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Steady–State Concentrations of Carbon Dioxide Absorbed into Phenyl Glycidyl Ether: An Optimal Homotopy Analysis Method

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Abstract

In this work, we extend our work in [1] for solving *the nonlinear coupled boundary value problems* that relates the concentrations of carbon dioxide CO_2 and phenyl glycidyl ether in solution. We first transform the coupled boundary value problems into an equivalent integral equations. We then apply the optimal homotopy analysis method for obtaining approximations to the solutions. The present method gives a rapidly convergent, easily computable, and readily verifiable sequence of analytic approximate solutions that is suitable for numerical parametric simulations. For speed up the calculations, we use the discrete averaged residual error to obtain optimal value of the adjustable parameters. The numerical results show that the optimal homotopy analysis method gives reliable algorithm for analytic approximate solutions of these systems. The error analysis of the sequence of the analytic approximate solutions has been performed by computing the residual error functions and the maximal residual error parameters, which demonstrate an approximate exponential rate of convergence.

1 Introduction

In practical life, there are many phenomena in chemistry, mechanics, biology, physics, chemical engineering and fluid dynamics can be represented by either linear or nonlinear

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differential equations. In Chemistry for example, the condensations of carbon dioxide and phenyl glycidyl ether and chemical kinetics problem are represented by systems of nonlinear ordinary differential equations.

Carbon dioxide is an important in the manufacturing of carbonated soft drinks, the powering of pneumatic systems in robots, plant photosynthesis, used in fire extinguishers, removing caffeine from coffee see details [2–4]. Carbon dioxide is a useful gas which is made of two oxygen atoms and one carbon atom [2]. Most recent, the chemical fixation of carbon dioxide has become a very crucial research area, due to the danger posed by global warming and that the conversion of carbon dioxide into valuable substances is an extremely attractive solution [3]. The kinetics of the reaction between CO_2 and phenyl glycidyl ether (PGE) in solution has attracted much interest. In [5, 6], they have investigated the chemical absorption of carbon dioxide into PGE solutions containing the catalyst $THACPMS41$ in a heterogeneous system. Several methods have been used to solve the system of condensations of carbon dioxide and phenyl glycidyl ether and obtained analytical approximate solutions such as Adomian Decomposition Method [2, 3], the variational iteration method [7], the iterative method [4] and TAM method [8] and the references cited therein.

We consider the system of nonlinear differential equations which arising in the study of the steady-state concentrations of CO_2 and PGE as

$$\begin{cases} u_i''(x) = \dfrac{\alpha_i u_1(x) u_2(x)}{1 + \beta_1 u_1(x) + \beta_2 u_2(x)}, \quad i = 1, 2, \quad \text{where } '' \equiv \dfrac{d^2}{dx^2} \\ u_1(0) = 1, \quad u_1(1) = k, \\ u_2'(0) = 0, \quad u_2(1) = 1. \end{cases} \tag{1.1}$$

Here $u_1(x)$ and $u_2(x)$ are denote the concentrations of CO_2 and PGE, respectively. The constants $\alpha_i,\ \beta_i, i = 1, 2$ are normalized parameters, x is the dimensionless distance as measured from the center, and k is the dimensionless concentration of CO_2 at the surface of the catalyst [3].

In this paper, we proposed a semi-numerical algorithm based on the optimal homotopy analysis method (OHAM) with Green's function to solve the system of nonlinear differential equations with boundary value problems in that relates the steady-state concentrations. We will show that using the integral form facilitates the computational work. The error analysis will be performed by using the residual error functions and the maximal error residual parameters, which demonstrate an approximate exponential rate of convergence.

2 Integral form of Lane-Emden equations and OHAM

We consider a general Lane-Emden equation with boundary conditions as

$$\begin{cases} u_i''(x) = f_i(u_1(x), u_2(x)) = 0, \qquad i = 1, 2 \\ u_1(0) = 1, \quad u_1(1) = k, \\ u_2'(0) = 0, \quad u_2(1) = 1, \end{cases} \tag{2.1}$$

Setting $f_i(u_1(x), u_2(x)) = \dfrac{\alpha_i u_1(x) u_2(x)}{1 + \beta_1 u_1(x) + \beta_2 u_2(x)}$, where $\alpha_i, \beta_i, i = 1, 2$ we recover the original model (1.1).

