alpha}_{2n},b^{-\alpha}_{2n},3a^{\beta}_{2n-1},b^{-\alpha}_{2n})_{297}$ oriented with $o(\alpha,2\beta,\alpha,-\alpha,3\beta,-\alpha)$ in table 2. There are six trivial link diagrams, which all together result in the lowest-degree term of z for $H(D_{297})$. A trivial knot D_1^T is obtained from the diagram D_{297} by switching n crossings of each for two b^{α}_{2n} and two $b^{-\alpha}_{2n}$. By using lemma 3.7, then

$$P_{D_1^T}\cdot H(D_1^T) = (v^{2n})^4\cdot 1 = v^{8n}.$$

Note that two b^{α}_{2n} and two $b^{-\alpha}_{2n}$ in the above process are all on the same component of D. Respectively smoothing the crossings of these four twist tangles.

First, we smooth n crossings of a b_{2n}^{α} to obtain the link diagram D_1' from D. Then D_1' is further changed into a trivial link D_2^T by switching n crossings of each for two $b_{2n}^{-\alpha}$ and two a_{2n-1}^{β}, and then switching $n-1$ crossings of the remaining two a_{2n-1}^{β}. By using lemma 3.7 and 3.10, then

$$P_{D_2^T} \cdot H(D_2^T) = vz\frac{v^{2n}-1}{v^2-1}(v^{2n})^2(v^{-2n})^2(v^{-(2n-2)})^2 \cdot \frac{v^{-1}-v}{z} = v^{4-4n} - v^{4-2n}.$$

Second, we smooth n crossings of the other b_{2n}^{α} and keep switching n crossings of the one b_{2n}^{α} to obtain the link diagram D_2' from D. Then D_2' is further changed into a trivial link D_3^T by switching n crossings of each for two $b_{2n}^{-\alpha}$ and a a_{2n-1}^{β}, and then switch $n-1$ crossings of another a_{2n-1}^{β}. By using the theorems 3.7 and 3.10, then

$$P_{D_3^T} \cdot H(D_3^T) = vz\frac{v^{2n}-1}{v^2-1}v^{2n}(v^{2n})^2v^{-2n}v^{-(2n-2)} \cdot \frac{v^{-1}-v}{z} = v^{2+2n} - v^{2+4n}.$$

Also, there is a $b_{2n}^{-\alpha}$ on the same component of the link diagram D_2'. We smooth n crossings of the $b_{2n}^{-\alpha}$ to obtain the link diagram D_1'' from D_2'. Then D_1'' is further changed into a trivial link D_4^T by switching n crossings of each for a $b_{2n}^{-\alpha}$ and two a_{2n-1}^{β}, and then switching $n-1$ crossings of the remaining two a_{2n-1}^{β}. Then

$$\begin{aligned} P_{D_4^T} \cdot H(D_4^T) &= vz\frac{v^{2n}-1}{v^2-1}(v^{2n})^2(v^{-2n})^2(v^{-(2n-2)})^2vz\frac{v^{2n}-1}{v^2-1} \cdot \left(\frac{v^{-1}-v}{z}\right)^2 \\ &= v^4 + v^{4-4n} - 2v^{4-2n}. \end{aligned}$$

Third, we smooth n crossings of a $b_{2n}^{-\alpha}$ and keep switching n crossings of each for two b_{2n}^{α} to obtain the link diagram D_3' from D. Then D_3' is further changed into a trivial link D_5^T by switching n crossings of each for two a_{2n-1}^{β}, and switching $n-1$ crossings of the remaining two a_{2n-1}^{β}. Then

$$P_{D_5^T} \cdot H(D_5^T) = vz\frac{v^{2n}-1}{v^2-1}(v^{2n})^2(v^{-2n})^2(v^{-(2n-2)})^2 \cdot \frac{v^{-1}-v}{z} = v^{4-4n} - v^{4-2n}.$$

At last, we smooth n crossings of the remaining twist tangle $b_{2n}^{-\alpha}$ and keep switching n crossings of each for two b_{2n}^{α} and a $b_{2n}^{-\alpha}$ to obtain the link diagram D_4' from D. Then D_4' is further changed into a trivial link D_6^T by switching n crossings of a a_{2n-1}^{β}, and switching $n-1$ crossings of another a_{2n-1}^{β}. Then

$$P_{D_6^T} \cdot H(D_6^T) = vz\frac{v^{2n}-1}{v^2-1}(v^{2n})^3v^{-2n}v^{-(2n-2)} \cdot \frac{v^{-1}-v}{z} = v^{2+2n} - v^{2+4n}.$$

Hence the lowest-degree term of z for $H(D(b_{2n}^{\alpha}, 2a_{2n-1}^{\beta}, b_{2n}^{\alpha}, b_{2n}^{-\alpha}, 3a_{2n-1}^{\beta}, b_{2n}^{-\alpha}))$ only contains the above six cases. According to the theorem 3.12, we have

$$min_z H(D_{297}) = v^4 + 3v^{4-4n} - 4v^{4-2n} + v^{8n} + 2v^{2+2n} - 2v^{2+4n}.$$

(10) There are two OTP link diagrams oriented with $o(\alpha, 2\beta, \alpha, \beta, \alpha, 3\gamma)$ in table 2, whose subscribes are numbered by 320 and 322.

For the link diagram $D_{320} = D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{320}$, there is only one lowest degree term of z for $H(D_{320})$. First, the link diagram D_{320} is changed into a trivial link diagram D^T_1 with three components by switching the crossings of all oriented twist tangles. In this process, we switch n crossings of each for two a^{α}_{2n}, a b^{α}_{2n}, two a^{β}_{2n-1} and two b^{γ}_{2n-1}, and switch $n-1$ crossings of each for the remaining a a^{β}_{2n-1} and a b^{γ}_{2n-1}. By using the lemmas 3.7-3.10 and theorem 3.12, then

$$min_z H(D_{320}) = (v^{-2n})^4 (v^{2n})^3 v^{-(2n-2)} v^{2n-2} \cdot \left(\frac{v^{-1}-v}{z}\right)^2 = (v^{-2-2n} + v^{2-2n} - 2v^{-2n}) z^{-2}.$$

Similarly, for the link diagram D_{322}, the lowest-order terms of the HOMFLY polynomials can be given in table 2.

(11) There are two OTP link diagrams oriented with $o(\alpha, 2\beta, \alpha, 3\beta, \alpha, \beta)$ in table 2, whose subscribes are numbered by 327 and 329.

For the link diagram $D_{329} = D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\beta}_{2n-1})_{329}$, there is only one lowest degree term of z for $H(D_{329})$. First, the link diagram D_{329} is changed into a trivial link diagram D^T_1 with three components by switching some crossings. In this process, we switch n crossings of each for a a^{α}_{2n}, two b^{α}_{2n} and three a^{β}_{2n-1}, and switch $n-1$ crossings of each for the remaining three a^{β}_{2n-1}. By using the lemmas 3.7, 3.8, 3.10 and theorem 3.12, then

$$min_z H(D_{329}) = v^{-2n} (v^{2n})^2 (v^{-2n})^3 (v^{-(2n-2)})^3 \cdot \left(\frac{v^{-1}-v}{z}\right)^2 = (v^{4-10n} - 2v^{6-10n} + v^{8-10n}) z^{-2}.$$

Similarly, for the link diagram D_{327}, the lowest-order terms of the HOMFLY polynomials can be given in table 2.

(12) There is only one OTP link diagram $D_{338} = D(a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1}, a^{-\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1})_{338}$ oriented with $o(\alpha, 3\gamma, -\alpha, \gamma, \alpha, 2\gamma)$ in table 2. There is only one lowest degree term of z for $H(D_{338})$. First, the link diagram D_{338} is changed into a trivial link diagram D^T_1 of three components by switching the crossings of all oriented twist tangles except a $a^{-\alpha}_{2n}$. In this process, we switch n crossings of each for two a^{α}_{2n} and three b^{γ}_{2n-1}, and switch $n-1$ crossings of each for the remaining three b^{γ}_{2n-1}. By using the lemmas 3.8, 3.9 and theorem 3.12, then

$$min_z H(D_{338}) = (v^{-2n})^2 (v^{2n})^3 (v^{2n-2})^3 \left(\frac{v^{-1}-v}{z}\right)^2 = (v^{-8+8n} - 2v^{-6+8n} + v^{-4+8n}) z^{-2}.$$

(13) There is only one OTP link diagram $D_{346} = D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 3b^{\gamma}_{2n-1})_{346}$ oriented with $o(3\alpha, 3\beta, 3\gamma)$ in table 2. There is only one lowest degree term of z for $H(D_{346})$. First, the link diagram D_{346} is changed into a trivial link diagram D^T_1 with three components by switching some crossings. In this process, we switch n crossings of each for two a^{α}_{2n}, a b^{γ}_{2n-1} and a a^{β}_{2n-1}, and switch $n-1$ crossings of each for the remaining two b^{γ}_{2n-1} and two a^{β}_{2n-1}. By using the lemmas 3.8-3.10 and theorem 3.12, then

$$\begin{aligned} &min_z H(D_{346}) \\ &= (v^{-2n})^2 v^{2n} v^{-2n} (v^{2n-2})^2 (v^{-(2n-2)})^2 \left(\frac{v^{-1}-v}{z}\right)^2 = (v^{-2-4n} + v^{2-4n} - 2v^{-4n}) z^{-2}. \end{aligned}$$

(14) There is only one OTP link diagram $D_{348} = D(3b^{\gamma}_{2n-1}, 3a^{-\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{348}$ oriented with $o(3\gamma, 3(-\alpha), 3\gamma)$ in table 2. There is only one lowest degree term of z for $H(D_{348})$. First, the link diagram D_{348} is changed into a trivial link diagram D^T_1 with three components by switching some crossings. In this process, we switch n crossings of each for three $a^{-\alpha}_{2n}$ and three b^{γ}_{2n-1}, and switch $n-1$ crossings of each for the remaining three b^{γ}_{2n-1}. By using the lemmas 3.8, 3.9 and theorem 3.12, then

$$min_z H(D_{348}) = (v^{-2n})^3 (v^{2n})^3 (v^{2n-2})^3 \cdot \left(\frac{v^{-1}-v}{z}\right)^2 = (v^{-8+6n} - 2v^{-6+6n} + v^{-4+6n}) z^{-2}.$$

(15) There is only one OTP link diagram $D_{364} = D(a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\beta}_{2n-1}, b^{\gamma}_{2n-1}, a^{\beta}_{2n-1})_{364}$ oriented with $o(\alpha, \gamma, \alpha, 2\gamma, 2\beta, \gamma, \beta)$ in table 2. There is only one lowest degree term of z for $H(D_{364})$. First, the link diagram D_{364} is changed into a trivial link diagram D^T_1 with three components by switching some crossings. In this process, we switch n crossings of each for two a^{α}_{2n}, a b^{γ}_{2n-1} and a a^{β}_{2n-1}, and switch $n-1$ crossings of each for two b^{γ}_{2n-1} and two a^{β}_{2n-1}. By using the lemmas 3.8-3.10, and theorem 3.12, then

$$min_z H(D_{364}) = (v^{-2n})^2 v^{2n} v^{-2n} (v^{2n-2})^2 (v^{-(2n-2)})^2 \left(\frac{v^{-1}-v}{z}\right)^2 = \frac{v^{-2-4n} + v^{2-4n} - 2v^{-4n}}{z^2}.$$

(16) There is only one OTP link diagram $D_{352} = D(b^{\alpha}_{2n}, 5a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, 2a^{\beta}_{2n-1})_{352}$ oriented with $o(\alpha, 5\beta, -\alpha, 2\beta)$ in table 2. There are thirteen trivial link diagrams, which all together result in the lowest-degree term of z for $H(D_{352})$. A trivial knot D^T_1 is obtained from the diagram D_{352} by switching some crossings. In this process, we switch n crossings of each for a $b^{-\alpha}_{2n}$ and three a^{β}_{2n-1}, and switch $n-1$ crossings of each for three a^{β}_{2n-1}. By using the lemmas 3.7 and 3.10, then

$$P_{D^T_1} \cdot H(D^T_1) = v^{2n} (v^{-2n})^3 (v^{-(2n-2)})^3 \cdot 1 = v^{-10n+6}.$$

Note that six a_{2n-1}^{β} and a $b_{2n}^{-\alpha}$ in the above process are all on the same component of D. Respectively smoothing the crossings of these seven twist tangles.

