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Some Results on Constructing of Graphs Which Have the Same Randić Energy

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Abstract

The Randić energy $RE(G)$ of a graph G is defined as the sum of the absolute values of the eigenvalues of its Randić matrix. In [8], Rojo et.al. has obtained the construction of bipartite graph having the same Randić energy by using a new graph operation. In this paper we define a graph operation for any graph and construct the graphs with the same Randić energy. Also, we obtain bipartite graphs which have the same Randić energy with path graphs from a new concept (mixed extension of graphs) introduced by Haemers in [6].

1 Introduction

Let G be a simple connected graph on the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. For $v_i \in V$, the degree of the vertex v_i is the number of the vertices adjacent to v_i and denoted by d_i. Given a matrix M, the characteristic polynomial of M is $\phi(M, x) = det(xI_n - M)$. The spectrum of a matrix M is the multiset of roots of $\phi_M(x)$ including multiplicity and denoted by $Sp\{M\}$. Some of graph matrices are the following: The adjacency matrix of a graph G, denoted $A(G)$, is the symmetric matrix indexed by the ordered set $(v_1, \ldots, v_n)$ of vertices of G with $a_{ij} = 1$ if $v_iv_j \in E(G)$ in G and 0 otherwise. The diagonal degree matrix of G is $D(G) = diag(d_1, \ldots, d_n)$. The Laplacian matrix of G is $L(G) = D(G) - A(G)$ and the normalized Laplacian matrix of G, $\mathcal{L}(G)$ is given by

$$l_{ij} = \begin{cases} 1 & \text{if } v_i = v_j \\ -\frac{1}{\sqrt{d_i\, d_j}} & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

and the Randić matrix $R(G) = (r_{ij})_{n\times n}$ is defined as [1,2]

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i\, d_j}} & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, it is well known [3] if G has no isolated vertices,

$$R = D^{-\frac{1}{2}} A D^{-\frac{1}{2}} \tag{1}$$

and

$$\mathcal{L}(\mathcal{G}) = D^{-\frac{1}{2}} L D^{-\frac{1}{2}} \tag{2}$$

Denote the eigenvalues of the Randić matrix $R = R(G)$ and normalized Laplacian matrix $\mathcal{L}(\mathcal{G})$ by $1 = \rho_1 \geq \rho_2 \ldots \geq \rho_n$ and $\tilde{\mu_1} \geq \ldots \geq \tilde{\mu_n}$, respectively. Therefore, from the equalities in (1) and (2), it can be seen that $\mathcal{L} = I - R$, also

$$\tilde{\mu_i} = 1 - \rho_i \tag{3}$$

for any i.

Given any matrix M with the eigenvalues λ_i for $1 \leq i \leq n$, the energy of matrix M can be defined as

$$E(M) = \sum_{i=1}^{n} |\lambda_i - \frac{trace(M)}{n}|$$

Therefore we can define the energies related to graph matrices. That's; the normalized Laplacian energy $\mathcal{L}E(G)$ and Randić energy $RE(G)$ are

$$\mathcal{L}E(G) = \sum_{i=1}^{n} |\tilde{\mu_i} - 1|$$

and

$$RE(G) = \sum_{i=1}^{n} |\rho_i|$$

respectively. Notice that from (3), we get

$$\mathcal{L}E(G) = RE(G)$$

For several lower and upper bounds on Randić energy, see the references in [2, 5, 7].

In [8], authors get the bipartite graphs having the same Randić energy by defining a graph operation. Starting from this fact, in the main section of this paper, we define another graph operation and get the general graphs with same Randić energy. Moreover, by using definition of mixed extension in [6] we find the bipartite graphs which have the same Randić energy with P_n.

2 Main Results

Definition 2.1. [8] Let G_1^r be the graph obtained from r copies of G by identifying the vertices in $V_1 = \{1, 2, \ldots, n_1\}$ and let G_2^r be the graph obtained from r copies of G by identifying the vertices in $V_2 = \{1, 2, \ldots, n_2\}$.

Theorem 2.2. *[8] Let G be a bipartite graph. Then*

$$RE(G_1^r) = RE(G_2^r) = RE(G) \tag{4}$$

Now let's define a new graph operation.

Definition 2.3. Let G be any graph. $G_{\sim}^{rc}$ is obtained by taking r-copies of the graph which the neighbourhood of each vertices in G must be the same in $G_{\sim}^{rc}$.

