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A Low Bound on Graph Energy in Terms of Minimum Degree

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Abstract

The energy $E(G)$ of a graph G is the sum of the absolute values of all eigenvalues of G. This article is motivated by [25] (*MATCH Commun. Math. Comput. Chem.* 55 (2006) 91-94), where Zhou initiated the study of bounding the energy of a graph in terms of the minimum degree together with other parameters and he obtained an interesting inequality. However, his result only holds for quadrangle-free graphs. Thus we are encouraged to bound the graph energy (from below) in terms of the minimum degree for an arbitrary graph. For a connected graph G with minimum degree δ, it is proved that $E(G) \geq 2\delta$, and the equality holds if and only if G is a complete multipartite graphs with equal size of chromatic sets. By applying the main result, we prove that K_{q+1} and $K_{q,q}$ are the only q-regular graphs with energy $2q$ if and only if q is a prime number or $q = 1$.

1 Introduction

Let G be an undirected graph without multiple edges and loops. The energy $E(G)$ of G is defined to be the sum of the absolute values of all eigenvalues of $A(G)$, where $A(G)$ denotes the adjacency matrix of G. The motivation for the definition of $E(G)$ comes from chemistry, where the first results on $E(G)$ were obtained as early as the 1940s [5]. However, in the last two decades research on graph energy has much intensified, resulting in a large number of publications. For detailed results on graph energy we refer the reader to book [16], where the authors summarized important results involving graph energy.

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Here, we only introduce some known results related to bounds of graph energy. Caporossi et. al. [4] proved that $E(G) \geq 2\sqrt{m}$ for all graphs G with m edges. Rada [19] extended the above result to a diagraph D by proving that if a digraph D has c_2 closed walks of length 2 then the energy of D is not less than $\sqrt{2c_2}$. Rada and Tineo [20] proved that $E(G) \geq 2m\sqrt{\frac{m}{q}}$ for a bipartite graph G with m edges and with q being half of the spectra moment of fourth order. Akbari, Ghorbani and Zare [1] established some lower bounds for $E(G)$ by using the rank and chromatic number of G. Gutman [12] studied conditions under which biregular graphs G of order n satisfy $E(G) \geq n$. McClelland [17] proved that $E(G) \leq \sqrt{2mn} \leq 2m$ for a graph with n vertices and m edges. Koolen and Moulton [15] bounded the graph energy in terms of vertex number and edge number. Zhou [26] gave an upper bound by using the vertex number, the edge number and the vertex degree sequence. Yu, Lu and Tian [24] improved Zhou's bound by adding 2-degree sequence. Further, Hou, Teng and Woo [13] bounded the graph energy in terms of k-degree of the graph. Recently, L. Wang, X. Ma [22] established an upper bound and a lower bound on graph energy in terms of vertex cover number. For additional bounds for graph energy or skew energy of oriented graphs see the recent papers [2,3,8–11,14,18], and the references cited therein.

This article is motivated by [25], where Zhou initiated the study of bounding the graph energy in terms of the minimum degree (together with other parameters) and he obtained an interesting inequality as follows.

PROPOSITION 1.1 (Zhou, [25]) *Let G be a quadrangle-free (n, m)-graph with minimum degree $\delta \geq 1$ and maximum degree Δ. Then*

$$E(G) \geq \frac{2\sqrt{2\delta\Delta}}{2(\delta + \Delta) - 1}\sqrt{2mn}.$$

However, his result only holds for quadrangle-free graphs. The problem of bounding the graph energy (from below) in terms of the minimum degree for an arbitrary graph is left open for more than ten years. In this paper, we are interested in solving this problem, obtain a result as follows.