Following Singh et al. [9, 10], we transform the system of boundary value problems (2.1) into an equivalent integral equations as

$$u_i(x) = g_i(x) + \int_0^1 K_i(x, s) f_i(u_1(x), u_2(x)) ds, \qquad i = 1, 2 \tag{2.2}$$

where $K_i(x, s)$ and $g_i(x)$ are given below:

$$g_1(x) = 1 + (k-1)x, \quad K_1(x, s) = \begin{cases} x(1-s), & x \le s, \\ s(1-x), & s \le x, \end{cases} \tag{2.3}$$

and

$$g_2(x) = 1, \quad K_2(x, s) = \begin{cases} (s-1), & x \le s, \\ (x-1), & s \le x. \end{cases} \tag{2.4}$$

The derivation of the green's functions $K_1(x, s)$ and $K_2(x, s)$ given in Appendix-I and Appendix-II.

Basic idea of homotopy analysis method for solving different scientific models can be found in [11, 12]. According to HAM [13], the zero-order deformation equation may be written as

$$(1-p)[\phi_i(x, p) - u_{i0}] = p h_{i0} T_i[\phi_i(x, p)], \qquad i = 1, 2, \tag{2.5}$$

where $p \in [0, 1]$ is an embedding parameter, u_{i0} are initial guesses, $h_{i0} \neq 0$ are convergence control parameters, $\phi_i(x, p)$ are unknown functions and $T_i[\phi_i(x, p)]$ are defined as

$$T_i[\phi_i(x, p)] = \phi_i(x, p) - g_i(x) - \int_0^1 K_i(x, s) f_i(\phi_1(s, p), \phi_2(s, p)) ds = 0, \quad i = 1, 2. \tag{2.6}$$

When $p=0$, the zero-order deformation equations (2.5) reduce into the form

$$\phi_i(x,0) = u_{i0}(x), \qquad i=1,2, \tag{2.7}$$

and when $p=1$, they lead to the following form

$$T_i[\phi_i(x,1)] = 0, \qquad i=1,2, \tag{2.8}$$

which is exactly the same as (2.2) provided that $\phi_i(x,1) = u_i(x)$. Thus, as the parameter p increasing form 0 to 1, $\phi_i(x,p)$ move from $u_{i0}(x)$ to $u_i(x)$.

We expand $\phi_i(x,p)$ in a Taylor series with respect to p to get

$$\phi_i(x,p) = u_{i0}(x) + \sum_{m=1}^{\infty} u_{im}(x)p^m, \qquad i=1,2, \tag{2.9}$$

where

$$u_{im}(x) = \frac{1}{m!}\frac{\partial^m \phi_i(x,p)}{\partial p^m}\bigg|_{p=0}, \qquad i=1,2. \tag{2.10}$$

If the convergence parameter $h_{i0} \neq 0$ are chosen properly, the series (2.9) converges for $p=1$ and it becomes

$$\phi_i(x,1) \equiv u_i(x) = \sum_{m=0}^{\infty} u_{im}(x), \qquad i=1,2, \tag{2.11}$$

which will be the solutions of the problem (2.2).

Define the vector

$$\overrightarrow{u}_{im} = \{u_{i0}(x), u_{i1}(x), \ldots, u_{im}(x)\}, \qquad i=1,2.$$

