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Conjectures on Resolvent Energy of Graphs

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Abstract

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of a simple graph G of order n. A graph-spectrum-based invariant, put forward by Gutman et al. [Resolvent energy of graphs, *MATCH Commun. Math. Comput. Chem.* **75** (2016) 279–290], is defined as $ER(G) = \sum_{i=1}^{n} (n - \lambda_i)^{-1}$. In the same paper the authors proposed several conjectures. In this paper we partially prove two conjectures.

1 Introduction

Let $G = (V, E)$ be a simple graph of order n with m edges, where $|V(G)| = n$ and $|E(G)| = m$. If the vertices v_i and v_j are adjacent, we write $v_iv_j \in E(G)$. The adjacency matrix $A = A(G)$ of the graph G is defined so that its (i, j)-entry is equal to 1 if $v_iv_j \in E(G)$ and 0 otherwise. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $A(G)$. When more than one graph is under consideration, then we write $\lambda_i(G)$ instead of λ_i. In what follows, the adjacency spectrum of the graph G, i.e., the multiset $\{\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)\}$ will be denoted by $S(G)$. If G has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ with multiplicities $k_1, k_2, \ldots, k_r$ respectively, we shall write $\left\{\lambda_1^{(k_1)}, \lambda_2^{(k_2)}, \ldots, \lambda_r^{(k_r)}\right\}$ for the spectrum of G (We often omit those k_i equal to 1).

Recently, Gutman et al. introduced the resolvent energy [8], and it is defined by

$$ER(G) = \sum_{i=1}^{n} \frac{1}{n - \lambda_i}. \tag{1}$$

For its basic mathematical properties, including various lower and upper bounds, see [1, 5–9] and the references therein. Also, its Laplacian spectrum version was recently put forward [2].

The k-th spectral moment of the graph G is defined as

$$M_k = M_k(G) = \sum_{i=1}^{n} \lambda_i^k. \tag{2}$$

As usual, P_n, C_n, and S_n denote, respectively, the path, the cycle, and the star graph on n vertices. Let P_n^* be a tree of order n obtained from a path $P_{n-1} : v_1 v_2 \cdots v_{n-2} v_{n-1}$ by attaching a new pendant edge $v_{n-2}v_n$ at v_{n-2}. A tree is called a double star $DS_{p,\,q}$ ($p \geq q \geq 1$, $p + q + 2 = n$) if it is obtained from S_{p+1} and S_{q+1} by connecting the center of S_{p+1} with that of S_{q+1} via an edge. Let S_n^* be a tree of order n with maximum degree $n-2$. In particular, $S_n^* \cong DS_{n-3,\,1}$ (The symbol $\cong$ means 'is isomorphic to'). Since $S(S_n) = \left\{\sqrt{n-1},\, 0^{(n-2)}, -\sqrt{n-1}\right\}$, we have

$$ER(S_n) = \frac{2n}{n^2 - n + 1} + \frac{n-2}{n}. \tag{3}$$

Gutman et al. [8] mentioned the following two conjectures:

Conjecture 1. *Among trees of order n, the tree P_n^* has second smallest and the tree S_n^* second-greatest resolvent energy.*

Conjecture 2. *The inequality $ER(S_n) < ER(C_n)$ holds for all $n \geq 4$. Consequently, any tree has smaller ER-value than any unicyclic graph of the same order.*

Farrugia discussed about the increase in the resolvent energy of a graph due to the addition of a new edge in [6]. In [1], Allem et al. presented some results on the extremal resolvent energy of unicyclic graphs, bicyclic graphs and tricyclic graphs. Ghebleh et al. [7] proved that the tree $P_{n-1}(a)$ has the a-th smallest resolvent energy ($P_{n-1}(a)$ is a tree obtained by attaching a pendant vertex at position a of the $(n-1)$-vertex path P_{n-1}). So, one part of the Conjecture 1 has been proved in [7], and here we confirm the remaining part of this conjecture. That is, $ER(T) < ER(S_n^*) < ER(S_n)$ for any tree T ($\ncong S_n,\, S_n^*$). Moreover we give lower bounds on $ER(C_n)$ in terms of n and it is better than the previous lower bound given by Du [5]. Finally we prove that $ER(C_n) > ER(S_n)$ for even n.