Remark 2.4. *It is clear that G and $G_{\sim}^{rc}$ have the same structural properties because of the fixed neighbourhood in the graphs. For instance, if G is a bipartite, so do $G_{\sim}^{rc}$.*

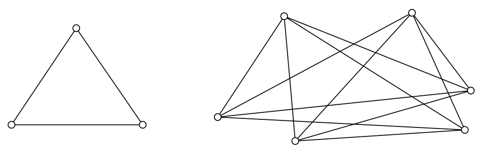

Fig. 1. K_3 and $2-copy$

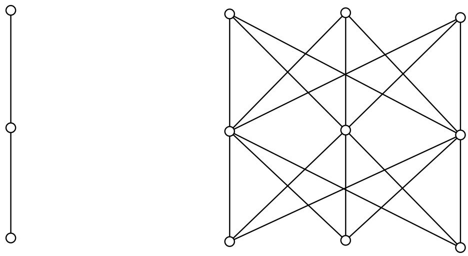

Fig. 2. P_3 and $3-copy$

Theorem 2.5. *Let $G^{rc}_{\sim}$ be the r-copies of G with n vertices. Then*

$$RE(G) = RE(G^{rc}_{\sim}). \tag{5}$$

Proof.

$$R(G^{rc}_{\sim}) = \left[\begin{array}{c|c|c} \frac{1}{r}R(G) & \cdots & \frac{1}{r}R(G) \\ \hline \vdots & & \vdots \\ \hline \frac{1}{r}R(G) & \cdots & \frac{1}{r}R(G) \end{array}\right]_{rn\times rn}$$

$$= \frac{1}{r}\left[\begin{array}{c|c|c} R(G) & \cdots & R(G) \\ \hline \vdots & & \vdots \\ \hline R(G) & \cdots & R(G) \end{array}\right]_{rn\times rn}$$

$$= \frac{1}{r}\big(R(G)_{nxn} \otimes J_{r\times r}\big)$$

Since $Sp\{J\} = \{r^{(1)}, 0^{(r-1)}\}$ and the property of Kronecker product of matrices, we get

$$Sp\{G^{rc}_{\sim}\} = Sp\{R(G)\} \bigcup \{0^{(r-1)}\} \tag{6}$$

Therefore, we have

$$RE(G^{rc}_{\sim}) = RE(G) \tag{7}$$

■

Remark 2.6. *The result in Theorem 2.2 is true for bipartite graphs, whereas our result is provided for any graph.*

Definition 2.7. *[6] Consider a graph G with vertex set $\{1, 2, \ldots, n\}$. Let $V_1, \ldots, V_n$ be mutually disjoint nonempty finite sets. We define a graph H with vertex set the union of $V_1, \ldots, V_n$ as follows. For each i, the vertices of V_i are either all mutually adjacent (V_i is* *a clique), or all mutually nonadjacent (V_i is a coclique). When $i \neq j$, a vertex of V_i is adjacent to a vertex of V_j if and only if i and j are adjacent in G. We call H a mixed extension of G.*

Let P_n be a path graph with vertex set $\{1, 2, \ldots, n\}$. We consider V_i coclique or clique for each $i \in \{1, 2, \ldots, n\}$ such that $|V_i| = k_i$. In this paper, we will present the mixed extension of P_n with its type. If V_i is coclique for all i, then the mixed extension of P_n will be shown type of $\big(k_1, \ldots, k_n\big)$. If at least V_i is a clique, (for instance, let V_k be a clique) , the mixed extension of P_n will be shown type of $\big(k_1, \ldots, k_{k-1}, K_k, k_{k+1}, \ldots, k_n\big)$. For some examples, see the following figures.

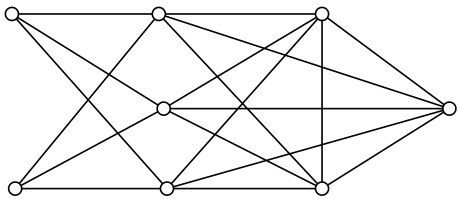

Fig. 3. The mixed extension of P_3 with type $(2, 3, K_3)$

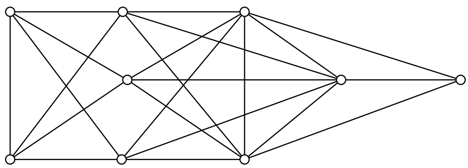

Fig. 4. The mixed extension of P_4 with type $(K_2, 3, K_3, 1)$

In Fig3., by type $(2, 3, K_3)$ it is understood that it will be taken 2 and 3 coclique, 3 clique instead of the each vertex of P_3, respectively. Similarly, for the mentioned type in Fig4., it will be taken 2 clique, 3 coclique, 3 clique and 1 coclique (also clique) instead of the each vertex of P_4, respectively.