First, we smooth $n-1$ crossings of a a_{2n-1}^{β} to obtain the link diagram D_1' from D. Then D_1' is further changed into a trivial link D_2^T by switching n crossings of each for a b_{2n}^{α}, a $b_{2n}^{-\alpha}$ and two a_{2n-1}^{β}, and then switching $n-1$ crossings of two a_{2n-1}^{β}. By using the lemmas 3.7 and 3.10, then

$$P_{D_2^T}\cdot H(D_2^T)=\left(-v^{-1}z\frac{v^{-(2n-2)}-1}{v^{-2}-1}\right)(v^{2n})^2(v^{-2n})^2(v^{-(2n-2)})^2\cdot\frac{v^{-1}-v}{z}=-v^{6-6n}+v^{4-4n}.$$

Also, there is only one $b_{2n}^{-\alpha}$ on the same component of D_1'. We smooth n crossings of the $b_{2n}^{-\alpha}$ to obtain the link diagram D_1'' from D_1'. Then D_1'' is further changed into a trivial link D_3^T by switching n crossings of each for a b_{2n}^{α} and three a_{2n-1}^{β}, and then switching $n-1$ crossings of the remaining three a_{2n-1}^{β}. Then

$$\begin{aligned}P_{D_3^T}\cdot H(D_3^T)&=\left(-v^{-1}z\frac{v^{-(2n-2)}-1}{v^{-2}-1}\right)vz\frac{v^{2n}-1}{v^2-1}v^{2n}(v^{-2n})^3(v^{-(2n-2)})^3\cdot\left(\frac{v^{-1}-v}{z}\right)^2\\&=-v^{8-12n}+v^{6-10n}+v^{8-10n}-v^{6-8n}.\end{aligned}$$

Second, we smooth n crossings of a $b_{2n}^{-\alpha}$ and keep switching $n-1$ crossings of a a_{2n-1}^{β} to obtain the link diagram D_2' from D. Then D_2' is further changed into a trivial link D_4^T by switching n crossings of each for a b_{2n}^{α} and three a_{2n-1}^{β}, and then switching $n-1$ crossings of two a_{2n-1}^{β}. Then

$$P_{D_4^T}\cdot H(D_4^T)=vz\frac{v^{2n}-1}{v^2-1}v^{2n}(v^{-2n})^3(v^{-(2n-2)})^3\frac{v^{-1}-v}{z}=v^{6-10n}-v^{6-8n}.$$

Also, there is a a_{2n-1}^{β} and a b_{2n}^{α} in the above process on the same component of D_2'. Respectively smoothing the crossings of these two twist tangles. On the one hand, we smooth n crossings of the a_{2n-1}^{β} to obtain the link diagram D_1'' from D_2'. Then D_1'' is further changed into a trivial link D_5^T by switching n crossings of each for a b_{2n}^{α} and three a_{2n-1}^{β}, and then switching $n-1$ crossings of the remaining two a_{2n-1}^{β}. By using the theorems 3.7 and 3.10, then

$$\begin{aligned}P_{D_5^T}\cdot H(D_5^T)&=vz\frac{v^{2n}-1}{v^2-1}\left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right)v^{2n}(v^{-2n})^3(v^{-(2n-2)})^3\cdot\left(\frac{v^{-1}-v}{z}\right)^2\\&=-v^{6-12n}+2v^{6-10n}-v^{6-8n}.\end{aligned}$$

On the other hand, we smooth n crossings of the b_{2n}^{α} and keep switching n crossings of the a_{2n-1}^{β} to obtain the link diagram D_2'' from D_2'. Then D_2'' is further changed into a

trivial link D_6^T by switching n crossings of each for two a_{2n-1}^{β}, and then switching $n-1$ crossings of two a_{2n-1}^{β}. Then

$$P_{D_6^T} \cdot H(D_6^T) = \left(vz\frac{v^{2n}-1}{v^2-1}\right)^2 (v^{-2n})^3(v^{-(2n-2)})^3 \cdot \left(\frac{v^{-1}-v}{z}\right)^2 = v^{6-12n} - 2v^{6-10n} + v^{6-8n}.$$

Third, we smooth n crossings of another twist tangle a_{2n-1}^{β}, and keep switching n crossings of a $b_{2n}^{-\alpha}$ and then switching $n-1$ crossings of a a_{2n-1}^{β} to obtain the link diagram D_3' from D. Then D_3' is further changed into a trivial link D_7^T by switching n crossings of each for a b_{2n}^{α} and two a_{2n-1}^{β}, and then switching $n-1$ crossings of another two a_{2n-1}^{β}. Then

$$P_{D_7^T} \cdot H(D_7^T) = \left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right)(v^{2n})^2(v^{-2n})^2(v^{-(2n-2)})^3 \cdot \frac{v^{-1}-v}{z} = v^{6-6n} - v^{6-8n}.$$

Fourth, we smooth $n-1$ crossings of the third twist tangle a_{2n-1}^{β} and keep switching n crossings of each for a $b_{2n}^{-\alpha}$ and a a_{2n-1}^{β} and then switching $n-1$ crossings of a a_{2n-1}^{β} to obtain the link diagram D_4' from D. Then D_4' is further changed into a trivial link D_8^T by switching n crossings of each for a b_{2n}^{α} and two a_{2n-1}^{β}, and then switch $n-1$ crossings of each for the remaining two a_{2n-1}^{β}. Then

$$P_{D_8^T} \cdot H(D_8^T) = \left(-v^{-1}z\frac{v^{-(2n-2)}-1}{v^{-2}-1}\right)(v^{2n})^2(v^{-2n})^3(v^{-(2n-2)})^3 \cdot \frac{v^{-1}-v}{z} = v^{6-8n} - v^{8-10n}.$$

Also, there is only one b_{2n}^{α} on the same component of D_4'. We smooth n crossings of the b_{2n}^{α} to obtain the link diagram D_1'' from D_4'. Then D_1'' is further changed into a trivial link D_9^T by switching n crossings of each for two a_{2n-1}^{β}, and then switching $n-1$ crossings of each for the remaining two a_{2n-1}^{β}. Then

$$\begin{aligned} P_{D_9^T} \cdot H(D_9^T) &= \left(-v^{-1}z\frac{v^{-(2n-2)}-1}{v^{-2}-1}\right) vz\frac{v^{2n}-1}{v^2-1}v^{2n}(v^{-2n})^3(v^{-(2n-2)})^3 \cdot \left(\frac{v^{-1}-v}{z}\right)^2 \\ &= -v^{8-12n} + v^{6-10n} + v^{8-10n} - v^{6-8n}. \end{aligned}$$

Fifth, we smooth n crossings of the fourth twist tangle a_{2n-1}^{β} and keep switching n crossings of each for a $b_{2n}^{-\alpha}$ and a a_{2n-1}^{β} and then switching $n-1$ crossings of each for two a_{2n-1}^{β} to obtain the link diagram D_5' from D. Then D_5' is further changed into a trivial link D_{10}^T by switching n crossings of each for a b_{2n}^{α} and a a_{2n-1}^{β}. Then

$$P_{D_{10}^T} \cdot H(D_{10}^T) = \left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right)(v^{2n})^2(v^{-2n})^2(v^{-(2n-2)})^2 \cdot \frac{v^{-1}-v}{z} = -v^{4-6n} + v^{4-4n}.$$

Also, there is only one b_{2n}^{α} on the same component of D_5'. We smooth n crossings of the b_{2n}^{α} to obtain the link diagram D_1'' from D_5'. Then D_1'' is further changed into a trivial

link D_{11}^T by switching n crossings of each for two a_{2n-1}^{β}, and then switch $n-1$ crossings of the remaining a_{2n-1}^{β}. Then

$$\begin{aligned} P_{D_{11}^T} \cdot H(D_{11}^T) &= \left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right) vz\frac{v^{2n}-1}{v^2-1}v^{2n}(v^{-2n})^3(v^{-(2n-2)})^3 \cdot \left(\frac{v^{-1}-v}{z}\right)^2 \\ &= -v^{6-12n}+2v^{6-10n}-v^{6-8n}. \end{aligned}$$

Sixth, we smooth $n-1$ crossings of the fifth twist tangle a_{2n-1}^{β} and keep switching n crossings of each for a $b_{2n}^{-\alpha}$ and two a_{2n-1}^{β} and then switching $n-1$ crossings of each for two a_{2n-1}^{β} to obtain the link diagram D_6' from D. Then D_6' is further changed into a trivial link D_{12}^T by switching n crossings of each for a b_{2n}^{α} and a a_{2n-1}^{β}. Then

$$P_{D_{12}^T} \cdot H(D_{12}^T) = \left(-v^{-1}z\frac{v^{-(2n-2)}-1}{v^{-2}-1}\right)(v^{2n})^2\,(v^{-2n})^3(v^{-(2n-2)})^2 \cdot \frac{v^{-1}-v}{z} = v^{4-6n}-v^{6-8n}.$$

At last, we smooth n crossings of the sixth twist tangle a_{2n-1}^{β} and keep switching n crossings of each for a $b_{2n}^{-\alpha}$ and two a_{2n-1}^{β} and then switching $n-1$ crossings of each for three a_{2n-1}^{β} to obtain the link diagram D_7' from D. Then D_7' is further changed into a trivial link D_{13}^T by switching n crossings of each for a b_{2n}^{α} and a a_{2n-1}^{β}. Then

$$P_{D_{13}^T} \cdot H(D_{13}^T) = \left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right)(v^{2n})^2(v^{-2n})^3(v^{-(2n-2)})^3 \cdot \frac{v^{-1}-v}{z} = v^{6-8n}-v^{6-10n}.$$

Hence the lowest-degree term of z for $H(D(b_{2n}^{\alpha}, 5a_{2n-1}^{\beta}, b_{2n}^{-\alpha}, 2a_{2n-1}^{\beta}))$ only contains the above thirteen cases. According to the theorem 3.12, we have

$$min_z H(D_{352}) = -v^{6-12n}-2v^{8-12n}+5v^{6-10n}+v^{8-10n}-4v^{6-8n}+2v^{4-4n}.$$

(17) There are two OTP link diagrams oriented with $o(\alpha, 3\beta, -\alpha, \beta, 3\gamma)$ in table 2, whose subscribes are numbered respectively by 357 and 358.