2 On Conjecture 1

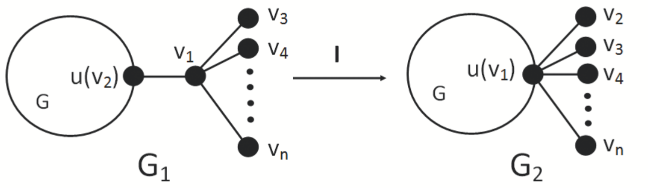

Figure 1. Transformation I.

In [4], Deng obtained the following result:

Lemma 3. [4] *Let u be a non-isolated vertex of a simple graph G. If G_1 and G_2 are the graphs obtained from G by identifying a leaf v_2 and the center v_1 of the n-vertex star S_n to u, respectively, depicted in Fig. 1, then $M_{2k}(G_1) < M_{2k}(G_2)$ for $n \geq 3$ and $k \geq 2$.*

Lemma 4. *Let $DS_{p,q}$ $(p \geq q \geq 1,\ p+q+2=n)$ be a double star. Then the spectrum of $DS_{p,q}$ is the following:*

$$S(DS_{p,q}) = \left\{ \pm\sqrt{\frac{p+q+1+\sqrt{(p-q)^2+2(p+q)+1}}{2}},\ \underbrace{0,\ 0,\ldots,\ 0}_{n-4}, \right.$$

$$\left. \pm\sqrt{\frac{p+q+1-\sqrt{(p-q)^2+2(p+q)+1}}{2}}, \right\}.$$

Proof. One can easily see that 0 is an eigenvalue of multiplicity $n-4$ and the remaining eigenvalues of $DS_{p,q}$ satisfy the following equations:

$$\lambda x_1 = px_3 + x_2,\ \lambda x_2 = qx_4 + x_1,\ \lambda x_3 = x_1,\ \lambda x_4 = x_2,$$

that is,

$$\lambda^4 - (p+q+1)\lambda^2 + pq = 0,$$

that is,

$$\lambda = \pm\sqrt{\frac{p+q+1 \pm \sqrt{(p-q)^2+2(p+q)+1}}{2}}.$$

■

Lemma 5. *Let $DS_{p,q}$ $(p \geq q \geq 1,\ p+q+2=n)$ be a double star. Then*

$$ER(DS_{p,q}) < ER(DS_{p+1,q-1}) < \cdots < ER(DS_{p+q-1,1}) = ER(S_n^*).$$

Proof. Let $a = n-1 = p+q+1$ and $b = \sqrt{(p-q)^2 + 2(p+q) + 1}$. By Lemma 4, the spectrum of $DS_{p,q}$ is the following:

$$S(DS_{p,q}) = \left\{ \pm\sqrt{\frac{a+b}{2}}, \pm\sqrt{\frac{a-b}{2}}, \underbrace{0, 0, \ldots, 0}_{n-4} \right\}.$$

Then

$$\begin{aligned} ER(DS_{p,q}) &= \frac{n-4}{n} + \frac{2n}{n^2 - \frac{a+b}{2}} + \frac{2n}{n^2 - \frac{a-b}{2}} \\ &= \frac{n-4}{n} + \frac{4n(n^2 - a/2)}{(n^2 - a/2)^2 - b^2/4} \\ &= \frac{n-4}{n} + \frac{4n\left(n^2 - \frac{n-1}{2}\right)}{(n^2 - \frac{n-1}{2})^2 - \frac{1}{4}\left[(p-q)^2 + 2(p+q) + 1\right]}. \end{aligned}$$

Since $p \geq q$, the difference $ER(DS_{p+1,q-1}) - ER(DS_{p,q})$ is positive by noting that, after cancelling $\frac{n-4}{n}$, the denominator of $ER(DS_{p+1,q-1})$ is smaller than that of $ER(DS_{p,q})$ but the numerators of $ER(DS_{p+1,q-1})$ and $ER(DS_{p,q})$ are equal. Thus we have

$$ER(DS_{p,q}) < ER(DS_{p+1,q-1}),$$

that is,

$$ER(DS_{p,q}) < ER(DS_{p+1,q-1}) < \cdots < ER(DS_{p+q-1,1}) = ER(S_n^*).$$

This completes the proof. ■

We are now ready to prove the remaining part of Conjecture 1.

