MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **81** (2019) 371-381

ISSN 0340 - 6253

Bounds on the Entire Zagreb Indices of Graphs*

Ali Ghalavand, Ali Reza Ashrafi*

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan 87317–53153, I R Iran

(Received April 23, 2018)

Abstract

Suppose G is a simple graph with vertex set and edge set $V(G)$ and $E(G)$, respectively. Define $B(G)$ to be the set of all $\{x, y\}$ such that $\{x, y\} \subseteq V(G) \cup E(G)$ and members of $\{x, y\}$ are adjacent or incident to each other. Alwardi et al. in a recent paper, [A. Alwardi, A. Alqesmah, R. Rangarajan and I. N. Cangul, Entire Zagreb indices of graphs, *Discrete Math. Algorithm. Appl.* **10**(3) (2018) 1850037 (16 pages)] introduced the first and second entire Zagreb indices of G as $M_1^{\mathcal{E}}(G) = \sum_{x \in V(G)\cup E(G)} deg_G(x)^2$ and $M_2^{\mathcal{E}}(G) = \sum_{\{x,y\}\in B(G)} deg_G(x)deg_G(y)$, where $deg_G(u)$ denotes the degree of a vertex or edge u in G. In this paper, we continue this work to obtain the relationship between entire Zagreb indices with the Zagreb and reformulated Zagreb indices of graphs. Some bounds for the first and second entire Zagreb indices are obtained. Moreover, the first through the fifth smallest first entire Zagreb index among all connected graphs and the first through the eleventh smallest first entire Zagreb index among all trees are computed. Finally, the connected graphs with the first through the fifth smallest first entire Zagreb index and the trees with the first through the eleventh smallest first entire Zagreb index are all graphs with maximum degree at most four.

1 Introduction

Throughout this paper, only the finite, undirected and simple graphs will be considered. Let G be such a graph with vertex set and edge set $V(G)$ and $E(G)$, respectively. The degree of a vertex v in G, $deg_G(v)$, is the number of edges incident to v. The notation $N[v, G]$ is used for the set of all vertices adjacent to v. A pendant vertex is a vertex

*Corresponding author (ashrafi@kashanu.ac.ir)

of degree one and we use $\Delta = \Delta(G)$ and $\delta = \delta(G)$ to denote the maximum degree and minimum degree of vertices in G, respectively. The number of vertices of degree i will be denoted by n_i or $n_i(G)$. It is easy to see that $\sum_{i=0}^{\Delta(G)} n_i = |V(G)|$. In addition, the number of edges of degree i will be denoted by ε_i or $\varepsilon_i(G)$. It is easy to see that $\sum_{i=0}^{2\Delta(G)-2} \varepsilon_i = |E(G)|$. Also, $m_{i,j}(G)$ is the number of edges of G connecting a vertex of degree i with a vertex of degree j. The set of all n-vertex trees will be denoted by $\tau(n)$ and as usual, the path and cycle with n vertices are denoted by P_n and C_n, respectively.

The **first Zagreb index** $M_1(G)$, and the **second Zagreb index** $M_2(G)$, can be defined as follows:

$$\begin{aligned} M_1(G) &= \sum_{uv \in E(G)} [deg(u) + deg(v)] = \sum_{v \in V(G)} deg^2(v), \\ M_2(G) &= \sum_{uv \in E(G)} deg(u)deg(v). \end{aligned}$$

These graph invariant were introduced by Ivan Gutman and Nenad Trinajstić [9]. Furtula and Gutman [6], introduced the **forgotten index** of G, $F(G)$, as the sum of cubes of vertex degrees. It is easy to see that

$$F(G) = \sum_{v \in V(G)} deg(v)^3 = \sum_{e=uv \in E(G)} [deg(u)^2 + deg(v)^2].$$

Milićević et al. [10] introduced the **first and second reformulated Zagreb indices** of a graph G as edge counterpart of the first and second Zagreb indices, respectively. These numbers are defined as

$$\begin{aligned} EM_1(G) &= \sum_{e \sim f} [deg_G(e) + deg_G(f)] = \sum_{e \in E(G)} deg_G(e)^2, \\ EM_2(G) &= \sum_{e \sim f} deg_G(e)deg_G(f), \end{aligned}$$

where for $e = uv$, $deg_G(e) = deg_G(u) + deg_G(v) - 2$ denotes the degree of the edge e, and $e \sim f$ means that the edges e and f are incident. Very recently Alwardi et al. [2] introduced the **first and second entire Zagreb indices** of a graph G, as