Differentiating (2.5) m-times with respect to p, dividing them by $m!$, setting subsequently $p=0$, the mth-order deformation equation is obtained

$$u_{im}(x) - \chi_m\, u_{i(m-1)}(x) = h_{i0}\, R_{im}(\overrightarrow{u}_{i(m-1)}, x), \qquad i=1,2 \tag{2.12}$$

where χ_m is given by

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \tag{2.13}$$

and

$$R_{im}(\overrightarrow{u}_{i(m-1)}, x) = \frac{1}{(m-1)!}\frac{\partial^{m-1} T_i[\phi_i(x,p)]}{\partial p^{m-1}}\bigg|_{p=0} = \frac{1}{(m-1)!}\frac{\partial^{m-1} T_i\big(\sum_{k=0}^{\infty} u_{ik}p^k\big)}{\partial p^{m-1}}\bigg|_{p=0}$$
$$= u_{i(m-1)}(x) - (1-\chi_m)g_i(x) - \int_0^1 K_i(x,s)\, D_{i(m-1)} ds, \qquad i=1,2.$$

Thus we have

$$R_{im}(\overrightarrow{u}_{i(m-1)}, x) = u_{i(m-1)}(x) - (1-\chi_m)g_i(x) - \int_0^1 K_i(x,s)\ D_{i(m-1)}ds, \quad i = 1,2 \quad (2.14)$$

where D_{im} are given by

$$D_{i(m)} = \frac{1}{(m)!}\frac{\partial^m}{\partial q^m} f\left(\sum_{k=0}^{\infty} u_{1k}q^k, \sum_{k=0}^{\infty} u_{2k}q^k\right)\Bigg|_{q=0}, \qquad i = 1,2. \quad (2.15)$$

Using (2.12) and (2.14), the mth-order deformation equations are simplified as

$$u_{im} - \chi_m u_{i(m-1)} = h_{i0}\left(u_{i(m-1)} - (1-\chi_m)g_i(x) - \int_0^1 K_i(x,s)\ D_{i(m-1)}ds\right),\ i = 1,2. \quad (2.16)$$

Taking $u_{i0} = g_i(x),\ i = 1,2,$ the solution components will be computed as:

$$\begin{cases} u_{i1}(x) = h_{i0}\left(u_{i0}(x) - g_i(x) - \int_0^1 K_i(x,s)D_{i0}ds\right), \quad i = 1,2 \\ \vdots \\ u_{im}(x) = (1+h_{i0})\ u_{i(m-1)}(x) - h_{i0}\int_0^1 K_i(x,s)D_{i(m-1)}ds,\ m \geq 2,\ i = 1,2. \end{cases} \quad (2.17)$$

The nth-order approximate solutions of the problem (2.2) are given by

$$\phi_{in}(x, h_{i0}) = \sum_{m=0}^{n} u_{im}(x, h_{i0}), \quad i = 1,2. \quad (2.18)$$

We define the squared discrete averaged residual errors formula

$$E_{in}(h_{i0}) = \frac{1}{n}\sum_{j=1}^{n}\left(T_i[\phi_{in}(x_j, h_{i0})]\right)^2, \qquad i = 1,2, \quad (2.19)$$

where $x_j = jh$, $h = x_j - x_{j-1}$. The optimal values h_{i0} will be obtained by solving

$$\frac{\partial E_{in}}{\partial h_{i0}} = 0, \qquad i = 1,2 \quad (2.20)$$

and then those values will be substituted (2.18), to get the optimal approximate solutions.

3 Numerical simulations

We fix the parameters α_1, α_2, β_1, β_2 and k, then calculate the approximate solutions, the absolute residual error functions, and the maximum absolute residual errors. To examine the accuracy and applicability of the OHAM, we define the residual and maximum absolute residual errors as

$$Res_{in}(x) = \left|\phi''_{in}(x) - \frac{\alpha_i \phi_{1n}(x)\phi_{2n}(x)}{1+\beta_1\phi_{1n}(x)+\beta_2\phi_{2n}(x)}\right|, \qquad R_{in} = \max_{0\le x\le 1} Res_{in}(x), \tag{3.1}$$

$$res_{in}(x) = \left|\psi''_{in}(x) - \frac{\alpha_i \psi_{1n}(x)\psi_{2n}(x)}{1+\beta_1\psi_{1n}(x)+\beta_2\psi_{2n}(x)}\right|, \qquad r_{in} = \max_{0\le x\le 1} res_{in}(x), \quad i=1,2, \tag{3.2}$$

where ϕ_{in} are ψ_{in}, the optimal homotopy analysis method solutions and are the Adomian decomposition method solutions, respectively.