For the link diagram $D(a_{2n}^{\alpha}, 3a_{2n-1}^{\beta}, b_{2n}^{-\alpha}, a_{2n-1}^{\beta}, 3b_{2n-1}^{\gamma})_{357}$, there is only one lowest degree term of z for $H(D_{357})$. First, the link diagram D_{357} is changed into a trivial link diagram D_1^T with three components by switching some crossings. In this process, we switch n crossings of each for a a_{2n}^{α}, a $b_{2n}^{-\alpha}$, a b_{2n-1}^{γ} and two a_{2n-1}^{β}, and switch $n-1$ crossings of each for a b_{2n-1}^{γ} and the remaining two a_{2n-1}^{β}. By using the lemmas 3.7-3.10 and theorem 3.12, then

$$min_z H(D_{357}) = v^{-2n}v^{2n}v^{2n}(v^{-2n})^2v^{2n-2}(v^{-(2n-2)})^2\left(\frac{v^{-1}-v}{z}\right)^2 = \frac{-2v^{2-4n}+v^{4-4n}+v^{-4n}}{z^2}.$$

Similarly, for the other link diagram D_{358}, the lowest-order term of the HOMFLY polynomials can be given in table 2.

(18) There is only one OTP link diagram $D_{362} = D(2a^{\beta}_{2n-1}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\beta}_{2n-1}, b^{\gamma}_{2n-1})_{362}$ oriented with $o(2\beta, \gamma, \alpha, 2\gamma, 2\beta, \gamma)$ in table 2. There are twelve trivial link diagrams, which all together result in the lowest-degree term of z for $H(D_{362})$. A trivial knot D^T_1 is obtained from the diagram D_{362} by switching some crossings. In this process, we switch n crossings of each for a b^{γ}_{2n-1} and two a^{β}_{2n-1}, and switch $n-1$ crossings of each for a b^{γ}_{2n-1} and two a^{β}_{2n-1}. By using the lemmas 3.9 and 3.10, then

$$P_{D^T_1} \cdot H(D^T_1) = v^{2n}(v^{-2n})^2 v^{2n-2}(v^{-(2n-2)})^2 \cdot 1 = v^{-4n+2}.$$

Note that two b^{γ}_{2n-1} and four a^{β}_{2n-1} in the above process are all on the same component of D. Respectively smoothing the crossings of these six twist tangles.

First, we smooth $n-1$ crossings of a b^{γ}_{2n-1} to obtain the link diagram D'_1 from D. Then D'_1 is further changed into a trivial link D^T_2 by switching n crossings of each for a $a^{-\alpha}_{2n}$, a b^{γ}_{2n-1} and two a^{β}_{2n-1}, and then switching $n-1$ crossings of each for a b^{γ}_{2n-1} and two a^{β}_{2n-1}. By using the lemmas 3.8-3.10, then

$$P_{D^T_2} \cdot H(D^T_2) = vz\frac{v^{2n-2}-1}{v^2-1} v^{2n}(v^{-2n})^3 v^{2n-2}(v^{-(2n-2)})^2 \cdot \frac{v^{-1}-v}{z} = v^{2-6n} - v^{-4n}.$$

Also, there are two a^{β}_{2n-1} in the above process on the same component of D'_1. Respectively smoothing the crossings of these two a^{β}_{2n-1}. On the one hand, we smooth $n-1$ crossings of a a^{β}_{2n-1} to obtain the link diagram D''_1 from D'_1. Then D''_1 is further changed into a trivial link D^T_3 by switching n crossings of each for a $a^{-\alpha}_{2n}$, a b^{γ}_{2n-1} and a a^{β}_{2n-1}, and then switching $n-1$ crossings of each for the remaining two b^{γ}_{2n-1} and two a^{β}_{2n-1}. By using the theorems 3.8-3.10, then

$$\begin{aligned} P_{D^T_3} \cdot H(D^T_3) =& vz\frac{v^{2n-2}-1}{v^2-1}\left(-v^{-1}z\frac{v^{-(2n-2)}-1}{v^{-2}-1}\right) v^{2n}(v^{-2n})^2(v^{2n-2})^2 \\ & (v^{-(2n-2)})^2\left(\frac{v^{-1}-v}{z}\right)^2 = -v^{-2} - v^{2-4n} + 2v^{-2n}. \end{aligned}$$

On the other hand, we smooth n crossings of the other a^{β}_{2n-1} and keep switching $n-1$ crossings of a a^{β}_{2n-1} to obtain the link diagram D''_2 from D'_1. Then D''_2 is further changed into a trivial link D^T_4 by switching n crossings of each for a $a^{-\alpha}_{2n}$, a b^{γ}_{2n-1} and a a^{β}_{2n-1}, and then switching $n-1$ crossings of each for the remaining two b^{γ}_{2n-1} and a a^{β}_{2n-1}. Then

$$\begin{aligned} P_{D^T_4} \cdot H(D^T_4) =& vz\frac{v^{2n-2}-1}{v^2-1}\left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right) v^{2n}(v^{-2n})^2(v^{2n-2})^2(v^{-(2n-2)})^2\left(\frac{v^{-1}-v}{z}\right)^2 \\ =& -v^{-2} + v^{-2-2n} - v^{-4n} + v^{-2n}. \end{aligned}$$

Second, we smooth n crossings of a a^{β}_{2n-1} and keep switching $n-1$ crossings of a b^{γ}_{2n-1} to obtain the link diagram D'_2 from D. Then D'_2 is further changed into a trivial link D^T_5 by switching n crossings of each for a $a^{-\alpha}_{2n}$, two b^{γ}_{2n-1} and a a^{β}_{2n-1}, and then switching $n-1$ crossings of each for a b^{γ}_{2n-1} and a a^{β}_{2n-1}. Then

$$\begin{aligned} P_{D^T_5} \cdot H(D^T_5) &= \left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right)(v^{2n})^2(v^{-2n})^2(v^{2n-2})^2 v^{-(2n-2)}\frac{v^{-1}-v}{z} \\ &= -v^{-2}+v^{-2+2n}. \end{aligned}$$

Also, there is only one b^{γ}_{2n-1} on the same component of D'_2. We smooth n crossings of the b^{γ}_{2n-1} to obtain the link diagram D''_1 from D'_2. Then D''_1 is further changed into a trivial link D^T_6 by switching n crossings of each for a $a^{-\alpha}_{2n}$, a b^{γ}_{2n-1} and a a^{β}_{2n-1}, and then switching $n-1$ crossings of each for the remaining a b^{γ}_{2n-1} and two a^{β}_{2n-1}. Then

$$\begin{aligned} P_{D^T_6} \cdot H(D^T_6) &= \left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right) vz\frac{v^{2n}-1}{v^2-1} v^{2n}(v^{-2n})^2(v^{2n-2})^2(v^{-(2n-2)})^2 \cdot \left(\frac{v^{-1}-v}{z}\right)^2 \\ &= -1-v^{-4n}+2v^{-2n}. \end{aligned}$$

Third, we smooth $n-1$ crossings of another a^{β}_{2n-1} and keep switching n crossings of a a^{β}_{2n-1} and $n-1$ crossings of a b^{γ}_{2n-1} to obtain the link diagram D'_3 from D. Then D'_3 is further changed into a trivial link D^T_7 by switching n crossings of each for a $a^{-\alpha}_{2n}$ and two b^{γ}_{2n-1}, and then switching $n-1$ crossings of each for a b^{γ}_{2n-1} and a a^{β}_{2n-1}. Then

$$\begin{aligned} P_{D^T_7} \cdot H(D^T_7) &= \left(-v^{-1}z\frac{v^{-(2n-2)}-1}{v^{-2}-1}\right)(v^{2n})^2(v^{-2n})^2(v^{2n-2})^2 v^{-(2n-2)} \cdot \frac{v^{-1}-v}{z} \\ &= -1+v^{-2+2n}. \end{aligned}$$

Also, there are two b^{γ}_{2n-1} in the above process on the same component of D'_3. Respectively smoothing the crossings of these two b^{γ}_{2n-1}. On the one hand, we smooth $n-1$ crossings of a b^{γ}_{2n-1} to obtain the link diagram D''_1 from D'_3. Then D''_1 is further changed into a trivial link D^T_8 by switching n crossings of each for a $a^{-\alpha}_{2n}$ and a b^{γ}_{2n-1}, and then switching $n-1$ crossings of each for the remaining a b^{γ}_{2n-1} and two a^{β}_{2n-1}. Then

$$\begin{aligned} P_{D^T_8} \cdot H(D^T_8) &= \left(-v^{-1}z\frac{v^{-(2n-2)}-1}{v^{-2}-1}\right) vz\frac{v^{2n-2}-1}{v^2-1} v^{2n}(v^{-2n})^2(v^{2n-2})^2(v^{-(2n-2)})^2 \\ &\quad \cdot \left(\frac{v^{-1}-v}{z}\right)^2 - v^{-2}-v^{2-4n}+2v^{-2n}. \end{aligned}$$

On the other hand, we smooth n crossings of the other b^{γ}_{2n-1} and keep switching $n-1$ crossings of a b^{γ}_{2n-1} to obtain the link diagram D''_2 from D'_3. Then D''_2 is further changed

into a trivial link D_9^T by switching n crossings of each for a $a_{2n}^{-\alpha}$ and a b_{2n-1}^{γ}, and then switching $n-1$ crossings of each for the remaining two a_{2n-1}^{β}. Then

$$P_{D_9^T}\cdot H(D_9^T)=\left(-v^{-1}z\frac{v^{-(2n-2)}-1}{v^{-2}-1}\right)vz\frac{v^{2n}-1}{v^2-1}v^{2n}(v^{-2n})^2(v^{2n-2})^2(v^{-(2n-2)})^2$$
$$\cdot\left(\frac{v^{-1}-v}{z}\right)^2=-1-v^{2-4n}+v^{2-2n}+v^{-2n}.$$

Fourth, we smooth n crossings of the other b_{2n-1}^{γ} and keep switching n crossings of a a_{2n-1}^{β} and then switching $n-1$ crossings of each for a b_{2n-1}^{γ} and a a_{2n-1}^{β} to obtain the link diagram D_4' from D. Then D_4' is further changed into a trivial link D_{10}^T by switching n crossings of each for a $a_{2n}^{-\alpha}$ and a b_{2n-1}^{γ}, and then switching $n-1$ crossings of each for a b_{2n-1}^{γ} and a a_{2n-1}^{β}. Then

$$P_{D_{10}^T}\cdot H(D_{10}^T)=vz\frac{v^{2n}-1}{v^2-1}v^{2n}(v^{-2n})^2(v^{2n-2})^2(v^{-(2n-2)})^2\cdot\frac{v^{-1}-v}{z}=v^{-2n}-1.$$

Fifth, we smooth $n-1$ crossings of the third twist tangle a_{2n-1}^{β} and keep switching n crossings of each for a b_{2n-1}^{γ} and a a_{2n-1}^{β} and then switching $n-1$ crossings of each for a b_{2n-1}^{γ} and a a_{2n-1}^{β} to obtain the link diagram D_5' from D. Then D_5' is further changed into a trivial link D_{11}^T by switching n crossings of each for a $a_{2n}^{-\alpha}$, a b_{2n-1}^{γ} and a a_{2n-1}^{β}. Then

$$P_{D_{11}^T}\cdot H(D_{11}^T)=\left(-v^{-1}z\frac{v^{-(2n-2)}-1}{v^{-2}-1}\right)(v^{2n})^2(v^{-2n})^3v^{2n-2}v^{-(2n-2)}\frac{v^{-1}-v}{z}=v^{-2n}-v^{2-4n}.$$