Theorem 1. *Let T ($\ncong S_n, S_n^*$) be a tree of order n. Then*

$$ER(T) < ER(S_n^*) < ER(S_n).$$

Proof. If $T \cong DS_{p,q}$ ($p \geq q \geq 1$, $p+q+2 = n$), then $ER(T) = ER(DS_{p,q}) < ER(S_n^*) < ER(S_n)$ as $T \ncong S_n^*$. Otherwise, $T \ncong DS_{p,q}$ and $T \ncong S_n$. Repeating Transformation I as shown in Fig. 1, any n-vertex tree T can be changed into the n-vertex double star $DS_{p,q}$ ($T \cong T_1 \to T_2 \to \cdots \to T_k \cong DS_{p,q}$). By Lemma 3, we have

$$M_{2k}(T) = M_{2k}(T_1) < M_{2k}(T_2) < \cdots < M_{2k}(T_k) = M_{2k}(DS_{p,q}).$$

For tree T, we have $M_k(T) = 0$ for all odd values of k. It is well known that [8]:

$$ER(T) = \frac{1}{n}\sum_{k=0}^{\infty}\frac{M_k(T)}{n^k} = \frac{1}{n}\sum_{k=0}^{\infty}\frac{M_{2k}(T)}{n^{2k}}. \tag{4}$$

Using the above results, we have

$$ER(T) = ER(T_1) < ER(T_2) < \cdots < ER(T_k) = ER(DS_{p,q}).$$

By Lemma 3, we have $M_{2k}(S_n^*) < M_{2k}(S_n)$ and hence $ER(S_n^*) < ER(S_n)$, by (4). By Lemma 5, we conclude that

$$ER(T) = ER(T_1) < ER(T_2) < \cdots < ER(DS_{p,q}) < \cdots < ER(DS_{p+q-1,1}) =$$
$$ER(S_n^*) < ER(S_n).$$

This completes the proof of the theorem. ■

3 On Conjecture 2

In this section we prove $ER(S_n) < ER(C_n)$ holds for all $n \geq 4$ and n is even. Du [5] presented the following result:

Lemma 6. [5] *For $n \geq 3$,*

(i) if n is even, then

$$ER(C_n) \geq \frac{n}{\sqrt{n^2-4}} - \frac{4}{n^2-4},$$

(ii) if n is odd, then

$$ER(C_n) \geq \frac{n}{\sqrt{n^2-4}} - \frac{8}{n^2-4}.$$

Lemma 7. [3] *The adjacency spectrum of cycle C_n is*

$$2\cos\frac{2\pi j}{n},\ j = 0, 1, \ldots, n-1.$$

One can easily find the following trigonometric identity:

$$\sum_{k=0}^{n-1}\cos(a+kb) = \frac{\sin\left(\frac{nb}{2}\right)}{\sin\left(\frac{b}{2}\right)}\cos\left(a+(n-1)\frac{b}{2}\right). \tag{5}$$

It is very useful to prove our main result in this section.

We present a lower bound on $ER(C_n)$ for even n.

Theorem 2. *Let C_n be a cycle of order n (n is even). Then*

$$ER(C_n) > \begin{cases} \dfrac{2n}{n^2-4} + \dfrac{2}{n} + \dfrac{n\,(n-4)}{n^2-2} & \text{if } n = 4p, \\[2ex] \dfrac{2n}{n^2-4} + \dfrac{n\,(n-2)}{n^2-2+\frac{4}{n-2}} & \text{if } n = 4p+2, \end{cases}$$

where p is a positive integer.

Proof. Since n is even, we consider the following two cases:

Case (i) : $n = 4p$. By Lemma 7, the adjacency spectrum of cycle C_n is

$$S(C_n) = \left\{ \pm 2,\ 0^{(2)},\ \underbrace{\pm 2\cos\frac{2\pi}{n}^{(2)},\ \pm 2\cos\frac{4\pi}{n}^{(2)},\ldots,\ \pm 2\cos\frac{2(p-1)\pi}{n}^{(2)}}_{p-1} \right\}.$$