According to the optimal homotopy analysis method, we have the following iteration formulation for considered problems as

$$\begin{cases} u_{1m} - \chi_m u_{1(m-1)} = h_{10}\left\{u_{1(m-1)} - (1-\chi_m)g_1(x)\int\limits_0^1 K_1(x,s)D_{1(m-1)}ds\right\}, \\ u_{2m} - \chi_m u_{2(m-1)} = h_{20}\left\{u_{2(m-1)} - (1-\chi_m)g_2(x) - \int\limits_0^1 K_2(x,s)D_{2(m-1)}ds\right\}, \end{cases} \tag{3.3}$$

where $g_i(x)$ and $K_i(x,s), i=1,2$ are given in equations (2.3) and (2.4).

The numerical results of approximate solutions ($\phi_{i4}(x)$ and $\psi_{i4}(x)$), $i = 1,2$ and the absolute residual errors ($Res_{i4}(x)$ and $res_{i4}(x)$), $i = 1,2$) obtained by the the optimal homotopy analysis method and the Adomian decomposition method are given in Table 1, Table 2 and Table 3. The maximum absolute residual errors ($R_{i4}, r_{i4},\ i = 1,2$) obtained by the the optimal homotopy analysis method and the Adomian decomposition method are given in Tables 4, 5 and 6. From the tables, we observe that the the optimal homotopy analysis method gives stable and convergent solution. We plot the 4th-order approximations to the solutions OHAM $\phi_{14}(x)$, ADM $\psi_{14}(x)$, OHAM $\phi_{24}(x)$, and ADM $\psi_{24}(x)$, for different parameters $\alpha_1, \alpha_2,\ \beta_1,\ \beta_2$ and k in Figures 1 through 6.

Table 1 The OHAM and ADM approximations and residual errors for $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 1, \beta_2 = 3, k = 0.5$

x	ϕ_{14}	ψ_{14}	ϕ_{24}	ψ_{24}	Res_{24}	res_{24}	Res_{24}	res_{24}
0.0	1.0000000	1.0000000	0.8397515	0.8396746	2.85E-04	3.15E-04	6.42E-04	6.31E-04
0.1	0.9428972	0.9428979	0.8415794	0.8415027	8.72E-05	1.30E-04	3.30E-04	2.60E-04
0.2	0.8875699	0.8875704	0.8469568	0.8468806	9.38E-05	4.90E-05	3.21E-05	9.80E-05
0.3	0.8339414	0.8339413	0.8557293	0.8556549	2.64E-04	2.27E-04	2.70E-04	4.55E-04
0.4	0.7819371	0.7819361	0.8677478	0.8676770	4.27E-04	4.08E-04	5.85E-04	8.17E-04
0.5	0.7314845	0.7314823	0.8828670	0.8828019	5.79E-04	5.87E-04	9.10E-04	1.17E-03
0.6	0.6825121	0.6825087	0.9009442	0.9008874	7.10E-04	7.54E-04	1.23E-03	1.50E-03
0.7	0.6349490	0.6349449	0.9218381	0.9217922	8.05E-04	8.93E-04	1.52E-03	1.78E-03
0.8	0.5887240	0.5887200	0.9454074	0.9453750	8.43E-04	9.81E-04	1.74E-03	1.96E-03
0.9	0.5437652	0.5437627	0.9715097	0.9714928	7.97E-04	9.91E-04	1.84E-03	1.98E-03
1.0	0.4999991	0.5000000	1.0000000	1.0000000	6.32E-04	8.88E-04	1.76E-03	1.77E-03