THEOREM 1.2 *Let G be a connected graph with minimum degree δ. Then $E(G) \geq 2\delta$ and the equality holds if and only if G is a complete multipartite graph with equal size of chromatic sets.*

Remark 1.3 *Theorem 1.2 seems somewhat trivial if the minimum degree of the graph is small. However, if the minimum degree is large enough, especially when* 2δ *is larger than the order of the graph, the result turns out to be nontrival. See the energy of* $K_{3,3,3}$ *for an example, our lower bound for this graph is* 12*, which is better than all lower bounds established in [1], [4], [19], [20], or [22].*

It is well known that K_{q+1} and $K_{q,q}$ are q-regular graphs with energy $2q$. Another interesting problem is:

Problem: *For a positive integer* q*, are* K_{q+1} *and* $K_{q,q}$ *the only* q*-regular graphs with energy* $2q$*? If not, how many such graphs exist and what graphs are them?*

By applying the main result we can give this problem a positive answer.

Notation and some known results applied in our proof are introduced in Section 2, the proof for Theorem 1.2 is given in Section 3 and the answer for the above problem is given at the end of this article.

2 Notation and some known results related to our main result

Throughout, we consider simple graphs, i.e., undirected graphs without loops and multiple edges. Let G be a simple graph with vertex set $V(G)$ and adjacency matrix $A(G)$. By $n(G)$ and $m(G)$ we respectively denote the vertex number and the edge number of G. If $x, y \in V(G)$ are adjacent we write $x \sim y$ and denote by xy the edge joining x and y. A subgraph H of G is called an induced subgraph if two vertices of $V(H)$ are adjacent in H if and only if they are adjacent in G. The set of neighbors of a vertex x in G, written as $N(x)$, is defined as

$$N(x) = \{y \in V(G) : y \sim x\}.$$

The number of vertices in $N(x)$ is called the degree of x in G, which is written as $d(x)$. By $\delta(G)$ (resp., $\Delta(G)$) we denote the minimum degree (resp., maximum degree) of G, i.e.,

$$\delta(G) = \min\{d(x) : x \in V(G)\}; \quad \Delta(G) = \max\{d(x) : x \in V(G)\}.$$

For a subset U of $V(G)$, denote by $G - U$ the induced subgraph obtained from G by deleting the vertices of U together with all edges incident to them. When H is an induced subgraph of G, we denote by $G - H$ the induced subgraph with vertex set $V(G) - V(H)$, which is also called the complement of H in G. If F is a subset of the edge set of G, then $G-F$ will denote the spanning subgraph which is obtained from G by deleting the edges in F. If F is a set of edges of G such that $G-F$ is the disjoint union of two complementary induced subgraphs H and K, then F is called an edge cut of G and we write $G-F=H\cup K$.

By P_n, C_n, K_n we respectively denote the path, the cycle and the complete graph on n vertices. An r-partite graph is a graph whose vertices can be partitioned into r different independent (or chromatic) sets. Equivalently, it is a graph that can be colored with r colors, so that no two endpoints of an edge have the same color. A complete r-partite graph is an r-partite graph in which there is an edge between every pair of vertices from different independent sets. A complete multipartite graph is a graph that is complete r-partite for some r. A complete multipartite graph is described by notation with a capital letter K subscripted by a sequence of the sizes of each set in the partition. For instance, $K_{2,2,2}$ is the complete tripartite graph of a regular octahedron, which can be partitioned into three independent sets each consisting of two vertices.

The following lemma (see [6] or [23]) will be applied when we establish the inequality and characterize the extremal graphs whose energy is precisely double of its minimum degree.

Lemma 2.1 (Lemma 2.1, [23]) *For the complete r-partite graph $K_{p_1,p_2,\ldots,p_r}$ on n vertices, the characteristic polynomial of this graph is*

$$P(x)=x^{n-r}(1-\sum_{i=1}^{r}\frac{p_i}{x+p_i})\prod_{i=1}^{r}(x+p_i).$$

Day and So studied how the energy of a graph changes when edges are deleted and they obtained an interesting result, which is of key importance when we establish the lower bound for graph energy.

Lemma 2.2 (Theorem 3.4, [7]) *If F is an edge cut of a simple graph G then $E(G-F)\leq E(G)$.*

3 Proof of Theorem 1.2

In this section, we give a proof for Theorem 1.2. Before doing that we consider the energy of a complete multipartite graph.