Using the arithmetic-harmonic-mean inequality, we have

$$\begin{aligned} ER(C_n) &= \sum_{i=1}^{n} \frac{1}{n-\lambda_i} \\ &= \frac{1}{n-2} + \frac{1}{n+2} + \frac{2}{n} + 2\sum_{i=1}^{p-1}\left[\frac{1}{n-2\cos\frac{2i\pi}{n}} + \frac{1}{n+2\cos\frac{2i\pi}{n}}\right] \\ &= \frac{2n}{n^2-4} + \frac{2}{n} + 2\sum_{i=1}^{p-1}\frac{2n}{n^2-4\cos^2\frac{2i\pi}{n}} \\ &\geq \frac{2n}{n^2-4} + \frac{2}{n} + \frac{4n\,(p-1)^2}{n^2(p-1) - 4\sum\limits_{i=1}^{p-1}\cos^2\frac{2i\pi}{n}} &(6) \\ &= \frac{2n}{n^2-4} + \frac{2}{n} + \frac{4n\,(p-1)^2}{n^2(p-1) - 2\sum\limits_{i=1}^{p-1}(1+\cos\frac{4i\pi}{n})}. &(7) \end{aligned}$$

By (5), we get

$$\sum_{i=1}^{p-1}\cos\frac{4i\pi}{n} = 0 \quad \text{as } n = 4p.$$

By the arithmetic-harmonic-mean inequality, one can easily see that the inequality in (6) is strict. Moreover, using the above result in (7), we have

$$ER(C_n) > \frac{2n}{n^2-4} + \frac{2}{n} + \frac{4n\,(p-1)^2}{n^2(p-1)-2(p-1)} = \frac{2n}{n^2-4} + \frac{2}{n} + \frac{n\,(n-4)}{n^2-2}.$$

Case (ii) : $n = 4p + 2$. By Lemma 7, the adjacency spectrum of cycle C_n is

$$S(C_n) = \left\{ \pm 2,\ \underbrace{\pm 2\cos\frac{2\pi}{n}^{(2)}, \pm 2\cos\frac{4\pi}{n}^{(2)}, \ldots, \pm 2\cos\frac{2p\pi}{n}^{(2)}}_{p} \right\}.$$

Similarly to the Case (i), using the arithmetic-harmonic-mean inequality, we have

$$\begin{aligned} ER(C_n) &= \frac{1}{n-2} + \frac{1}{n+2} + 2\sum_{i=1}^{p}\left[\frac{1}{n - 2\cos\frac{2i\pi}{n}} + \frac{1}{n + 2\cos\frac{2i\pi}{n}}\right] \\ &= \frac{2n}{n^2-4} + 2\sum_{i=1}^{p}\frac{2n}{n^2 - 4\cos^2\frac{2i\pi}{n}} \\ &\geq \frac{2n}{n^2-4} + \frac{4n\,p^2}{n^2p - 4\sum\limits_{i=1}^{p}\cos^2\frac{2i\pi}{n}} \qquad (8) \\ &= \frac{2n}{n^2-4} + \frac{4n\,p^2}{n^2p - 2\sum\limits_{i=1}^{p}(1+\cos\frac{4i\pi}{n})}. \qquad (9) \end{aligned}$$

By (5), we get

$$\sum_{i=1}^{p}\cos\frac{4i\pi}{n} = \frac{\sin\left(\frac{2p\pi}{n}\right)}{\sin\left(\frac{2\pi}{n}\right)}\cos\left(\frac{2p\pi}{n} + \frac{2\pi}{n}\right) = -\frac{\cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{2\pi}{n}\right)} = -\frac{1}{2} \quad \text{as } n = 4p+2.$$

By the arithmetic-harmonic-mean inequality, one can easily see that the inequality in (8) is strict. Moreover, using the above result in (9), we have

$$ER(C_n) > \frac{2n}{n^2-4} + \frac{4n\,p^2}{n^2p - 2p + 1} = \frac{2n}{n^2-4} + \frac{n\,(n-2)}{n^2 - 2 + \frac{4}{n-2}}.$$

■

Remark 8. *The bound in Theorem 2 is always better than the bound in Lemma 6 (i).*

Proof. One can easily obtain

$$\frac{n}{\sqrt{n^2-4}} = \left(1 - \frac{4}{n^2}\right)^{-1/2} < 1 + \frac{3}{n^2}. \qquad (10)$$