Theorem 2.8. *[11] Let G be a graph and $\tilde{\mu_1} \geq \ldots \geq \tilde{\mu_n}$ are the eigenvalues of $L(G)$. G is a bipartite graph if and only if for each $\widetilde{\mu}_i$, $2 - \widetilde{\mu}_i = \widetilde{\mu}_j$ for some j.*

Lemma 2.9. *[10] Let G be a bipartite graph of order n. Then*

$$\rho_i = -\rho_{n-i+1}\,,\ i = 1,\, 2, \ldots,\, \left\lceil \frac{n}{2} \right\rceil .$$

Remark 2.10. *It is easy to see the converse of the proposition in Lemma 2.9 is also true from equation in (3) and Theorem 2.8.*

Theorem 2.11. *Let H be the mixed extension of P_n with any type. The non-zero eigenvalues of graphs H and P_n are the same if and only if the extension must be type of $(k, k, \ldots, k)$ and $(k, l, k, l, \ldots, k, l)$ for any k, l.*

Proof. Let H be the mixed extension of P_n with any type. We consider the non-zero eigenvalues of graphs H and P_n are the same. Then we can say H must be a bipartite graph from Remark 2.10. Hence the extension can not contain any clique. That's: it must be with type $(k_1, k_2, \ldots, k_n)$. Then the entries of Randić block matrix $R(H)$

$$r'_{ij} = \begin{cases} \frac{1}{\sqrt{(k_i+k_{j-1})(k_j+k_{i+1})}}J & \text{if } i-j=1 \\ \frac{1}{\sqrt{(k_j+k_{i-1})(k_i+k_{j+1})}}J & \text{if } i-j=-1 \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

such that J is all one matrix and $k_0 = k_{n+1} = 0$. It can be easily seen that this matrix is a tridiagonal block matrix and it has $k_1, \ldots, k_n$ identical rows. Hence the spectrum of H contains 0 eigenvalue with algebraic multiplicity $\sum_{i=1}^{n} k_i$. The given partition of $R(H)$ is equitable with quotient matrix $Q = (q_{ij})$ such that

$$q_{ij} = \begin{cases} \frac{k_j}{\sqrt{(k_j+k_{i-1})(k_i+k_{j+1})}} & \text{if } i-j=-1 \\ \frac{k_j}{\sqrt{(k_i+k_{j-1})(k_j+k_{i+1})}} & \text{if } i-j=1 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Then, the characteristic matrix of Q is

$$\phi_n(Q,x) = det(xI_n - Q) = \begin{vmatrix} x & \frac{-k_2}{\sqrt{(k_1+k_3)k_1}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{-k_1}{\sqrt{(k_1+k_3)k_1}} & x & \frac{-k_3}{\sqrt{(k_1+k_3)(k_2+k_4)}} & \ddots & \ddots & 0 & 0 \\ 0 & \frac{-k_2}{\sqrt{(k_1+k_3)(k_2+k_4)}} & x & \ddots & \ddots & \frac{-k_{n-1}}{\sqrt{(k_{n-1}+k_{n-3})(k_n+k_{n-2})}} & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & x & \frac{-k_n}{\sqrt{k_{n-1}(k_n+k_{n-2})}} \\ 0 & 0 & \cdots & \cdots & \cdots & \frac{-k_{n-1}}{\sqrt{k_{n-1}(k_n+k_{n-2})}} & x \end{vmatrix} \tag{10}$$

By expanding it along the first row, one gets:

$$\phi_n = x\phi_{n-1} - \frac{k_n}{k_n + k_{n-2}}\phi_{n-2} \tag{11}$$

with $\phi_1 = x$ and $\phi_2 = x^2 - \frac{k_3}{k_3+k_1}$. The characteristic equation of difference equation in (11) is $r^2 - xr + \frac{k_n}{k_n+k_{n-2}} = 0$. Hence the roots are

$$r_{1,2} = \frac{x \pm \sqrt{x^2 - \frac{4k_n}{k_n+k_{n-2}}}}{2}$$

Then, the initial conditions imply

$$c_1\Big(\frac{x + \sqrt{x^2 - \frac{4k_n}{k_n+k_{n-2}}}}{2}\Big) + c_2\Big(\frac{x - \sqrt{x^2 - \frac{4k_n}{k_n+k_{n-2}}}}{2}\Big) = x$$

$$c_1\Big(\frac{x + \sqrt{x^2 - \frac{4k_n}{k_n+k_{n-2}}}}{2}\Big)^2 + c_2\Big(\frac{x - \sqrt{x^2 - \frac{4k_n}{k_n+k_{n-2}}}}{2}\Big)^2 = x^2 - \frac{k_3}{k_3+k_1}$$

which yields

$$c_1 = s_n + \frac{z_n x}{\sqrt{x^2 - v_n}}$$

and

$$c_2 = s_n - \frac{z_n x}{\sqrt{x^2 - v_n}}$$

where $s_n = \frac{k_3(k_n+k_{n-2})}{2k_n(k_1+k3)}$, $z_n = \frac{2k_1k_n+k_3k_n-k_3k_{n-2}}{2k_n(k_1+k3)}$ and $v_n = \frac{4k_n}{k_n+k_{n-2}}$