Table 2 The OHAM and ADM approximations and residual errors for $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 2, \beta_2 = 4, k = 2$

x	ϕ_{14}	ψ_{14}	ϕ_{24}	ψ_{24}	Res_{24}	res_{24}	Res_{24}	res_{24}
0.0	1.0000000	1.0000000	0.8397570	0.8406713	3.44E-05	2.90E-04	1.40E-04	5.80E-04
0.1	1.0924330	1.0927067	0.8411048	0.8420177	3.71E-05	1.89E-04	1.48E-04	3.79E-04
0.2	1.1862694	1.1868154	0.8452601	0.8461680	1.31E-04	1.53E-04	5.23E-04	3.06E-04
0.3	1.2815920	1.2824085	0.8523901	0.8532870	2.65E-04	2.23E-04	8.95E-04	4.47E-04
0.4	1.3784839	1.3795674	0.8626625	0.8635377	3.88E-04	4.40E-04	1.17E-03	8.80E-04
0.5	1.4770298	1.4783735	0.8762466	0.8770828	4.53E-04	8.39E-04	1.29E-03	1.67E-03
0.6	1.5773154	1.5789083	0.8933144	0.8940852	4.16E-04	1.45E-03	1.18E-03	2.90E-03
0.7	1.6794288	1.6812539	0.9140409	0.9147092	2.39E-04	2.30E-03	7.68E-04	4.60E-03
0.8	1.7834600	1.7854938	0.9386061	0.9391218	1.11E-04	3.41E-03	5.94E-06	6.83E-03
0.9	1.8895020	1.8917132	0.9671952	0.9674937	6.66E-04	4.80E-03	1.14E-03	9.61E-03
1.0	1.9976509	2.0000000	1.0000000	1.0000000	1.44E-03	6.47E-03	2.72E-03	1.29E-02

Table 3 The OHAM and ADM approximations and residual errors for $\alpha_1 = 2, \alpha_2 = 3, \beta_1 = 1, \beta_2 = 3, k = 3$

x	ϕ_{14}	ψ_{14}	ϕ_{24}	ψ_{24}	Res_{24}	res_{24}	Res_{24}	res_{24}
0.0	1.0000000	1.0000000	0.6393384	0.6482133	2.41E-03	4.81E-03	8.19E-04	7.22E-03
0.1	1.1773922	1.1782417	0.6419059	0.6507546	6.58E-03	2.84E-03	6.82E-03	4.26E-03
0.2	1.3585337	1.3601665	0.6500667	0.6588204	1.07E-02	1.70E-03	1.24E-02	2.55E-03
0.3	1.5438814	1.5462063	0.6645009	0.6730588	1.40E-02	1.96E-03	1.67E-02	2.94E-03
0.4	1.7338918	1.7367854	0.6858864	0.6941062	1.59E-02	4.13E-03	1.87E-02	6.20E-03
0.5	1.9290268	1.9323267	0.7149070	0.7225968	1.59E-02	8.64E-03	1.77E-02	1.29E-02
0.6	2.1297584	2.1332581	0.7522611	0.7591726	1.34E-02	1.57E-02	1.30E-02	2.36E-02
0.7	2.3365749	2.3400194	0.7986711	0.8044931	8.14E-03	2.57E-02	4.09E-03	3.86E-02
0.8	2.5499857	2.5530689	0.8548915	0.8592460	1.77E-04	3.87E-02	9.28E-03	5.80E-02
0.9	2.7705278	2.7728903	0.9217187	0.9241569	1.16E-02	5.45E-02	2.72E-02	8.18E-02
1.0	2.9987708	3.0000000	1.0000000	1.0000000	2.63E-02	7.31E-02	4.98E-02	1.09E-01

Table 4 The maximum absolute residual errors when $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 1, \beta_2 = 3, k = 1/2$

M	R_{1M}	r_{1M}	R_{2M}	r_{2M}
2	2.29E-02	8.89E-02	5.10E-02	1.77E-01
3	4.64E-03	4.36E-03	8.12E-03	8.71E-03
4	8.43E-04	9.99E-04	1.84E-03	1.99E-03
5	1.89E-05	3.53E-05	1.80E-05	7.07E-05
6	3.84E-07	3.38E-06	2.65E-06	6.77E-06

Table 5 The maximum absolute residual errors when $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 2, \beta_2 = 4, k = 2$