Now,

$$\frac{2}{n} + \frac{n\,(n-4)}{n^2-2} - \frac{n\,(n-2)}{n^2 - 2 + \frac{4}{n-2}} \quad = \quad \frac{2}{n} + \frac{n\,(n-4)}{n^2-2} - \frac{n\,(n-2)^2}{n^3 - 2n^2 - 2n + 8}$$

$$= \frac{8n-32}{(n^3-2n)(n^3-2n^2-2n+8)} \geq 0 \quad \text{as } n \geq 4.$$

From the above, we have

$$\frac{2n}{n^2-4} + \frac{2}{n} + \frac{n\,(n-4)}{n^2-2} \geq \frac{2n}{n^2-4} + \frac{n\,(n-2)}{n^2-2+\frac{4}{n-2}}. \tag{11}$$

Therefore we have to prove that

$$\frac{2n}{n^2-4} + \frac{n\,(n-2)}{n^2-2+\frac{4}{n-2}} > \frac{n}{\sqrt{n^2-4}} - \frac{4}{n^2-4}.$$

Using (10), we have to prove that

$$\frac{2n+4}{n^2-4} + \frac{n\,(n-2)^2}{n^3-2n^2-2n+8} > 1 + \frac{3}{n^2},$$

that is,

$$\frac{2n+4}{n^2-4} + \frac{-2n^2+6n-8}{n^3-2n^2-2n+8} > \frac{3}{n^2},$$

that is,

$$\frac{6n^3-12n^2-16n+64}{(n^2-4)(n^3-2n^2-2n+8)} > \frac{3}{n^2},$$

that is,

$$3n^5-6n^4+2n^3+16n^2-24n+96 > 0,$$

which is always true for $n \geq 4$. ■

We now obtain a lower bound on $ER(C_n)$ for odd n.

Theorem 3. *Let C_n be a cycle of order n (n is odd). Then*

$$ER(C_n) > \begin{cases} \dfrac{1}{n-2} + \dfrac{(n-1)\,(n^2-n+2)}{n\,(n^2-n+4)} & \text{if } n = 4p+1, \\[2ex] \dfrac{1}{n-2} + \dfrac{(n-3)\,(n^2-3n+2)}{n\,(n^2-3n+4)} + \dfrac{2n}{n^2+\pi} & \text{if } n = 4p+3, \end{cases}$$

where p is a non-negative integer.

Proof. If $p = 0$, then $n = 3$ and hence $E(C_3) > 1 + \frac{6}{9+\pi}$ holds. Otherwise, $p \geq 1$. Since n is odd, we consider the following two cases:

Case (i) : $n = 4p+1$. By Lemma 7, the adjacency spectrum of cycle C_n is

$$S(C_n) = \Bigg\{2,\ \underbrace{2\cos\frac{2\pi}{n}^{(2)},\ 2\cos\frac{4\pi}{n}^{(2)},\ \ldots,\ 2\cos\frac{2p\pi}{n}^{(2)}}_{p},$$

$$\underbrace{-2\cos\frac{\pi}{n}^{(2)}, -2\cos\frac{3\pi}{n}^{(2)}, \ldots, -2\cos\frac{(2p-1)\pi}{n}^{(2)}}_{p}\Bigg\}.$$

By the arithmetic-harmonic-mean inequality, we have

$$\begin{aligned} ER(C_n) &= \frac{1}{n-2} + 2\sum_{i=1}^{p}\left[\frac{1}{n-2\cos\frac{2i\pi}{n}} + \frac{1}{n+2\cos\frac{(2i-1)\pi}{n}}\right] \\ &\geq \frac{1}{n-2} + \frac{2\,p^2}{np - 2\sum\limits_{i=1}^{p}\cos\frac{2i\pi}{n}} + \frac{2\,p^2}{np + 2\sum\limits_{i=1}^{p}\cos\frac{(2i-1)\pi}{n}}. \end{aligned}$$