LEMMA 3.1 *Let G be the complete r-partite graph $K_{p_1,p_2,\ldots,p_r}$, where $r \geq 2$ and the size of each partite set is arranged such that $p_1 \geq p_2 \geq \ldots \geq p_r \geq 1$. Then we have*

$$E(G) \geq 2\delta(G) = 2\sum_{i=2}^{r} p_i$$

and the equality holds if and only if $p_1 = p_2 = \ldots = p_r$.

Proof. We proceed by induction on the order of the graph to prove the inequality. If the order of G is 2, then $G = P_2$ and thus $E(G) = 2 = 2\delta(G)$, as required. Suppose the inequality holds for complete multipartite graphs of order at most $n-1$ and consider the complete r-partite graph $G = K_{p_1,p_2,\ldots,p_r}$ with order $n = \sum_{i=1}^{r} p_i$. Let W be a set of r vertices of $V(G)$ coming from r different partite sets and let $K = G[W]$ be the induced subgraph with vertex set W. Then K is a complete graph on r vertices. If $p_1 = 1$, then $G = K$ with energy $E(G) = 2(r-1)$, double of its minimum degree, and thus the assertion holds for G. If $p_1 \geq 2$ and $p_2 = \ldots = p_r = 1$, then the characteristic polynomial of this graph is

$$P(x) = x^{n-r}(x+1)^{r-2}[x^2 - (r-2)x - p_1(r-1)],$$

which has nonzero eigenvalues

$$-1, -1, \ldots, -1, \frac{(r-2) + \sqrt{(r-2)^2 + 4p_1(r-1)}}{2}, \frac{(r-2) - \sqrt{(r-2)^2 + 4p_1(r-1)}}{2},$$

where the multiplicity of eigenvalue -1 is $r-2$. Consequently,

$$E(G) = (r-2) + \sqrt{(r-2)^2 + 4p_1(r-1)},$$

which is strictly larger than double of the minimum degree $r-1$.

Assume that $p_i \geq 2$ for $i = 1, 2, \ldots, s$ and $p_{s+1} = \ldots = p_r = 1$, where $2 \leq s \leq r$, and let $H = G - K$. Then H is a complete s-partite graph isomorphic to $K_{p_1-1,p_2-1,\ldots,p_s-1}$. Let F be the set of edges between K and H. Then F is an edge cut of G such that $G - F = K \cup H$. By Lemma 2.2 we have

$$E(G) \geq E(K) + E(H) \tag{1}$$

Since K is a complete graph on r vertices,

$$E(K) = 2(r-1). \tag{2}$$

The induction hypothesis to H implies that

$$E(H) \geq 2\sum_{i=2}^{s}(p_i - 1) = 2\sum_{i=2}^{r}(p_i - 1). \tag{3}$$

Substituting (2) and (3) to (1), we have

$$E(G) \geq E(K) + E(H) \geq 2(r-1) + 2\sum_{i=2}^{r}(p_i - 1) = 2\sum_{i=2}^{r} p_i = 2\delta(G), \tag{4}$$

which proves the inequality for G.

If $p_1 = p_2 = \ldots = p_r$ for $G = K_{p_1,p_2,\ldots,p_r}$, then the characteristic polynomial of G is

$$P(x) = x^{n-r}(x + p_1)^{r-1}(x + p_1 - rp_1),$$

with nonzero eigenvalues

$$-p_1, -p_1, \ldots, -p_1, p_1(r-1),$$

where the multiplicity of $-p_1$ is $r-1$. In this case, $E(G) = 2p_1(r-1)$, which is precisely double of its minimum degree.