M	R_{1M}	r_{1M}	R_{2M}	r_{2M}
2	5.95E-02	7.93E-02	1.10E-01	1.58E-01
3	1.96E-03	2.26E-02	3.91E-03	4.53E-02
4	4.55E-04	6.47E-03	1.29E-03	1.29E-02
5	9.45E-05	1.85E-03	1.77E-05	3.70E-03
6	2.92E-05	5.28E-04	4.34E-05	1.05E-03

Table 6 The maximum absolute residual errors when $\alpha_1 = 2, \alpha_2 = 3, \beta_1 = 1, \beta_2 = 3, k = 3$

M	R_{1M}	r_{1M}	R_{2M}	r_{2M}
2	4.22E-01	4.57E-01	5.57E-01	6.85E-01
3	2.53E-02	1.88E-01	3.32E-01	2.82E-01
4	1.62E-02	7.31E-02	1.88E-02	1.09E-01
5	1.66E-03	4.73E-03	1.04E-03	7.11E-03
6	2.08E-04	1.70E-03	3.47E-04	4.55E-03

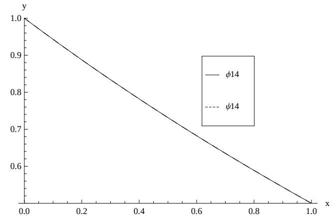

Figure 1 The OHAM and ADM solutions for $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 1, \beta_2 = 3, k = 1/2$

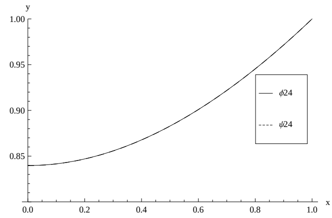

Figure 2 The OHAM and ADM solutions for $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 1, \beta_2 = 3, k = 1/2$

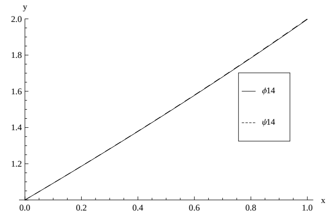

Figure 3 The OHAM and ADM solutions for $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 2, \beta_2 = 4, k = 2$

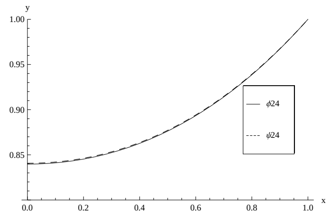

Figure 4 The OHAM and ADM solutions for $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 2, \beta_2 = 4, k = 2$

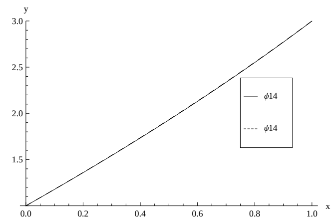

Figure 5 The OHAM and ADM solutions for $\alpha_1 = 2, \alpha_2 = 3, \beta_1 = 1, \beta_2 = 3, k = 3$

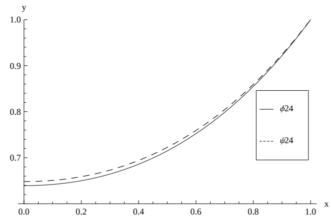

Figure 6 The OHAM and ADM solutions for $\alpha_1 = 2, \alpha_2 = 3, \beta_1 = 1, \beta_2 = 3, k = 3$

Case-I:- $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 1, \beta_2 = 3, k = 1/2$

Using (3.3) with initial guesses $u_{10} = 1, u_{20} = 1$, we obtain 4th-order approximations $\phi_{14}(x, h_{10}, h_{20})$ and $\phi_{24}(x, h_{10}, h_{20})$. Applying (2.19) and (2.20), we obtain optimal values $h_{10} = -1.01203, h_{20} = -0.97988$ and hence the optimal homotopy analysis approximations to the solutions are obtained as

$$\phi_{14}(x) = 1 - 0.58016x + 0.09276x^2 - 0.01364x^3 + 0.00162x^4 - 0.00059x^5 + 0.000014x^6.$$

$$\phi_{24}(x) = 0.83975 + 0.18549x^2 - 0.02743x^3 + 0.00331x^4 - 0.00114x^5 + 0.000027x^6.$$

and by setting $h_{10} = h_{20} = -1$, the Adomian decomposition method approximations to the solutions are obtained as

$$\psi_{14}(x) = 1 - 0.58016x + 0.09274x^2 - 0.01367x^3 + 0.00166x^4 - 0.00059x^5 + 0.000012x^6.$$
$$\psi_{24}(x) = 0.83967 + 0.18549x^2 - 0.02734x^3 + 0.00333x^4 - 0.00118x^5 + 0.000024x^6.$$