$$M_1^{\mathcal{E}}(G) = \sum_{x \in V(G) \cup E(G)} deg_G(x)^2 \quad \text{and} \quad M_2^{\mathcal{E}}(G) = \sum_{\{x,y\} \in B(G)} deg_G(x)deg_G(y),$$

where $B(G)$ denotes the set of all $2-$element subsets $\{x, y\}$ such that $\{x, y\} \subseteq V(G) \cup E(G)$ and members of $\{x, y\}$ are adjacent or incident to each other. They proved that for any $k-$regular graph on p vertices,

$$M_1^{\mathcal{E}}(G) = pk(2k^2 - 3k + 2) \quad \text{and} \quad M_2^{\mathcal{E}}(G) = pk(2k^3 - \frac{7}{2}k^2 + 4k - 2).$$

Moreover, the authors proved that the first and second entire Zagreb indices can be computed by the following formulas:

$$\begin{aligned}
M_1^{\mathcal{E}}(G) &= 4|E(G)| - 3M_1(G) + 2M_2(G) + \frac{1}{2}\sum_{u\in V(G)} (deg_G(u))^3,\\
M_2^{\mathcal{E}}(G) &= 4|E(G)| - 2M_1^{\mathcal{E}}(G) - 2M_1(G) + M_2(G) + \frac{1}{2}\sum_{u\in V(G)} (deg_G(u))^4\\
&+ \sum_{u\in V(G)} (deg_G(u))^2 \sum_{v\in N[u,G]} deg_G(v) + \frac{1}{2}\sum_{u\in V(G)}\Big(\sum_{v\in N[u,G]} deg_G(u)\Big)^2.
\end{aligned}$$

The aim of this paper is to continue this work by extending last equalities and apply them to find extremal graphs with respect to these graph parameters. We refer to a recent paper of Ali et al. [1] for more information on related topics.

2 Main Results

In this paper, the relationship between entire Zagreb indices with the Zagreb and reformulated Zagreb indices of graphs are investigated. Some bounds for the first and second entire Zagreb indices are obtained. Moreover, the first through the fifth smallest first entire Zagreb index among all connected graphs and the first through the eleventh smallest first entire Zagreb index among all trees are computed. Finally, the connected graphs with the first through the fifth smallest first entire Zagreb index and the trees with the first through the eleventh smallest first entire Zagreb index are all graphs with maximum degree at most four.

Theorem 2.1. *Let G be a connected graph with n vertices and m edges. Then*

1. $M_1^{\mathcal{E}}(G) = M_1(G) + EM_1(G)$.
2. $M_2^{\mathcal{E}}(G) = 3M_2(G) + EM_2(G) + F(G) - 2M_1(G)$.

Proof. To prove (1), we note that by our definitions,

$$\begin{aligned}
M_1^{\mathcal{E}}(G) &= \sum_{x\in V(G)\cup E(G)} deg_G(x)^2\\
&= \sum_{x\in V(G)} deg_G(x)^2 + \sum_{x\in E(G)} deg_G(x)^2\\
&= M_1(G) + EM_1(G),
\end{aligned}$$

as desired. To prove (2), we note that by definition:

$$\begin{aligned}
M_2^{\mathcal{E}}(G) &= \sum_{\{x,y\}\in B(G)} deg_G(x)deg_G(y) \\
&= \sum_{xy\in E(G)} deg_G(x)deg_G(y) + \sum_{e,f\in E(G),e\sim f} deg_G(e)deg_G(f) \\
&+ \sum_{x\in V(G)} \sum_{xy\in E(G)} deg_G(x)deg_G(xy).
\end{aligned}$$

Therefore,

$$\begin{aligned}
M_2^{\mathcal{E}}(G) &= M_2(G) + EM_2(G) + \sum_{x\in V(G)} \sum_{xy\in E(G)} deg_G(x)[deg_G(x) + deg_G(y) - 2] \\
&= M_2(G) + EM_2(G) + \sum_{x\in V(G)} \sum_{xy\in E(G)} deg_G(x)^2 \\
&+ \sum_{x\in V(G)} \sum_{xy\in E(G)} deg_G(x)deg_G(y) - \sum_{x\in V(G)} \sum_{xy\in E(G)} 2deg_G(x) \\
&= M_2(G) + EM_2(G) + \sum_{x\in V(G)} deg_G(x)^3 \\
&+ 2\sum_{xy\in E(G)} deg_G(x)deg_G(y) - \sum_{x\in V(G)} 2deg_G(x)^2 \\
&= 3M_2(G) + EM_2(G) + F(G) - 2M_1(G).
\end{aligned}$$