The general solution of (11) is thus

$$\phi_n = c_1 r_1^n + c_2 r_2^n \tag{12}$$

Let $x = \sqrt{v_n} cos\theta$ and from the equations (12), then the general solution is obtained as

$$\phi_n = (\sqrt{v_n}e^{i\theta})^n\big(s_n + z_n\frac{e^{i\theta}+e^{-i\theta}}{e^{i\theta}-e^{-i\theta}}\big) + \big(\sqrt{v_n}e^{-i\theta}\big)^n\big(s_n - z_n\frac{e^{i\theta}+e^{-i\theta}}{e^{i\theta}-e^{-i\theta}}\big)$$

Now we consider $u^{1/2} = e^{i\theta}$. Therefore, by simplification and computation we get

$$\phi_n(\sqrt{v_n}(u^{1/2}+u^{-1/2})) = \frac{\sqrt{v_n}(u^{1/2}+u^{-1/2})u^{1/2}\big(\frac{1}{2}\sqrt{v_n}\big)^{n-1}\Big[u^{\frac{n}{2}} - u^{\frac{-n}{2}} + y_{n-1}\big(u^{\frac{n-2}{2}} - u^{\frac{2-n}{2}}\big)\Big]}{u-1} - \frac{\sqrt{v_n\big(\frac{1}{2}\sqrt{v_n}\big)^{n-2}}u^{1/2}\Big[u^{\frac{n-1}{2}} - u^{\frac{1-n}{2}} + y_{n-2}\big(u^{\frac{n-3}{2}} - u^{\frac{3-n}{2}}\big)\Big]}{u-1} \tag{13}$$

from (12). Since the quotient matrix in (9) is a generalization of the Randić matrix of P_n, we also get the characteristic polynomial of P_n

$$\phi_n(\sqrt{2}(u^{1/2}+u^{-1/2})) = \frac{\sqrt{2}(u^{1/2}+u^{-1/2})u^{1/2}\big(\frac{1}{2}\sqrt{2}\big)^{n-1}\Big[u^{\frac{n}{2}} - u^{\frac{-n}{2}}\Big]}{u-1} - \frac{\sqrt{2}\big(\frac{1}{2}\sqrt{2}\big)^{n-2}u^{1/2}\Big[u^{\frac{n-1}{2}} - u^{\frac{1-n}{2}}\Big]}{u-1} \tag{14}$$

from (13) such that $k_i = 1$ for all i. Since they have the same non-zero parts of characteristic polynomial, the coefficients of the polynomials in (13) and (14) must be the same. From this fact, we get

$$k_n = k_{n-2}$$

and

$$\frac{k_3}{k_1} = \frac{k_{n-1}}{k_{n-3}}$$

for all n. Hence the types which imply above relations must be $(k, k, \ldots, k)$ and $(k, l, k, l, \ldots, k, l)$ for any k, l.

Conversely, it is clear to see the non zero eigenvalues are the same from the characteristic matrices of P_n and H with type $(k, k, \ldots, k)$ and $(k, l, \ldots, k, l)$. ■

Corollary 2.12. *Let H be the mixed extension of P_n with types $(k, k, \ldots, k)$ and $(k, l, k, l, \ldots, k, l)$ for any k, l. Then*

$$RE(H) = RE(P_n)$$

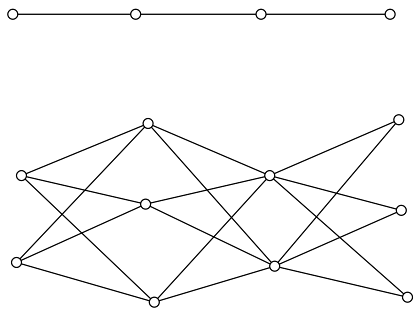

Fig. 5. P_4 and the mixed extension with type $(2, 3, 2, 3)$

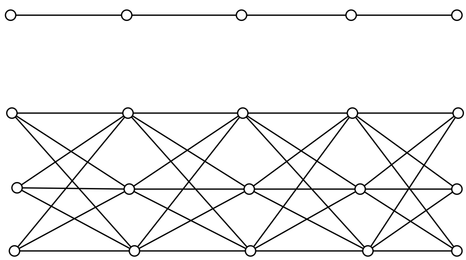

Fig. 6. P_5 and the mixed extension with type $(3, 3, 3, 3, 3)$

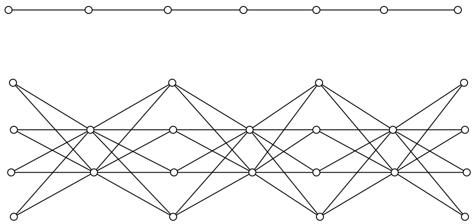

Fig. 7. P_7 and the mixed extension with type $(4, 2, 4, 2, 4, 2, 4)$

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