Conversely, suppose that $G = K_{p_1,p_2,\ldots,p_r}$, with $p_1 \geq p_2 \geq \ldots p_r \geq 1$, is a complete r-partite graph with energy double of its minimum degree, i.e., $E(G) = 2\delta(G) = 2\sum_{i=2}^{r} p_i$. We proceed by induction on the order of G to prove that $p_1 = p_2 = \ldots = p_r$. If the order of G is 2, the assertion holds trivially. Assume the assertion holds for complete multipartite graphs of order at most $n-1$ and consider $G = K_{p_1,p_2,\ldots,p_r}$ with order $n = \sum_{i=1}^{r} p_i$. If $p_1 = 1$, then $p_1 = p_2 = \ldots = p_r = 1$ and we are done. If $p_1 > p_2 = p_3 = \ldots = p_r = 1$, we have proved that $E(G) = (r-2) + \sqrt{(r-2)^2 + 4p_1(r-1)}$, which is strictly larger than double of the minimum degree $r-1$, a contradiction. Assume that $p_i \geq 2$ for $i = 1, 2, \ldots, s$ and $p_{s+1} = \ldots = p_r = 1$, where $2 \leq s \leq r$. As above, let $K = G[W]$ be the induced subgraph on an r-vertices set W, in which every vertex comes from r different partite sets of G. Set $H = G - K$, and let F be the set of edges between K and H. The condition $E(G) = 2\delta(G)$ will forces all inequality involved in (4) to turn into equalities. Thus we have

$$E(H) = 2\sum_{i=2}^{s}(p_i - 1) = 2\delta(H). \tag{5}$$

The induction hypothesis to H implies $p_1 - 1 = \ldots = p_s - 1$, or equivalently, $p_1 = \ldots = p_s$.

To complete the proof, it suffices to prove $s = r$. The characteristic polynomial of $G = K_{p_1,p_2,\ldots,p_r}$, in which $p_1 = p_2 = \ldots = p_s > 1$ and $p_{s+1} = \ldots = p_r = 1$, is

$$P(x) = x^{n-r}(x + p_1)^{s-1}(x+1)^{r-s-1}[x^2 - (sp_1 + r - p_1 - s - 1)x + p_1(1-r)],$$

which has nonzero eigenvalues

$$-p_1, \dots, -p_1, -1, \dots, -1, \frac{b + \sqrt{b^2 + 4(r-1)p_1}}{2}, \frac{b - \sqrt{b^2 + 4(r-1)p_1}}{2},$$

where the multiplicity of $-p_1$ is $s - 1$, the multiplicity of -1 is $r - s - 1$ and $b = sp_1 + r - p_1 - s - 1$. The energy of this graph is

$$E(G) = b + \sqrt{b^2 + 4(r-1)p_1}.$$

From $E(G) = 2\delta(G)$ it follows

$$b + \sqrt{b^2 + 4(r-1)p_1} = 2(b+1),$$

which further leads to

$$(p_1 - 1)(r - s) = 0.$$

As $p_1 > 1$, we have $r = s$, as required. This completes the proof for Lemma 3.1. ■

Now, we are ready to give a proof for Theorem 1.2

Proof of Theorem 1.2

If G is a complete multipartite graph, the result has been proved by Lemma 3.1. Let G be an arbitrary connected graph. We proceed by induction on the order of G to prove the inequality $E(G) \geq 2\delta(G)$ and to characterize the extremal graphs with energy double of $\delta(G)$. If the order of G is 2, then $E(G) = 2 = 2\delta(G)$. Suppose the result holds for all connected graphs with order at most $n - 1$ and let G be a connected graph with order $n \geq 3$.

If G is a tree with $m \geq 1$ edges, then $\delta(G) = 1$. As $E(G) \geq 2\sqrt{m} \geq 2$, we have $E(G) \geq 2\delta(G)$ and the equality holds if and only if $G = K_2$.

Suppose G is not a tree. Let g be the girth of G and let C be a cycle in G of size g. Then C is an induced subgraph of G. If $G = C$, we have $E(G) = 4 = 2\delta(G)$ for $g = 3$ or 4 and $E(G) \geq 2\sqrt{g} > 4 = 2\delta(G)$ for $g \geq 5$. Suppose G is not a cycle. Let $H = G - C$ and let F be the set of edges between C and H. Then F is an edge cut of G such that $G - F = C \cup H$. By Lemma 2.2, we have

$$E(G) \geq E(C) + E(H). \tag{6}$$

If $g \geq 5$, each vertex of H has at most one neighbor in C. Thus

$$\delta(H) \geq \delta(G) - 1. \tag{7}$$

The induction hypothesis to every component of H implies that

$$E(H) \geq 2\delta(H). \tag{8}$$

Substituting (7) and (8) to (6) and recalling that $E(C) \geq 2\sqrt{g} > 4$ we have