At last, we smooth n crossings of the remaining twist tangle a_{2n-1}^{β} and keep switching n crossings of each for a b_{2n-1}^{γ} and a a_{2n-1}^{β} and then switching $n-1$ crossings of each for a b_{2n-1}^{γ} and two a_{2n-1}^{β} to obtain the link diagram D_6' from D. Then D_6' is further changed into a trivial link D_{12}^T by switching n crossings of a $a_{2n}^{-\alpha}$ and then switching $n-1$ crossings of a b_{2n-1}^{γ}. Then

$$P_{D_{12}^T}\cdot H(D_{12}^T)=\left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right)v^{2n}(v^{-2n})^2(v^{2n-2})^2(v^{-(2n-2)})^2\frac{v^{-1}-v}{z}=v^{-2n}-v^{-4n}.$$

Hence the lowest-degree term of z for $H(D(2a_{2n-1}^{\beta},b_{2n-1}^{\gamma},a_{2n}^{\alpha},2b_{2n-1}^{\gamma},2a_{2n-1}^{\beta},b_{2n-1}^{\gamma}))$ only contains the above twelve cases. According to the theorem 3.12, we have

$$\begin{aligned}&min_zH(D_{362})\\&=-4-4v^{-2}+v^{2-6n}-3v^{2-4n}+v^{-2-2n}+v^{2-2n}-4v^{-4n}+11v^{-2n}+2v^{-2+2n}.\end{aligned}$$

(19) There is only one triangle link diagram $D_{351}=D(3b_{2n-1}^{\gamma},3a_{2n-1}^{\beta},3b_{2n-1}^{\gamma})_{351}$ oriented with $o(3\gamma,3\beta,3\gamma)$ in table 2. There are thirteen trivial link diagrams, which all

together result in the lowest-degree term of z for $H(D_{351})$. A trivial knot D_1^T is obtained from the diagram D_{351} by switching some crossings. In this process, we switch n crossings of each for three b_{2n-1}^{γ} and a a_{2n-1}^{β}, and switch $n-1$ crossings of each for two b_{2n-1}^{γ} and a a_{2n-1}^{β}. By using the lemmas 3.9 and 3.10, then

$$P_{D_1^T} \cdot H(D_1^T) = (v^{2n})^3 v^{-2n} (v^{2n-2})^2 v^{-(2n-2)} \cdot 1 = v^{6n-2}.$$

Note that five b_{2n-1}^{γ} and two a_{2n-1}^{β} in the above process are all on the same component of D. Respectively smoothing the crossings of these seven twist tangles.

First, we smooth $n-1$ crossings of a b_{2n-1}^{γ} to obtain the link diagram D_1' from D. Then D_1' is further changed into a trivial link D_2^T by switching n crossings of each for two b_{2n-1}^{γ} and a a_{2n-1}^{β}, and then switching $n-1$ crossings of each for two b_{2n-1}^{γ} and a a_{2n-1}^{β}. Then

$$P_{D_2^T} \cdot H(D_2^T) = vz\frac{v^{2n-2}-1}{v^2-1}(v^{2n})^2 v^{-2n}(v^{2n-2})^2 v^{-(2n-2)} \cdot \frac{v^{-1}-v}{z} = v^{-2+4n} - v^{-4+6n}.$$

Also, there are two b_{2n-1}^{γ} in the above process on the same component of D_1'. Respectively smoothing the crossings of these two b_{2n-1}^{γ}. On the one hand, we smooth $n-1$ crossings of a b_{2n-1}^{γ} to obtain the link diagram D_1'' from D_1'. Then D_1'' is further changed into a trivial link D_3^T by switching n crossings of each for a b_{2n-1}^{γ} and a a_{2n-1}^{β}, and then switching $n-1$ crossings of each for two b_{2n-1}^{γ} and two a_{2n-1}^{β}. Then

$$\begin{aligned} P_{D_3^T} \cdot H(D_3^T) = &\left(vz\frac{v^{2n-2}-1}{v^2-1}\right)^2 v^{2n} v^{-2n} (v^{2n-2})^2 (v^{-(2n-2)})^2 \\ &\left(\frac{v^{-1}-v}{z}\right)^2 = 1 - 2v^{-2+2n} + v^{-4+4n}. \end{aligned}$$

On the other hand, we smooth n crossings of the other b_{2n-1}^{γ} and keep switching $n-1$ crossings of a b_{2n-1}^{γ} to obtain the link diagram D_2'' from D_1'. Then D_2'' is further changed into a trivial link D_4^T by switching n crossings of each for a b_{2n-1}^{γ} and a a_{2n-1}^{β}, and then switching $n-1$ crossings of each for the remaining two b_{2n-1}^{γ} and two a_{2n-1}^{β}. Then

$$\begin{aligned} P_{D_4^T} \cdot H(D_4^T) =& vz\frac{v^{2n-2}-1}{v^2-1} vz\frac{v^{2n}-1}{v^2-1} v^{2n} v^{-2n} (v^{2n-2})^3 (v^{-(2n-2)})^2 \cdot \left(\frac{v^{-1}-v}{z}\right)^2 \\ =& v^{-2+2n} - v^{-4+4n} - v^{-2+4n} + v^{-4+6n}. \end{aligned}$$

Second, we smooth n crossings of another b_{2n-1}^{γ} and keep switching $n-1$ crossings of a b_{2n-1}^{γ} to obtain the link diagram D_2' from D. Then D_2' is further changed into a trivial

link D_5^T by switching n crossings of each for two b_{2n-1}^{γ} and a a_{2n-1}^{β}, and then switching $n-1$ crossings of each for a b_{2n-1}^{γ} and a a_{2n-1}^{β}. Then

$$P_{D_5^T}\cdot H(D_5^T)=vz\frac{v^{2n}-1}{v^2-1}(v^{2n})^2v^{-2n}(v^{2n-2})^2v^{-(2n-2)}\cdot\frac{v^{-1}-v}{z}=v^{-2+4n}-v^{-2+6n}.$$

Also, there is only one b_{2n-1}^{γ} on the same component of D_2'. We smooth n crossings of the b_{2n-1}^{γ} to obtain the link diagram D_1'' from D_2'. Then D_1'' is further changed into a trivial link D_6^T by switching n crossings of each for a b_{2n-1}^{γ} and a a_{2n-1}^{β}, and then switching $n-1$ crossings of each for the remaining two b_{2n-1}^{γ} and two a_{2n-1}^{β}. Then

$$\begin{aligned}P_{D_6^T}\cdot H(D_6^T)=&(vz\frac{v^{2n}-1}{v^2-1})^2v^{2n}v^{-2n}(v^{2n-2})^3(v^{-(2n-2)})^2\left(\frac{v^{-1}-v}{z}\right)^2\\=&v^{-2+2n}-2v^{-2+4n}+v^{-2+6n}.\end{aligned}$$

Third, we smooth $n-1$ crossings of a a_{2n-1}^{β} and keep switching n crossings of a b_{2n-1}^{γ} and $n-1$ crossings of a b_{2n-1}^{γ} to obtain the link diagram D_3' from D. Then D_3' is further changed into a trivial link D_7^T by switching n crossings of each for a b_{2n-1}^{γ} and a a_{2n-1}^{β}, and then switching $n-1$ crossings of each for two b_{2n-1}^{γ} and a a_{2n-1}^{β}. Then

$$\begin{aligned}P_{D_7^T}\cdot H(D_7^T)=&\left(-v^{-1}z\frac{v^{-(2n-2)}-1}{v^{-2}-1}\right)(v^{2n})^2v^{-2n}(v^{2n-2})^3v^{-(2n-2)}\frac{v^{-1}-v}{z}\\=&v^{-4+6n}-v^{-2+4n}.\end{aligned}$$

Fourth, we smooth n crossings of the third twist tangle b_{2n-1}^{γ} and keep switching n crossings of a b_{2n-1}^{γ} and then switching $n-1$ crossings of each for a b_{2n-1}^{γ} and a a_{2n-1}^{β} to obtain the link diagram D_4' from D. Then D_4' is further changed into a trivial link D_8^T by switching n crossings of each for a b_{2n-1}^{γ} and a a_{2n-1}^{β}, and then switching $n-1$ crossings of a b_{2n-1}^{γ}. Then

$$P_{D_8^T}\cdot H(D_8^T)=vz\frac{v^{2n}-1}{v^2-1}(v^{2n})^2v^{-2n}(v^{2n-2})^2v^{-(2n-2)}\cdot\frac{v^{-1}-v}{z}=v^{-2+4n}-v^{-2+6n}.$$

Also, there are two b_{2n-1}^{γ} in the above process on the same component of D_4'. Respectively smoothing the crossings of these two b_{2n-1}^{γ}. On the one hand, we smooth $n-1$ crossings of a b_{2n-1}^{γ} to obtain the link diagram D_1'' from D_4'. Then D_1'' is further changed into a trivial link D_9^T by switching n crossings of a a_{2n-1}^{β}, and then switching $n-1$ crossings of each for a b_{2n-1}^{γ} and a a_{2n-1}^{β}. Then

$$\begin{aligned}P_{D_9^T}\cdot H(D_9^T)=&vz\frac{v^{2n}-1}{v^2-1}vz\frac{v^{2n-2}-1}{v^2-1}v^{2n}v^{-2n}(v^{2n-2})^2(v^{-(2n-2)})^2\left(\frac{v^{-1}-v}{z}\right)^2\\=&1-v^{2n}-v^{-2+2n}+v^{-2+4n}.\end{aligned}$$

On the other hand, we smooth n crossings of the other b^{γ}_{2n-1} and keep switching $n-1$ crossings of a b^{γ}_{2n-1} to obtain the link diagram D''_2 from D'_4. Then D''_2 is further changed into a trivial link D^T_{10} by switching n crossings of a a^{β}_{2n-1}, and then switching $n-1$ crossings of each for the remaining a b^{γ}_{2n-1} and a a^{β}_{2n-1}. Then

$$\begin{aligned} P_{D^T_{10}} \cdot H(D^T_{10}) &= \left(vz\frac{v^{2n}-1}{v^2-1}\right)^2 v^{2n}v^{-2n}(v^{2n-2})^3(v^{-(2n-2)})^2 \cdot \left(\frac{v^{-1}-v}{z}\right)^2 \\ &= v^{-2+2n} - 2v^{-2+4n} + v^{-2+6n}. \end{aligned}$$

Fifth, we smooth $n-1$ crossings of the fourth twist tangle b^{γ}_{2n-1} and keep switching n crossings of each for two b^{γ}_{2n-1} and then switching $n-1$ crossings of each for a b^{γ}_{2n-1} and a a^{β}_{2n-1} to obtain the link diagram D'_5 from D. Then D'_5 is further changed into a trivial link D^T_{11} by switching n crossings of a a^{β}_{2n-1} and then switching $n-1$ crossings of each for a b^{γ}_{2n-1} and a a^{β}_{2n-1}. Then

$$P_{D^T_{11}} \cdot H(D^T_{11}) = vz\frac{v^{2n-2}-1}{v^2-1}(v^{2n})^2v^{-2n}(v^{2n-2})^2(v^{-(2n-2)})^2\frac{v^{-1}-v}{z} = v^{2n} - v^{-2+4n}.$$

Sixth, we smooth n crossings of the other a^{β}_{2n-1} and keep switching n crossings of each for two b^{γ}_{2n-1} and then switching $n-1$ crossings of each for two b^{γ}_{2n-1} and a a^{β}_{2n-1} to obtain the link diagram D'_6 from D. Then D'_6 is further changed into a trivial link D^T_{12} by switching n crossings of each for a b^{γ}_{2n-1} and a a^{β}_{2n-1}. Then

$$\begin{aligned} P_{D^T_{12}} \cdot H(D^T_{12}) &= \left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right)(v^{2n})^3v^{-2n}(v^{2n-2})^2v^{-(2n-2)}\frac{v^{-1}-v}{z} \\ &= -v^{-2+4n} + v^{-2+6n}. \end{aligned}$$