By (5), we get

$$\begin{aligned} 2\sum_{i=1}^{p}\cos\frac{2i\pi}{n} &= \frac{2\,\sin\left(\frac{p\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}\cos\left(\frac{(p-1)\pi}{n} + \frac{2\pi}{n}\right) \\ &= \frac{2\,\sin\left(\frac{p\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}\cos\left(\frac{p\pi}{n} + \frac{\pi}{n}\right) \\ &= \frac{\cos\left(\frac{\pi}{2n}\right)}{\sin\left(\frac{\pi}{n}\right)} - 1 = \frac{1}{2\,\sin\left(\frac{\pi}{2n}\right)} - 1 \end{aligned}$$

and

$$\begin{aligned} 2\sum_{i=1}^{p}\cos\frac{(2i-1)\pi}{n} = 2\,\frac{\sin\left(\frac{p\pi}{n}\right)\,\cos\left(\frac{p\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)} = \frac{\sin\left(\frac{2p\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)} &= \frac{\cos\left(\frac{\pi}{2n}\right)}{\sin\left(\frac{\pi}{n}\right)} \quad \text{as} \quad n = 4p+1 \\ &= \frac{1}{2\,\sin\left(\frac{\pi}{2n}\right)}. \end{aligned}$$

From the above results, we have

$$\begin{aligned} ER(C_n) &\geq \frac{1}{n-2} + \frac{2\,p^2}{np - \frac{1}{2\,\sin\left(\frac{\pi}{2n}\right)} + 1} + \frac{2\,p^2}{np + \frac{1}{2\,\sin\left(\frac{\pi}{2n}\right)}} \\ &= \frac{1}{n-2} + \frac{2p^2\,(2np+1)}{n^2p^2 + np + \frac{1}{2\,\sin\left(\frac{\pi}{2n}\right)} - \frac{1}{4\,\sin^2\left(\frac{\pi}{2n}\right)}} \\ &> \frac{1}{n-2} + \frac{2p^2\,(2np+1)}{n^2p^2 + np} \quad \text{as} \quad 2\,\sin\left(\frac{\pi}{2n}\right) < 1 \\ &= \frac{1}{n-2} + \frac{(n-1)\,(n^2-n+2)}{n\,(n^2-n+4)} \quad \text{as} \quad n = 4p+1. \end{aligned}$$

Case (ii) : $n = 4p+3$. By Lemma 7, the adjacency spectrum of cycle C_n is

$$S(C_n) = \Bigg\{2, \underbrace{2\cos\frac{2\pi}{n}^{(2)}, 2\cos\frac{4\pi}{n}^{(2)}, \ldots, 2\cos\frac{2p\pi}{n}^{(2)}}_{p},$$

$$\underbrace{-2\cos\frac{\pi}{n}^{(2)}, -2\cos\frac{3\pi}{n}^{(2)}, \ldots, -2\cos\frac{(2p+1)\pi}{n}^{(2)}}_{p+1}\Bigg\}.$$

By the arithmetic-harmonic-mean inequality, we have

$$\begin{aligned} ER(C_n) &= \frac{1}{n-2} + 2\sum_{i=1}^{p}\frac{1}{n-2\cos\frac{2i\pi}{n}} + 2\sum_{i=1}^{p+1}\frac{1}{n+2\cos\frac{(2i-1)\pi}{n}} \\ &\geq \frac{1}{n-2} + \frac{2\,p^2}{np-2\sum\limits_{i=1}^{p}\cos\frac{2i\pi}{n}} + \frac{2\,p^2}{n\,p+2\sum\limits_{i=1}^{p}\cos\frac{(2i-1)\pi}{n}} + \frac{2}{n+2\,\cos\frac{(2p+1)\,\pi}{n}}. \end{aligned}$$

Using the results in **Case (i)** with $\sin x < x$ and $n = 4p+3$, from the above, we get

$$\begin{aligned} ER(C_n) &\geq \frac{1}{n-2} + \frac{2\,p^2}{np - \frac{1}{2\,\sin\left(\frac{\pi}{2n}\right)} + 1} + \frac{2\,p^2}{n\,p + \frac{1}{2\,\sin\left(\frac{\pi}{2n}\right)}} + \frac{2}{n+2\,\sin\frac{\pi}{2n}} \\ &> \frac{1}{n-2} + \frac{2p^2\,(2np+1)}{n^2p^2+np} + \frac{2n}{n^2+\pi} \\ &= \frac{1}{n-2} + \frac{(n-3)\,(n^2-3n+2)}{n\,(n^2-3n+4)} + \frac{2n}{n^2+\pi}. \end{aligned}$$