$$E(G) \geq E(C) + E(H) > 4 + 2\delta(H) > 2\delta(G).$$

Now we consider the case when $g = 4$. Let P be a path on two adjacent vertices x_0, y_0 and let $Q = G - P$. Then $G - F_1 = P \cup Q$, where F_1 is the set of edges between P and Q. Lemma 2.2 says

$$E(G) \geq E(P) + E(Q). \tag{9}$$

Since each vertex of Q has at most one neighbor in P (recalling that $g = 4$). Thus

$$\delta(Q) \geq \delta(G) - 1. \tag{10}$$

The induction hypothesis applying to every component of Q implies that

$$E(Q) \geq 2\delta(Q). \tag{11}$$

Substituting (10) and (11) to (9) and recalling that $E(P) = 2$ we have

$$E(G) \geq E(P) + E(Q) \geq 2 + 2\delta(Q) \geq 2\delta(G), \tag{12}$$

which proves the inequality. If $E(G) = 2\delta(G)$, then all inequalities involved in (12) turn into equalities and thus we have $\delta(Q) = \delta(G) - 1$ and $E(Q) = 2\delta(Q)$. From $E(Q) = 2\delta(Q)$ it is easy to see that Q is connected. The induction hypothesis to Q implies that Q is a complete bipartite graph with equal chromatic sets. Assume $Q = K_{q,q}$ with two partite sets X, Y, each has q vertices. Then $\delta(Q) = q$ and thus $\delta(G) = q + 1$. Consequently, $d(x_0) \geq q + 1$. Noting that x_0 cannot be adjacent to two vertices in $X \cup Y$ from different partite sets, we have $d(x_0) \leq q + 1$. Thus $d(x_0) = q + 1$. Similarly, $d(y_0) = q + 1$. Consequently, x_0 is adjacent to all vertices in Y (or X) and y_0 is adjacent to all vertices in X (or Y). Hence, $G = K_{q+1,q+1}$ is a complete bipartite graph with equal chromatic sets.

Next, we consider the case when $g = 3$. Let Y be a maximal clique in G with r vertices. As $g = 3$, we have $r \geq 3$. If $G = Y$ then we have $E(G) = 2(r - 1) = 2\delta(G)$, and we are done. Suppose $G \neq Y$. Let $Z = G - Y$ and let F_2 be the set of edges between Y and Z. Then F_2 is an edge cut of G such that $G - F_2 = Y \cup Z$. Lemma 2.2 implies

$$E(G) \geq E(Y) + E(Z). \tag{13}$$

Noting each vertex of Z has at most $r - 1$ neighbors in K_r. Thus

$$\delta(Z) \geq \delta(G) - r + 1. \tag{14}$$

The induction hypothesis to every component of Z gives that

$$E(Z) \geq 2\delta(Z). \tag{15}$$

Substituting (14) and (15) to (13) and recalling that $E(Y) = 2(r - 1)$ we have

$$E(G) \geq E(Y) + E(Z) \geq 2(r - 1) + 2\delta(Z) \geq 2\delta(G), \tag{16}$$

which proves the inequality. If $E(G) = 2\delta(G)$, then all inequalities involved in (16) turn into equalities and thus we have $\delta(Z) = \delta(G) - r + 1$ and $E(Z) = 2\delta(Z)$. By $E(Z) = 2\delta(Z)$ we easily see that Z is connected. The induction hypothesis to Z implies that Z is a complete multipartite graph with equal chromatic sets. Assume that Z is isomorphic to $K_{s,s,\ldots,s}$, a complete k-partite graph in which each partite set has s vertices. Then $\delta(Z) = s(k - 1)$ and thus $\delta(G) = s(k - 1) + r - 1$. Consequently, every $z \in Z$ has precisely $r - 1$ neighbors in Y and misses exactly one vertex in Y. It is easy to see that two vertices from distinct partite set of Z miss two distinct vertices in Y (otherwise, such two vertices together with their $r-1$ common neighbors in Y form a clique of order $r+1$, a contradiction to the choice of Y).