At last, we smooth n crossings of the remaining twist tangle b^{γ}_{2n-1} and keep switching n crossings of each for two b^{γ}_{2n-1} and a a^{β}_{2n-1} and then switching $n-1$ crossings of each for two b^{γ}_{2n-1} and a a^{β}_{2n-1} to obtain the link diagram D'_7 from D. Then D'_7 is further changed into a trivial link D^T_{13} by switching n crossings of each for the remaining a b^{γ}_{2n-1} and a a^{β}_{2n-1}. Then

$$P_{D^T_{13}} \cdot H(D^T_{13}) = vz\frac{v^{2n}-1}{v^2-1}(v^{2n})^3(v^{-2n})^2(v^{2n-2})^2v^{-(2n-2)}\frac{v^{-1}-v}{z} = v^{-2+4n} - v^{-2+6n}.$$

Hence the lowest-degree term of z for $H(D(3b^{\gamma}_{2n-1}, 3a^{\beta}_{2n-1}, 3b^{\gamma}_{2n-1}))$ only contains the above thirteen cases. According to the theorem 3.12, we have

$$min_z H(D_{351}) = 2 - 3v^{-2+4n} + v^{-4+6n} + v^{-2+6n}.$$

■

Theorem 3.14. *Let $smin_zH(D)$ be the second lowest degree term of z in $H(D(2b^\alpha_{2n}, 4a^\alpha_{2n}, 2b^\alpha_{2n}, a^\alpha_{2n})_{64})$. Then*

$$smin_zH(D_{64}) = -3 + v^{-10n} - 2v^{-6n} + 2v^{-4n} + v^{-2n} + v^{8n}.$$

Proof. There is only one lowest-degree term of z in $H(D(2b^\alpha_{2n}, 4a^\alpha_{2n}, 2b^\alpha_{2n}, a^\alpha_{2n})_{64})$. First, the link diagram $D_{64} = D(2b^\alpha_{2n}, 4a^\alpha_{2n}, 2b^\alpha_{2n}, a^\alpha_{2n})_{64}$ is changed into a trivial link diagram D^T_0 of three components by switching the crossings of all oriented twist tangles except a a^α_{2n}. In this process, we switch n crossings of each for four a^α_{2n} and four b^α_{2n}. By using the lemmas 3.7 and 3.8, then

$$P_{D^T_0} \cdot H(D^T_0) = (v^{-2n})^4(v^{2n})^4 \cdot \left(\frac{v^{-1} - v}{z}\right)^2 = (-2 + v^{-2} + v^2)z^{-2}.$$

In addition, for the link diagram D, all twist tangles except an unused a^α_{2n} are composed of two different components. For each such twist tangle, smoothing its any crossing don't result in a lowest-degree term of z in $H(D)$. By using theorem 3.12, we have

$$min_zH(D_{64}) = (-2 + v^{-2} + v^2)z^{-2}.$$

In the following, the second lowest-degree of z in $H(D_{64})$ are calculated. Let c_1 be a crossing of D. Without loss of generality, we assume $s(c_1) = +1$. Smoothing the crossing c_1 will produce a term vz. In the meantime, if the crossing c_1 is composed of two different components of D, smoothing it will enable the original component number to decrease by one. Hence the degree of z is two higher than the degree of z in $min_zH(D)$. Hence the second lowest-degree term of z in $H(D)$ can be produced by firstly smoothing a crossing of a twist tangle on two different components and then switching some of the remaining crossings of D.

There are eleven trivial link diagrams, which all together result in the second lowest-degree term of z in $H(D(2b^\alpha_{2n}, 4a^\alpha_{2n}, 2b^\alpha_{2n}, a^\alpha_{2n}))$. For the link diagram $D_{64} = D(2b^\alpha_{2n}, 4a^\alpha_{2n}, 2b^\alpha_{2n}, a^\alpha_{2n})_{64}$, we note that four b^α_{2n} and four a^α_{2n} are composed of two different components. For each such twist tangle, smoothing its crossings only once will result in a second lowest-order term of z in $H(D)$.

First, we smooth n crossings of a b^α_{2n} to obtain the link diagram D'_1 from D. Then D'_1 is further changed into a trivial link D^T_1 by switching n crossings of each twist tangle a^α_{2n}. By using the lemmas 3.7 and 3.8, then

$$P_{D^T_1} \cdot H(D^T_1) = vz\frac{v^{2n} - 1}{v^2 - 1}(v^{-2n})^5 \cdot \frac{v^{-1} - v}{z} = v^{-10n} - v^{-8n}.$$

Also, there is only one a_{2n}^{α} on the same component of D_1'. We smooth n crossings of the a_{2n}^{α} to obtain the link diagram D_1'' from D_1'. Then D_1'' is further changed into a trivial link D_2^T by switching n crossings of each for two b_{2n}^{α} and four a_{2n}^{α}. Then

$$\begin{aligned} P_{D_2^T} \cdot H(D_2^T) &= vz\frac{v^{2n}-1}{v^2-1}\left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right)(v^{2n})^2(v^{-2n})^4\left(\frac{v^{-1}-v}{z}\right)^2 \\ &= -v^{-6n}+2v^{-4n}-v^{-2n}. \end{aligned}$$

Second, we smooth n crossings of another b_{2n}^{α} and keep switching n crossings of a b_{2n}^{α} to obtain the link diagram D_2' from D. Then D_2' is further changed into a trivial link D_3^T by switching n crossings of each for a b_{2n}^{α} and four a_{2n}^{α}. Then

$$P_{D_3^T} \cdot H(D_3^T) = vz\frac{v^{2n}-1}{v^2-1}(v^{2n})^2(v^{-2n})^4 \cdot \frac{v^{-1}-v}{z} = v^{-4n}-v^{-2n}.$$

Also, there is only one b_{2n}^{α} on the same component of D_2'. We smooth n crossings of the b_{2n}^{α} to obtain the link diagram D_1'' from D_2'. Then D_1'' is further changed into a trivial link D_4^T by switching n crossings of each twist tangle a_{2n}^{α}. Then

$$P_{D_4^T} \cdot H(D_4^T) = \left(vz\frac{v^{2n}-1}{v^2-1}\right)^2 v^{2n}(v^{-2n})^5 \cdot \left(\frac{v^{-1}-v}{z}\right)^2 = v^{-8n}-2v^{-6n}+v^{-4n}.$$

Third, we smooth n crossings of the third twist tangle b_{2n}^{α} and keep switching n crossings of each for two b_{2n}^{α} to obtain the link diagram D_3' from D. Then D_3' is further changed into a trivial link D_5^T by switching n crossings of each for a b_{2n}^{α} and four a_{2n}^{α}. Then

$$P_{D_5^T} \cdot H(D_5^T) = vz\frac{v^{2n}-1}{v^2-1}(v^{2n})^3(v^{-2n})^4 \cdot \frac{v^{-1}-v}{z} = -1+v^{-2n}.$$

Also, there is only one b_{2n}^{α} on the same component of D_3'. We smooth n crossings of the b_{2n}^{α} to obtain the link diagram D_1'' from D_3'. Then D_1'' is further changed into a trivial link D_6^T by switching n crossings of each twist tangle a_{2n}^{α}. Then

$$P_{D_6^T} \cdot H(D_6^T) = \left(vz\frac{v^{2n}-1}{v^2-1}\right)^2 (v^{2n})^2(v^{-2n})^5 \cdot \left(\frac{v^{-1}-v}{z}\right)^2 = v^{-6n}-2v^{-4n}+v^{-2n}.$$

Fourth, we smooth n crossings of the other b_{2n}^{α} and keep switching n crossings of each for three b_{2n}^{α} to obtain the link diagram D_4' from D. Then D_4' is further changed into a trivial link D_7^T by switching n crossings of each for four a_{2n}^{α}. Then

$$P_{D_7^T} \cdot H(D_7^T) = vz\frac{v^{2n}-1}{v^2-1}(v^{2n})^3(v^{-2n})^4 \cdot \frac{v^{-1}-v}{z} = -1+v^{-2n}.$$

Fifth, we smooth n crossings of a a_{2n}^{α} and keep switching n crossings of each twist tangle b_{2n}^{α} to obtain the trivial link diagram D_8^T from D. Then

$$P_{D_8^T} \cdot H(D_8^T) = \left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right)(v^{2n})^4 \cdot \frac{v^{-1}-v}{z} = -v^{6n}+v^{8n}.$$

Sixth, we smooth n crossings of another a_{2n}^{α} and keep switching n crossings of each twist tangle b_{2n}^{α} and a a_{2n}^{α} to obtain the trivial link diagram D_9^T from D. Then

$$P_{D_9^T} \cdot H(D_9^T) = \left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right)(v^{2n})^4 v^{-2n} \cdot \frac{v^{-1}-v}{z} = -v^{4n}+v^{6n}.$$

Seventh, we smooth n crossings of the third twist tangle a_{2n}^{α} and keep switching n crossings of each for four b_{2n}^{α} and two a_{2n}^{α} to obtain the trivial link diagram D_{10}^T from D. Then

$$P_{D_{10}^T} \cdot H(D_{10}^T) = \left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right)(v^{2n})^4 (v^{-2n})^2 \cdot \frac{v^{-1}-v}{z} = -v^{2n}+v^{4n}.$$

At last, we smooth n crossings of the remaining twist tangle a_{2n}^{α} and keep switching n crossings of each for four b_{2n}^{α} and three a_{2n}^{α} to obtain the trivial link diagram D_{11}^T from D. Then

$$P_{D_{11}^T} \cdot H(D_{11}^T) = \left(-v^{-1}z\frac{v^{-2n}-1}{v^{-2}-1}\right)(v^{2n})^4 (v^{-2n})^3 \cdot \frac{v^{-1}-v}{z} = -1+v^{2n}.$$

Hence the second lowest-degree term of z in $H(D(2b_{2n}^{\alpha}, 4a_{2n}^{\alpha}, 2b_{2n}^{\alpha}, a_{2n}^{\alpha}))$ only contains the above eleven cases, hence we have

$$smin_z H(D_{64}) = -3+v^{-10n}-2v^{-6n}+2v^{-4n}+v^{-2n}+v^{8n}.$$

■

Theorem 3.15. *The OTP link diagrams of 366 link types shown in Table 1 (Appendix A) are all chiral.*

Proof. By using theorem 3.2 and 3.5, we only need to proof that 77 OTP links are chiral, which are indicated by ‘ $*$ ’ in Table 1 (Appendix A). The lowest-degree term or second lowest-degree term of HOMFLY polynomials of these 77 OTP links are given by the theorems 3.13 and 3.14. Hence for the link $D(3a_{2n}^{\alpha}, 6b_{2n}^{\alpha})_{23}$, we have

$$min_z H(D_{23}) = (6v^{6n}+v^{-4+6n}-4v^{-2+6n}-4v^{2+6n}+v^{4+6n})z^{-4}.$$

By using the definition (3) of HOMFLY polynomial, we obtain

$$\begin{aligned} min_z H(D_{23}^*) &= min_z H(D_{23})(-v^{-1}, z) \\ &= (6v^{-6n} + v^{4-6n} - 4v^{2-6n} - 4v^{-2-6n} + v^{-4-6n})z^{-4}. \end{aligned}$$

If D_{23} is achiral, we have

$$min_z H(D_{23}^*) = min_z H(D_{23}).$$

But we have

$$min_z H(D_{23}) \neq min_z H(D_{23})(-v^{-1}, z).$$

Clearly, D_{23} must be chiral link. Similarly, we can show that the remaining 76 links are all chiral by using the lowest-degree terms or second lowest-degree terms of their HOMFLY polynomials. ■

4 Conclusion

In the present paper, we show that 366 OTP links are all chiral by calculating their invariants such as component number, crossing number, writhe number and HOMFLY polynomial. Among these links, there are 100 pair of systemical links have the same values for the above invariants. Moreover, there are 178 OTP links with even number of components, which include one link with six components, 25 links with four components, and 152 links with two components. The chirality of these links are determined by their component number. For the remaining 188 links with odd number of components, they include 4 link with five components, 94 links with three components, and 90 links with one component. Evidently, the links with two components have the most links than the other ones. Moreover, these results show that for any i $(1 \leq i \leq 6)$, there exists at least an oriented triangle link with i components such that each edge consist of two twisted strands in antiparallel orientation.