■

Remark 9. *The bound in Theorem 3 is always better than the bound in Lemma 6 (ii).*

Proof. Now,

$$\begin{aligned} &\frac{(n-1)\,(n^2-n+2)}{(n^2-n+4)} - \frac{(n-3)\,(n^2-3n+2)}{(n^2-3n+4)} - \frac{2n^2}{n^2+\pi} \\ &= \frac{2n^4-8n^3+22n^2-32n+16}{(n^2-n+4)(n^2-3n+4)} - \frac{2n^2}{n^2+\pi} \\ &= \frac{\pi(2n^4-8n^3+22n^2-32n+16)-16n^2}{(n^2-n+4)(n^2-3n+4)(n^2+\pi)} > 0 \quad \text{as } n \geq 3. \end{aligned}$$

From the above result, we have

$$\frac{1}{n-2} + \frac{(n-1)\,(n^2-n+2)}{n\,(n^2-n+4)} > \frac{1}{n-2} + \frac{(n-3)\,(n^2-3n+2)}{n\,(n^2-3n+4)} + \frac{2n}{n^2+\pi}.$$

Therefore we have to prove that

$$\frac{1}{n-2} + \frac{(n-3)\,(n^2-3n+2)}{n\,(n^2-3n+4)} + \frac{2n}{n^2+\pi} > \frac{n}{\sqrt{n^2-4}} - \frac{8}{n^2-4}.$$

By (10), we have to prove that

$$\frac{1}{n-2} + \frac{(n-3)\,(n^2-3n+2)}{n\,(n^2-3n+4)} + \frac{2n}{n^2+\pi} > 1 + \frac{3}{n^2} - \frac{8}{n^2-4},$$

that is,

$$\frac{1}{n-2}+\frac{-3n^2+7n-6}{n^3-3n^2+4n}+\frac{2n}{n^2+\pi}+\frac{5n^2+12}{n^2(n^2-4)}>0,$$

that is,

$$\frac{4n^3-4n^2+\pi(-2n^3+10n^2-16n+12)2n}{(n^2+\pi)(n^4-5n^3+10n^2-8n)}+\frac{5}{(n^2-4)}>0,$$

that is,

$$5n^6-(2\pi+21)n^5+(15\pi+46)n^4-(33\pi+56)n^3+(22\pi+16)n^2+24\pi n-48\pi>0,$$

which is always true for $n\geq 3$. ■

Here we prove Conjecture 2 when n is even.

Theorem 4. *The inequality $ER(S_n)<ER(C_n)$ holds for all $n\geq 4$ (n is even).*

Proof. By Theorem 2 with (11), we have

$$ER(C_n)>\frac{2n}{n^2-4}+\frac{2}{n}+\frac{n\,(n-4)}{n^2-2}\geq\frac{2n}{n^2-4}+\frac{n\,(n-2)}{n^2-2+\frac{4}{n-2}}\quad\text{for even } n.$$

We have to prove that $ER(S_n)<ER(C_n)$, for all $n\geq 4$ (n is even), that is,

$$ER(C_n)>\frac{2n}{n^2-4}+\frac{n\,(n-2)}{n^2-2+\frac{4}{n-2}}>\frac{2n}{n^2-n+1}+\frac{n-2}{n}=ER(S_n),$$

that is,

$$\frac{n\,(n-2)^2}{n^3-2n^2-2n+8}-\frac{n-2}{n}>\frac{2n(n-5)}{(n^2-4)\,(n^2-n+1)},$$

that is,

$$\frac{n^2-6n+8}{n^4-2n^3-2n^2+8n}>\frac{n(n-5)}{(n^2-4)\,(n^2-n+1)},$$

that is,

$$3n^4-4n^3-12n^2+56n-32>0,$$

which is always true for $n\geq 4$. This completes the proof of the theorem. ■

Remark 10. *Still Conjecture 2 is open for odd n.*

Remark 11. *From Theorem 3, we have $ER(C_{11})>0.99979$ and $ER(S_{11})=1.01638$, by (3). Hence our lower bound on $ER(C_n)$ in Theorem 3 is not enough to prove Conjecture 2 completely. We need to find better lower bound on $ER(C_n)$ when n is odd.*

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