To complete the proof, we need to prove $r = k$. Suppose on the contrary that $r > k$. Let $W = \{z_1, z_2, \ldots, z_k\}$ be a subset of $V(Z)$ inducing a complete subgraph of Z. Then the induced subgraph with vertex set $V(Y) \cup W$, written as K', is isomorphic to $K_{2,2,\ldots,2,1,\ldots,1}$, where the multiplicities of 2 and 1 are respectively k and $r - k$. Let $H' = G - K'$. Then H' is isomorphic to the complete k-partite graph $K_{s-1,s-1,\ldots,s-1}$. Let F_3 be the set of edges between K' and H'. Then $G - F_3 = K' \cup H'$ and thus

$$E(G) \geq E(K') + E(H'). \tag{17}$$

By Lemma 3.1, we have

$$E(K') > 2\delta(K') = 2(r + k - 2), \tag{18}$$

and

$$E(H') = 2\delta(H') = 2(s-1)(k-1). \tag{19}$$

Substituting (18) and (19) to (17), we have

$$E(G) > 2(r+k-2) + 2(s-1)(k-1) = 2[s(k-1)+r-1] = 2\delta(G),$$

which is a contradiction and thus we have $r = k$.

Now, partition $V(Z)$ into disjoint union of r chromatic sets as

$$V(Z) = W_1 \cup W_2 \cup \ldots \cup W_r,$$

and suppose

$$W_i = \{z_{i1}, z_{i2}, \ldots, z_{is}\} \text{ for } i = 1, 2, \ldots, r.$$

Then $z_{ij} \sim z_{kl}$ if and only if $i \neq k$. Let

$$U_j = \{z_{1j}, z_{2j}, \ldots, z_{rj}\} \text{ for } j = 1, 2, \ldots, s.$$

Then every U_j induces a complete graph, written as Z_j, on r vertices. We have shown that every vertex in U_1 has $r-1$ neighbors in Y and misses exactly one vertex in Y. In addition distinct vertex in U_1 misses distinct vertex of Y. Label the vertices of Y as $V(Y) = \{y_1, y_2, \ldots, y_r\}$ such that $y_i \nsim z_{i1}$ and $y_i \sim z_{j1}$ for $j \neq i$. Since $y_i \sim z_{j1}$ for all $j \neq i$, we have $y_i \nsim z_{il}$ for any l (otherwise we obtain a clique of order $r+1$). Consider the degree of y_i, we confirm that $y_i \sim z_{jl}$ for any $j \neq i$ and any l. Finally, we conclude that G is isomorphic to the complete r-partite graph $K_{s+1,s+1,\ldots s+1}$. ■

With Theorem 1.2 in hand, we can solve the question posted at the first section. Let q be a positive integer and $xy - y = q$ an equation on x, y. For a positive integer solution (r, s) of $xy - y = q$, i.e., $rs - s = q$, we have a corresponding complete r-partite graph $K_{s,s,\ldots,s}$, which is q-regular and has energy $2q$ (thanks to Lemma 3.1). By Theorem 1.2, every q-regular graph with energy $2q$ must be such 0form. Thus we have:

Corollary 3.2 *Let $q \geq 2$ be an integer. An q-regular graph G has energy $2q$ if and only if G is isomorphic to a complete r-partite graph $K_{s,s,\ldots,s}$, where a pair of positive integers (r, s) is a solution of the equation $xy - y = q$. Particularly, K_{q+1} and $K_{q,q}$ are the only q-regular graphs with energy $2q$ if and only q is a prime number.*

Example: When $q = 6$, the equation $xy - y = q$ has four positive integer solutions:

$$(7, 1),\ (2, 6),\ (3, 3),\ (4, 2).$$

Thus there are four 6-regular graphs with energy 12. They are $K_7, K_{6,6}, K_{3,3,3}, K_{2,2,2,2}$.

We believe that the lower bound on graph energy of a connected graph G can be improved from 2δ to $2\overline{\delta}$, where $\overline{\delta}$ is the average degree of G, and we believe the equality holds if and only if G is a complete multipartite graphs with equal size of chromatic sets. This conjecture is left as a further research problem.

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