Furthermore, for these 366 links, the largest number of crossings is $18n$ and there are 104 such links. This means that these links have at least a complete twist on each edge. On the other hand, the smallest number of crossings is $18n - 9$ and there is only one link $D(3b_{2n-1}^{\gamma}, 3a_{2n-1}^{\beta}, 3b_{2n-1}^{\gamma})_{351}$. This means the link has at least half twist on each edge. These results provide a possibility to adjust the length of DNA triangle links by considering their twist number. Also, the writhe number of oriented triangle link diagrams

are given in table 1, which, together with the crossing number, identify the chirality of 111 triangle links.

There are 77 oriented triangle links whose lowest-degree terms of HOMFLY polynomials are given as a general formula in term of the twist number n in table 2. Also, for each link type, there exists an infinite family of OTP links obtained by changing the crossing number on each edge. Hence these polynomials also give a general formula for the lowest-degree terms of the HOMFLY polynomials for each family of links. In special, there is an OTP link $D(2b^{\alpha}_{2n}, 4a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n})_{64}$ whose second lowest-degree term of HOMFLY polynomial is also given. These results show that these links are all chiral by using the asymmetry of these polynomials over v. Here we note that the lowest-degree terms of HOMFLY polynomials of some OTP links have the same polynomials. However, this does not means that these links have the same HOMFLY polynomials, or belongs to the same link type.

Thus, our results show that the chirality of triangle prism molecules with DNA double edges can be determined by the 366 topological structures whether each building block is symmetry. Also, they confirms that the synthesized DNA triangle prism with one or two components are both chiral without considering the twist number on each edge [22, 23]. Our work provide a theoretical approach to synthesizing, control and study the chiral structures of DNA triangle prisms.

Acknowledgments: This work was supported by a grant from the National Natural Science Foundation of China (No. 11501454).

Appendix A The crossing number, component number and writhe number of 366 OT-link diagrams

D	c(D)	u(D)	w(D)	D^s
$D(9b^{\alpha}_{2n})_1$	18n	6	18n	
$D(9a^{\alpha}_{2n})_2$		5	-18n	
$D(a^{\alpha}_{2n}, 8b^{\alpha}_{2n})_3$		5	14n	
$D(3b^{\alpha}_{2n}, a^{\alpha}_{2n}, 5b^{\alpha}_{2n})_5$		5	14n	
$D(3a^{\alpha}_{2n}, 6b^{\alpha}_{2n})_{23}$ *		5	6n	
$D(b^{\alpha}_{2n}, 8a^{\alpha}_{2n})_4$		4	-14n	
$D(3a^{\alpha}_{2n}, b^{\alpha}_{2n}, 5a^{\alpha}_{2n})_6$		4	-14n	
$D(2a^{\alpha}_{2n}, 7b^{\alpha}_{2n})_7$		4	10n	
$D(a^{\alpha}_{2n}, 7b^{\alpha}_{2n}, a^{\alpha}_{2n})_9$		4	10n	$D(a^{\alpha}_{2n}, 6b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n})_{13}$
$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 5b^{\alpha}_{2n})_{11}$		4	10n	
$D(a^{\alpha}_{2n}, 5b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n})_{15}$		4	10n	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, 5b^{\alpha}_{2n})_{17}$		4	10n	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 5b^{\alpha}_{2n})_{19}$
$D(3b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 3b^{\alpha}_{2n})_{21}$		4	10n	

$D(3a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, 3a^{\alpha}_{2n})_{28}$		4	-6n	
$D(3a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n})_{40}$		4	-6n	
$D(4a^{\alpha}_{2n}, 5b^{\alpha}_{2n})_{57}$		4	2n	
$D(3a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n})_{61}$		4	2n	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 2b^{\alpha}_{2n})_{87}$		4	2n	
$D(2b^{\alpha}_{2n}, 7a^{\alpha}_{2n})_{8}$		3	-10n	
$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 5a^{\alpha}_{2n})_{12}$		3	-10n	
$D(b^{\alpha}_{2n}, 6a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n})_{14}$		3	-10n	$D(b^{\alpha}_{2n}, 7a^{\alpha}_{2n}, b^{\alpha}_{2n})_{10}$
$D(b^{\alpha}_{2n}, 5a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n})_{16}$		3	-10n	
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 5a^{\alpha}_{2n})_{18}$		3	-10n	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 5a^{\alpha}_{2n})_{20}$
$D(3a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 3a^{\alpha}_{2n})_{22}$		3	-10n	
$D(a^{\alpha}_{2n}, 6b^{\alpha}_{2n}, 2a^{\alpha}_{2n})_{25}$ *		3	6n	
$D(3b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, 3b^{\alpha}_{2n})_{27}$ *		3	6n	
$D(a^{\alpha}_{2n}, 5b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n})_{29}$ *		3	6n	$D(a^{\alpha}_{2n}, 5b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n})_{35}$
$D(3b^{\alpha}_{2n}, a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, 2a^{\alpha}_{2n})_{31}$ *		3	6n	$D(3b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n})_{45}$
$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 4b^{\alpha}_{2n}, a^{\alpha}_{2n})_{33}$ *		3	6n	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, 4b^{\alpha}_{2n}, a^{\alpha}_{2n})_{37}$ *		3	6n	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 4b^{\alpha}_{2n}, a^{\alpha}_{2n})_{43}$
$D(3b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n})_{39}$ *		3	6n	
$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 3b^{\alpha}_{2n})_{41}$ *		3	6n	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 3b^{\alpha}_{2n})_{53}$
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n})_{47}$ *		3	6n	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n})_{55}$
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 3b^{\alpha}_{2n})_{49}$ *		3	6n	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n})_{51}$ *		3	6n	
$D(4b^{\alpha}_{2n}, 5a^{\alpha}_{2n})_{58}$ *		3	-2n	
$D(2b^{\alpha}_{2n}, 4a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n})_{64}$ *		3	-2n	
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, 3a^{\alpha}_{2n})_{66}$ *		3	-2n	
$D(a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 3a^{\alpha}_{2n})_{72}$ *		3	-2n	
$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n})_{76}$ *		3	-2n	
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n})_{80}$ *		3	-2n	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n})_{92}$
$D(2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 3a^{\alpha}_{2n})_{82}$ *		3	-2n	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 3a^{\alpha}_{2n})_{84}$
$D(3b^{\alpha}_{2n}, 6a^{\alpha}_{2n})_{24}$		2	-6n	
$D(b^{\alpha}_{2n}, 6a^{\alpha}_{2n}, 2b^{\alpha}_{2n})_{26}$		2	-6n	
$D(b^{\alpha}_{2n}, 5a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n})_{30}$		2	-6n	$D(b^{\alpha}_{2n}, 5a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n})_{36}$
$D(3a^{\alpha}_{2n}, b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, 2b^{\alpha}_{2n})_{32}$		2	-6n	$D(3a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n})_{46}$
$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 4a^{\alpha}_{2n}, b^{\alpha}_{2n})_{34}$		2	-6n	
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 4a^{\alpha}_{2n}, b^{\alpha}_{2n})_{38}$		2	-6n	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 4a^{\alpha}_{2n}, b^{\alpha}_{2n})_{44}$
$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 3a^{\alpha}_{2n})_{42}$		2	-6n	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 3a^{\alpha}_{2n})_{54}$
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n})_{48}$		2	-6n	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n})_{56}$
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 3a^{\alpha}_{2n})_{50}$		2	-6n	
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n})_{52}$		2	-6n	
$D(2a^{\alpha}_{2n}, 5b^{\alpha}_{2n}, 2a^{\alpha}_{2n})_{59}$		2	2n	$D(2a^{\alpha}_{2n}, 4b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n})_{69}$
$D(2a^{\alpha}_{2n}, 4b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n})_{63}$		2	2n	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, 3b^{\alpha}_{2n})_{65}$		2	2n	
$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, 2a^{\alpha}_{2n})_{67}$		2	2n	$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n})_{85}$
$D(b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 3b^{\alpha}_{2n})_{71}$		2	2n	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, 2a^{\alpha}_{2n})_{73}$		2	2n	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n})_{97}$
$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n})_{75}$		2	2n	
$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, 2a^{\alpha}_{2n})_{77}$		2	2n	$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n})_{89}$
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n})_{79}$		2	2n	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n})_{91}$
$D(2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 3b^{\alpha}_{2n})_{81}$		2	2n	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 3b^{\alpha}_{2n})_{83}$
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n})_{93}$		2	2n	$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n})_{99}$
$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n})_{95}$		2	2n	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n})_{101}$
$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n})_{103}$		2	2n	
$D(2b^{\alpha}_{2n}, 5a^{\alpha}_{2n}, 2b^{\alpha}_{2n})_{60}$ *		1	-2n	$D(2b^{\alpha}_{2n}, 4a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n})_{70}$
$D(3b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n})_{62}$ *		1	-2n	
$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, 2b^{\alpha}_{2n})_{68}$ *		1	-2n	$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n})_{86}$
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, 2b^{\alpha}_{2n})_{74}$ *		1	-2n	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n})_{98}$
$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, 2b^{\alpha}_{2n})_{78}$ *		1	-2n	$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n})_{90}$
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\alpha}_{2n})_{88}$ *		1	-2n	
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n})_{94}$ *		1	-2n	$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n})_{100}$
$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n})_{96}$ *		1	-2n	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n})_{102}$
$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n})_{104}$ *		1	-2n	
$D(4b^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{107}$	18n-3	4	2n+3	
$D(6b^{\alpha}_{2n}, 3b^{\beta}_{2n-1})_{169}$		4	18n-3	
$D(4b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{105}$		3	6n+3	
$D(4a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{106}$		3	-18n+3	
$D(4b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{119}$ *		3	-2n+3	$D(3b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{129}$
$D(4a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{120}$		3	-10n+3	$D(3a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{130}$
$D(a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{125}$ *		3	-2n+3	
$D(3a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{132}$		3	-10n+3	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{134}$		3	-10n+3	
$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{139}$ *		3	-2n+3	$D(2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{141}$