Case-II:- $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 2, \beta_2 = 4, k = 2$

Using (3.3) with initial guesses $u_{10} = 1, u_{20} = 1$, we obtain 4th-order approximations $\phi_{14}(x, h_{10}, h_{20})$ and $\phi_{24}(x, h_{10}, h_{20})$.Applying (2.19) and (2.20), we obtain optimal values $h_{10} = -0.867067, h_{20} = -1.03525$ and hence the optimal homotopy analysis approximations to the solutions are obtained as

$$\phi_{14}(x) = 1 + 0.91758x + 0.0660x^2 + 0.0137x^3 - 0.000073x^4 + 0.00034x^5 + 3.73 \times 10^{-7}x^6$$
$$\phi_{24}(x) = 0.83975 + 0.13198x^2 + 0.02804x^3 - 0.000621x^4 + 0.00083071x^5 + 8.91 \times 10^{-7}x^6$$

and by setting $h_{10} = h_{20} = -1$, the Adomian decomposition method approximations to the solutions are obtained as

$$\begin{aligned}\psi_{14}(x) &= 1 + 0.92033x + 0.065917x^2 + 0.014092x^3 - 0.000778x^4 + 0.000428x^5 \\ &\quad + 4.049 \times 10^{-6}x^6 \\ \psi_{24}(x) &= 0.840671 + 0.13183x^2 + 0.02818x^3 - 0.00155x^4 + 0.000857x^5 + 8.098 \times 10^{-6}x^6\end{aligned}$$

Case-III:- $\alpha_1 = 2, \alpha_2 = 3, \beta_1 = 1, \beta_2 = 3, k = 3$

Using (3.3) with initial guesses $u_{10} = 1, u_{20} = 1$, we obtain 4th-order approximations $\phi_{14}(x, h_{10}, h_{20})$ and $\phi_{24}(x, h_{10}, h_{20})$. Applying (2.19) and (2.20), we obtain optimal values $h_{10} = -0.91498, h_{20} = -0.982747$ and hence the optimal homotopy analysis approximations to the solutions are obtained as

$$\begin{aligned}\phi_{14}(x) &= 1 + 1.75671x + 0.164388x^2 + 0.0777417x^3 - 0.00445005x^4 + 0.00426919x^5 \\ &\quad + 0.000111803x^6. \\ \phi_{24}(x) &= 0.639338 + 0.245178x^2 + 0.116475x^3 - 0.00804963x^4 + 0.00687807x^5 \\ &\quad + 0.000180125x^6.\end{aligned}$$

and by setting $h_{10} = h_{20} = -1$, the Adomian decomposition method approximations to the solutions are obtained as

$$\psi_{14}(x) = 1 + 1.76548x + 0.16192x^2 + 0.0758222x^3 - 0.00850667x^4 + 0.005136x^5 + 0.000152889x^6.$$

$$\psi_{24}(x) = 0.648213 + 0.24288x^2 + 0.113733x^3 - 0.01276x^4 + 0.007704x^5 + 0.000229333x^6.$$

4 Conclusion

We have examined a system of nonlinear differential equations with boundary conditions, that relates the steady-state concentrations of carbon dioxide and PGE in solution. The proposed approach depends mainly on combining the optimal homotopy analysis method combined with the Green's function strategy. Our approach generated a rapidly convergent approximations of the concentrations of carbon dioxide and PGE to a high degree of accuracy. The evaluated approximations show enhancements over existing techniques. The obtained results were supported by proper figures to show the power of the method and to show the enhancements over exiting techniques.

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