$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{143}$ *	3	-2n+3	$D(2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{145}$
$D(a^{\alpha}_{2n}, 5b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{171}$	3	14n-3	
$D(3b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{173}$	3	14n-3	
$D(3a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{174}$ *	3	-2n-3	
$D(3a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{185}$ *	3	6n-3	
$D(2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{190}$ *	3	6n-3	
$D(3b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 3b^{-\alpha}_{2n})_{193}$	3	6n+3	
$D(3a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 3a^{-\alpha}_{2n})_{194}$	3	-18n+3	
$D(2b^{\alpha}_{2n}, a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 3a^{-\alpha}_{2n})_{198}$	3	-10n+3	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, 2b^{-\alpha}_{2n})_{199}$ *	3	-2n+3	
$D((b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{151}$	2	-6n+3	$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{163}$
$D((4a^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{108}$	2	-14n+3	
$D((4b^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{109}$	2	2n+3	$D(3b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{113}$
$D((4a^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{110}$	2	-14n+3	$D(3a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{114}$
$D((a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{111}$	2	2n+3	
$D((b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{112}$	2	-14n+3	
$D((2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{115}$	2	2n+3	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{117}$
$D((2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{116}$	2	-14n+3	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{118}$
$D((3a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{149}$	2	-6n+3	
$D((3b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{150}$	2	-6n+3	
$D(a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{152}$	2	-6n+3	$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{164}$
$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{153}$	2	-6n+3	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{165}$
$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, a^{\alpha}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{154}$	2	-6n+3	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{166}$
$D(2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{155}$	2	-6n+3	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{167}$
$D(2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{156}$	2	-6n+3	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{168}$
$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{157}$	2	-6n+3	$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{159}$
$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\alpha}_{2n-1})_{158}$	2	-6n+3	$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, a^{\alpha}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{160}$
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{161}$	2	-6n+3	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{162}$	2	-6n+3	
$D(6a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{170}$	2	-6n-3	
$D(2a^{\alpha}_{2n}, 4b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{175}$	2	10n-3	
$D(2b^{\alpha}_{2n}, 4a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{176}$	2	2n-3	
$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{177}$	2	10n-3	
$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{178}$	2	2n-3	
$D(3b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{179}$	2	10n-3	
$D(3a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{180}$	2	2n-3	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{181}$	2	10n-3	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{183}$
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{182}$	2	2n-3	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{184}$
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 3b^{-\alpha}_{2n})_{195}$	2	2n+3	
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 3a^{-\alpha}_{2n})_{196}$	2	-14n+3	
$D(3a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 3b^{-\alpha}_{2n})_{205}$	2	-6n+3	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 2a^{-\alpha}_{2n}, b^{-\alpha}_{2n})_{207}$	2	-6n+3	$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, b^{-\alpha}_{2n}, a^{-\alpha}_{2n})_{208}$
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, 2a^{-\alpha}_{2n})_{206}$	2	-6n+3	
$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{121}$ *	1	-2n+3	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{137}$
$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{122}$	1	-10n+3	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{138}$
$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{123}$ *	1	-2n+3	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{147}$
$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{124}$	1	-10n+3	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{148}$
$D(b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{126}$	1	-10n+3	
$D(a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{127}$ *	1	-2n+3	$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{135}$
$D(b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{128}$	1	-10n+3	$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{136}$
$D(3b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{131}$ *	1	-2n+3	
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{133}$ *	1	-2n+3	
$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{140}$	1	-10n+3	$D(2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{142}$
$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{144}$	1	-10n+3	$D(2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{146}$
$D(3b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{186}$ *	1	6n-3	
$D(b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{187}$ *	1	6n-3	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{191}$
$D(a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{188}$ *	1	6n-3	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{192}$
$D(2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{189}$ *	1	6n-3	
$D(b^{\alpha}_{2n}, 5a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{172}$ *	1	-2n-3	
$D(2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 3b^{-\alpha}_{2n})_{197}$ *	1	-2n+3	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 2b^{-\alpha}_{2n}, a^{-\alpha}_{2n})_{201}$ *	1	-2n+3	$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, a^{-\alpha}_{2n}, b^{-\alpha}_{2n})_{203}$
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, 2a^{-\alpha}_{2n})_{200}$	1	-10n+3	
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 2a^{-\alpha}_{2n}, b^{-\alpha}_{2n})_{202}$	1	-10n+3	$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, b^{-\alpha}_{2n}, a^{-\alpha}_{2n})_{204}$

$D(4a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{212}$	18n-4	4	-14n+4	
$D(3a^{\alpha}_{2n}, b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{219}$		4	-14n+4	
$D(2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{227}$		4	-6n+4	
$D(2b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2b^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{241}$		4	18n-4	
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{-\alpha}_{2n}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{270}$		4	-14n+4	
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, b^{-\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{276}$		4	-6n+4	
$D(4b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{209}$ *		3	-2n+4	
$D(2b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{228}$ *		3	-2n+4	
$D(4a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{211}$		3	-18n+4	
$D(b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{218}$		3	-10n+4	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{230}$
$D(3a^{\alpha}_{2n}, b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{220}$		3	-10n+4	
$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{221}$		3	-10n+4	
$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{223}$		3	-10n+4	
$D(2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{232}$		3	-10n+4	
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{233}$		3	-10n+4	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{237}$
$D(2b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2b^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{243}$		3	14n-4	
$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2b^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{245}$		3	14n-4	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2b^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{247}$
$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{258}$ *		3	6n-4	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{260}$
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, b^{-\alpha}_{2n}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{266}$		3	-10n+4	
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{-\alpha}_{2n}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{269}$		3	-18n+4	
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, b^{-\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{272}$ *		3	-2n+4	
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{-\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{274}$		3	-10n+4	
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{-\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{275}$		3	-10n+4	
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{-\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{279}$		3	-10n+4	
$D(4b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{210}$		2	2n+4	
$D(a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{213}$		2	-6n+4	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{225}$
$D(3b^{\alpha}_{2n}, a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{215}$		2	-6n+4	
$D(b^{\alpha}_{2n}, 3a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{217}$		2	-14n+4	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{229}$
$D(2a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{222}$		2	-6n+4	
$D(2b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{224}$		2	-6n+4	
$D(2a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{231}$		2	-14n+4	
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{234}$		2	-6n+4	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{238}$
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{236}$		2	-6n+4	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{240}$
$D(2a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{244}$		2	2n-4	
$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{246}$		2	2n-4	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{248}$
$D(2a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2b^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{249}$		2	10n-4	
$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2b^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{251}$		2	10n-4	$D(2b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{253}$
$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{255}$		2	10n-4	
$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{257}$		2	10n-4	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{259}$
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, b^{-\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{262}$		2	2n+4	
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{-\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{263}$		2	-6n+4	
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{-\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{271}$		2	-6n+4	
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{-\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{268}$		2	-6n+4	$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, b^{-\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{278}$
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{-\alpha}_{2n}, 2a^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{265}$		2	-14n+4	
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{-\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{273}$		2	-14n+4	
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{-\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{280}$		2	-6n+4	
$D(a^{\alpha}_{2n}, 3b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{214}$ *		1	-2n+4	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{226}$
$D(3b^{\alpha}_{2n}, a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{216}$ *		1	-2n+4	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{235}$		1	-10n+4	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{239}$
$D(2a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{242}$ *		1	-2n-4	
$D(2b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{250}$ *		1	6n-4	
$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{252}$ *		1	6n-4	$D(2a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{254}$
$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1})_{256}$ *		1	6n-4	
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{-\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{261}$ *		1	-2n+4	
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{-\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{264}$ *		1	-2n+4	
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{-\alpha}_{2n}, 2b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{267}$		1	-10n+4	$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{-\alpha}_{2n}, a^{\alpha}_{2n}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1})_{277}$
$D(2b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, a^{-\alpha}_{2n})_{281}$	18n-5	3	14n-5	
$D(2a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, b^{-\alpha}_{2n})_{284}$ *		3	6n-5	
$D(2a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, a^{-\alpha}_{2n})_{287}$ *		3	6n-5	
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, b^{-\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{303}$		3	-10n+5	$D(a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, a^{\beta}_{2n-1})_{313}$
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{-\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{305}$		3	-10n+5	$D(a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, a^{\beta}_{2n-1})_{315}$
$D(2b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, b^{-\alpha}_{2n})_{282}$		2	18n-5	

$D(2a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, a^{-\alpha}_{2n})_{283}$		2	2n-5	
$D(2b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, a^{-\alpha}_{2n})_{285}$		2	10n-5	
$D(2a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, b^{-\alpha}_{2n})_{288}$		2	10n-5	
$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, a^{-\alpha}_{2n})_{289}$		2	10n-5	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, a^{-\alpha}_{2n})_{291}$
$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, b^{-\alpha}_{2n})_{294}$		2	10n-5	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, b^{-\alpha}_{2n})_{296}$
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, b^{-\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{299}$		2	-6n+5	$D(a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, a^{\beta}_{2n-1})_{309}$
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{-\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{300}$		2	-14n+5	$D(b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, a^{\beta}_{2n-1})_{310}$
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{-\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{301}$		2	-6n+5	$D(b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, a^{\beta}_{2n-1})_{311}$
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{-\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{302}$		2	-14n+5	$D(a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, a^{\beta}_{2n-1})_{312}$
$D(2b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, b^{-\alpha}_{2n})_{286}$		1	14n-5	
$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, b^{-\alpha}_{2n})_{290}$		1	14n-5	$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, b^{-\alpha}_{2n})_{292}$
$D(b^{\alpha}_{2n}, a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, a^{-\alpha}_{2n})_{293}$ *		1	6n-5	$D(a^{\alpha}_{2n}, b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 4b^{\gamma}_{2n-1}, a^{-\alpha}_{2n})_{295}$
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, b^{-\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{297}$ *		1	-2n+5	$D(b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, a^{\beta}_{2n-1})_{307}$
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{-\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{298}$		1	-18n+5	$D(a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, a^{\beta}_{2n-1})_{308}$
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{-\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n})_{304}$		1	-10n+5	$D(b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, a^{\beta}_{2n-1})_{314}$
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, b^{-\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n})_{306}$		1	-10n+5	$D(b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, a^{\beta}_{2n-1})_{316}$
$D(a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1}, b^{-\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1})_{342}$	18n-6	4	10n-6	
$D(3a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 3b^{\gamma}_{2n-1})_{344}$		4	-6n	
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{320}$ *		3	-2n	
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\beta}_{2n-1})_{329}$ *		3	-10n+6	$D(b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n})_{335}$
$D(a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1}, a^{-\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1})_{338}$ *		3	6n-6	
$D(a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1}, b^{-\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1})_{339}$		3	14n-6	
$D(b^{\alpha}_{2n}, 2a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 3b^{\gamma}_{2n-1})_{346}$ *		3	-2n	
$D(3b^{\gamma}_{2n-1}, 3a^{-\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{348}$ *		3	6n-6	
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{318}$		2	-6n	
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{319}$		2	2n	
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{321}$		2	2n	$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{323}$
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\beta}_{2n-1})_{325}$		2	-6n+6	$D(b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n})_{331}$
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{\beta}_{2n-1})_{328}$		2	-14n+6	$D(b^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n})_{334}$
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{\beta}_{2n-1})_{330}$		2	-14n+6	$D(a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n})_{336}$
$D(b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1}, b^{-\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1})_{337}$		2	18n-6	
$D(b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1}, a^{-\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1})_{340}$		2	10n-6	
$D(a^{\alpha}_{2n}, 2b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 3b^{\gamma}_{2n-1})_{345}$		2	2n	
$D(3b^{\gamma}_{2n-1}, 3b^{-\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{347}$		2	18n-6	
$D(3b^{\gamma}_{2n-1}, b^{-\alpha}_{2n}, 2a^{-\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{350}$		2	10n-6	
$D(b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{317}$		1	6n	
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\beta}_{2n-1}, a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{322}$ *		1	-2n	$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{\beta}_{2n-1}, b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{324}$
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{\alpha}_{2n}, a^{\beta}_{2n-1})_{326}$		1	-18n+6	$D(a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, a^{\alpha}_{2n})_{332}$
$D(a^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{\alpha}_{2n}, a^{\beta}_{2n-1})_{327}$ *		1	-10n+6	$D(a^{\alpha}_{2n}, 4a^{\beta}_{2n-1}, b^{\alpha}_{2n}, 2a^{\beta}_{2n-1}, b^{\alpha}_{2n})_{333}$
$D(b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1}, a^{-\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1})_{341}$		1	14n-6	
$D(3b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, 3b^{\gamma}_{2n-1})_{343}$		1	6n	
$D(3b^{\gamma}_{2n-1}, a^{-\alpha}_{2n}, 2b^{-\alpha}_{2n}, 3b^{\gamma}_{2n-1})_{349}$		1	14n-6	
$D(a^{\alpha}_{2n}, 5a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, 2a^{\beta}_{2n-1})_{353}$	18n-7	3	-18n+7	
$D(a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, a^{\beta}_{2n-1}, 3b^{\gamma}_{2n-1})_{357}$ *		3	-2n+1	
$D(a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\beta}_{2n-1}, b^{\gamma}_{2n-1}, a^{\beta}_{2n-1})_{364}$ *		3	-2n-1	
$D(a^{\alpha}_{2n}, 5a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, 2a^{\beta}_{2n-1})_{354}$		2	-14n+7	
$D(b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, a^{\beta}_{2n-1}, 3b^{\gamma}_{2n-1})_{355}$		2	2n+1	
$D(a^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, a^{\beta}_{2n-1}, 3b^{\gamma}_{2n-1})_{356}$		2	-6n+1	
$D(a^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\beta}_{2n-1}, b^{\gamma}_{2n-1}, a^{\beta}_{2n-1})_{365}$		2	2n-1	$D(b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\beta}_{2n-1}, b^{\gamma}_{2n-1}, a^{\beta}_{2n-1})_{366}$
$D(b^{\alpha}_{2n}, 5a^{\beta}_{2n-1}, b^{-\alpha}_{2n}, 2a^{\beta}_{2n-1})_{352}$ *		1	-10n+7	
$D(b^{\alpha}_{2n}, 3a^{\beta}_{2n-1}, a^{-\alpha}_{2n}, a^{\beta}_{2n-1}, 3b^{\gamma}_{2n-1})_{358}$ *		1	-2n+1	
$D(b^{\alpha}_{2n}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\beta}_{2n-1}, b^{\gamma}_{2n-1}, a^{\beta}_{2n-1})_{363}$		1	6n-1	
$D(a^{\alpha}_{2n}, 3b^{\gamma}_{2n-1}, a^{\beta}_{2n-1}, 2b^{\gamma}_{2n-1}, 2a^{\beta}_{2n-1})_{360}$	18n-8	2	2n-2	
$D(2a^{\beta}_{2n-1}, b^{\gamma}_{2n-1}, b^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\beta}_{2n-1}, b^{\gamma}_{2n-1})_{361}$		2	2n	
$D(b^{\alpha}_{2n}, 3b^{\gamma}_{2n-1}, a^{\beta}_{2n-1}, 2b^{\gamma}_{2n-1}, 2a^{\beta}_{2n-1})_{359}$		1	6n-2	
$D(2a^{\beta}_{2n-1}, b^{\gamma}_{2n-1}, a^{\alpha}_{2n}, 2b^{\gamma}_{2n-1}, 2a^{\beta}_{2n-1}, b^{\gamma}_{2n-1})_{362}$ *		1	-2n	
$D(3b^{\gamma}_{2n-1}, 3a^{\beta}_{2n-1}, 3b^{\gamma}_{2n-1})_{351}$ *	18n-9	1	6n-3	

Table 1: ‘ $*$ ’ are used to indicate these link diagrams whose the lowest-degree terms of HOMFLY polynomials over z are given in table 2.

References

[1] L. Guo, H. Bai, J. Hao, S. Liu, A topological approach to assembling strands-based DNA triangular prisms, *MATCH Commun. Math. Comput. Chem.*, in press.

[2] W. J. Lough, I. W. Wainer, *Chirality in Natural and Applied Science*, CRC Press, Oxford, 2002.

[3] M. Liu, L. Zhang, T. Wang, Supramolecular chirality in self-assembled systems, *Chem. Rev.* **115** (2015) 7304–7397.

[4] N. C. Seeman, DNA in a material world, *Nature* **421** (2003) 427–431.

[5] A. Travers, G. Muskhelishvili, DNA structure and function, *FEBS J.* **282** (2015) 2279-2295.

[6] F. A. Aldaye, A. L. Palmer, H. F. Sleiman, Assembling materials with DNA as the guide, *Nature* **321** (2008) 1795–1799.

[7] Y. Li, C. Mao, Z. Deng, Supramolecular wireframe DNA polyhedra: assembly and applications, *Chin. J. Chem.* **35** (2017) 801–810.

[8] R. P. Goodman, R. M. Berry, A. J. Turberfield, The single-steps synthesis of a DNA tetrahedron, *Chem. Commun.* **12** (2004) 1372–1373.

[9] F. A. Aldaye, H. F. Sleiman, Modular access to structurally switchable 3D discrete DNA assemblies, *J. Am. Chem. Soc.* **129** (2007) 13376–13377.

[10] Z. Li, B. Wei, J. Nangreave, C. Lin, Y. Liu, Y. Mi, H. Yan, A replicable tetrahedral nanostructure self-assembled from a single DNA strand, *J. Am. Chem. Soc.* **131** (2009) 13093–13098.

[11] Y. He, T. Ye, M. Su, C. Zhang, A. E. Ribbe, W. Jiang, C. D. Mao, Hierarchical self-assembly of DNA into symmetric supramolecular polyhedral, *Nature* **452** (2008) 198–201.

[12] J. Chen, N. C. Seeman, The electroophoretic properties of a DNA cube and its substructure catenanes, *Nature* **350** (1991) 631–633.

[13] C. Zhang, S. H. Ko, M. Su, Y. J. Leng, A. E. Ribbe, W. Jiang, C. D. Mao, Symmetry controls the face geometry of DNA polyhedra, *J. Am. Chem. Soc.* **131** (2009) 1413–1415.

[14] W. M. Shih, J. D. Quispe, G. F. Joyce, A 1.7-kilobase single-stranded DNA that folds into a nanoscale octahedron, *Nature* **427** (2004) 618–621.

[15] J. Zimmermann, M. P. J. Cebulla, S. Möninghoff, G. V. Kiedrowski, Self-assembly of a DNA dodecahedron from 20 trisoligonucleotides with C3h linkers, *Angew. Chem. Int. Ed.* **47** (2008) 3626–3630.

[16] C. Zhang, M. Su, Y. He, X. Zhao, P. A. Fang, A. E. Ribbe, W. Jiang, C. D. Mao, Conformational flexibility facilitates self-assembly of complex DNA nanostructures, *Proc. Natl. Acad. Sci. U. S. A.* **105** (2008) 10665–10669.

[17] D. Bhatia, S. Mehtab, R. Krishnan, S. S. Indi, A. Basu, Y. Krishnan, Icosahedral DNA nanocapsules by modular assembly, *Angew. Chem. Int. Ed.* **48** (2009) 4134–4137.

[18] E. Flapan, *When Topology Meets Chemistry: A Topological Look at Molecular Chirality*, Cambridge Univ. Press, Cambridge, 2000.

[19] R. P. Goodman, I. A. T. Schaap, C. F. Tardin, C. M. Erben, R. M. Berry, C. F. Schmidt, A. J. Turberfield, Rapid chiral assembly of rigid DNA building blocks for molecular nanofabrication, *Science* **310** (2005) 1661–1665.

[20] Y. He, M. Su, P. Fang, C. Zhang, A. E. Ribbe, W. Jiang, C. Mao, On the chirality of self-assembled DNA octahedra, *Angew. Chem. Int. Ed.* **49** (2010) 748–751.

[21] C. Zhang, W. Wu, X. Li, C. Tian, H. Qian, G. Wang, W. Jiang, C. Mao, Controlling the chirality of DNA nanocages, *Angew. Chem. Int. Ed.* **51** (2012) 7999–8002.

[22] Z. Nie, X. Li, Y. Li, C. Tian, P. Wang, C. Mao, Self-assembly of DNA nanoprisms with only two component strands, *Chem. Commun.* **49** (2013) 2807–2809.

[23] X. He, L. Dong, W. Wang, N. Lin, Y. Mi, Folding single-stranded DNA to form the smallest 3D DNA triangular prism, *Chem. Commun.* **49** (2013) 2906–2908.

[24] S. Liu, L. Guo, H. Bai, J. Hao, A topological approach to assembling strands-based DNA tetrahedra, *MATCH Commun. Math. Comput. Chem.* **81** (2019) 209–244.

[25] W. Qiu, Z. Wang, G. Hu, *The Chemistry and Mathematics of DNA Polyhedra*, Nova, New York, 2010.

[26] X. Jin, A survey on several invariants of three types of polyhedral links, *MATCH Commun. Math. Comput. Chem.* **76** (2016) 569–594.

[27] M. Li, Q. Deng, X. Jin, A general method for computing the Homfly polynomial of DNA double crossover 3–regular links, *PloS One* **10** (2015) #e0125184.

[28] G. Hu, X. Zhai, D. Lu, W. Qiu, The architecture of Platonic polyhedral links, *J. Math. Chem.* **42** (2009) 592–603.

[29] X. Jin, X. Cheng, Topological chirality of a type of DNA and protein polyhedral links, *J. Math. Chem.* **53** (2015) 1791–1807.

[30] J. Duan, W. Qiu, Using dual polyhedral links models to predict characters of DNA polyhedra, *J. Mol. Struct.* **359** (2013) 233–236.

[31] S. Liu, H. Zhang, Some invariants of polyhedral links, *MATCH Commun. Math. Comput. Chem.* **70** (2013) 383–400.

[32] X. Jin, F. Zhang, The Jones polynomial for polyhedral links, *MATCH Commun. Math. Comput. Chem.* **65** (2011) 501–520.

[33] X. Jin, F. Zhang, The Homfly polynomial for even polyhedral links,*MATCH Commun. Math. Comput. Chem.* **63** (2010) 657–677.

[34] S. Liu, X. Cheng, H. Zhang, W. Qiu, The architecture of polyhedral links and their Homfly polynomials, *J. Math. Chem.* **48** (2010) 439–456.

[35] S. Jablan, L. Radović, R. Sazdanović, Knots and links derived from prismatic graphs, *MATCH Commun. Math. Comput. Chem.* **66** (2010) 65–92.

[36] S. Liu, H. Zhang, W. Qiu, The Homfly polynomial for a family of polyhedral links, *MATCH Commun. Math. Comput. Chem.* **67** (2012) 65–90.

[37] S. Liu, H. Zhang, The HOMFLY polynomials of odd polyhedral links, *J. Math. Chem.* **51** (2013) 1310—-1328.

[38] P. R. Cromwell, *Knots and Links*, Cambridge Univ. Press, Cambridge, 2004.

[39] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu, A new polynomial invariant of knots and links, *Bull. Am. Math. Soc.* **12** (1985) 239–146

[40] F. Jaeger, A combinatorial model for the Homfly polynomial, *Eur. J. Comb.* **11** (1990) 549–558.

[41] C. Cerf, Nullification writhe and chirality of alternating links, *J. Knot Theory Ramif.* **6** (1997) 621–632.

[42] L. H. Kauffman, State models and the Jones polynomial, *Topology* **26** (1987) 395–407.