This completes our argument. ■

Theorem 2.2. *For every simple graph G, $M_1^{\mathcal{E}}(G)$ is an even integer. Moreover, for all non-negative integers k, there exists at least one graph G with $M_1^{\mathcal{E}}(G) = 2k$.*

Proof. It is well-known that the number of vertices of odd degree in G and $L(G)$ are even. Hence a simple calculation shows that $M_1(G)$ and $EM_1(G)$ are even, and by Theorem 2.1, $M_1^{\mathcal{E}}(G)$ is also even. Finally, if G has k isolated edges, i.e. $\varepsilon_0(G) = |E(G)|$, then $M_1^{\mathcal{E}}(G) = 2k$, which proves that each non-negative even integer is the first entire Zagreb index of at least one graph. ■

Theorem 2.3. *Let G be a connected graph with n vertices and m edges. Then $M_1^{\mathcal{E}}(G) \geq \frac{(M_1(G))^2}{n+m}$. The equality holds if and only if G is isomorphic to C_n.*

Proof. By the Cauchy-Schwarz inequality,

$$\Big(\sum_{x\in V(G)\cup E(G)} deg_G(x)\Big)^2 \leq (n+m) \sum_{x\in V(G)\cup E(G)} deg_G(x)^2 = (n+m)M_1^{\mathcal{E}}(G).$$

On the other hand, $\sum_{x\in V(G)\cup E(G)} deg_G(x) = M_1(G)$. Therefore, $M_1^{\mathcal{E}}(G) \geq \frac{(M_1(G))^2}{n+m}$, and the equality holds if and only if G is isomorphic to C_n. ■

Corollary 2.4. *Let G be a connected graph with n vertices and m edges. Then $M_1^{\mathcal{E}}(G) \geq \frac{16m^4}{n^2(n+m)}$ with equality if and only if G is isomorphic to C_n.*

Proof. By a result of Yoon and Kim [11], we can see that,

$$M_1(G) \geq \frac{4m^2}{n}, \tag{2.1}$$

with equality if and only if G is a regular graph. Therefore, by Theorem 2.3, $M_1^{\mathcal{E}}(G) \geq \frac{16m^4}{n^2(n+m)}$ with equality if and only if G is isomorphic to C_n. ■

Lemma 2.5. *If G is a graph with n vertices, m edges, and without isolated edges, i.e. $\varepsilon_0(G) = 0$, then $\varepsilon_1(G) = 4m - M_1(G) + \sum_{i=3}^{2n-4} \varepsilon_i(G)(i-2)$ and $\varepsilon_2(G) = M_1(G) - 3m - \sum_{i=3}^{2n-4} \varepsilon_i(G)(i-1)$.*

Proof. The result follows from $\varepsilon_1(G) + \varepsilon_2(G) + \sum_{i=3}^{2n-4} \varepsilon_i(G) = m$ and $\varepsilon_1(G) + 2\varepsilon_2(G) + \sum_{i=3}^{2n-4} i\varepsilon_i(G) = M_1(G) - 2m$. ■

Theorem 2.6. (*See [8]) Let G be a graph with n vertices, m edges and without isolated vertices, i.e. $n_0(G) = 0$. Then $M_1(G) \geq 6m - 2n$. The equality holds if and only if $n_i(G) = 0$, for each i with $3 \leq i \leq n-1$.*

Lemma 2.7. *Let G be a graph with n vertices, m edges and without isolated edges (i. e. $\varepsilon_0(G) = 0$). Then $EM_1(G) = 3M_1(G) - 8m + \sum_{i=3}^{2n-4}(i-1)(i-2)\varepsilon_i(G)$.*

Proof. By Definition and Lemma 2.5, we have

$$\begin{aligned}
EM_1(G) &= \sum_{e \in E(G)} deg_G(e)^2 = \sum_{i=0}^{2n-4} i^2 \varepsilon_i(G) \\
&= \varepsilon_0(G) + \varepsilon_1(G) + 4\varepsilon_2(G) + \sum_{i=3}^{2n-4} i^2 \varepsilon_i(G) \\
&= 4m - M_1(G) + \varepsilon_0(G) + \sum_{i=3}^{2n-4} \varepsilon_i(G)(i-2) \\
&+ 4M_1(G) - 12m - \sum_{i=3}^{2n-4} 4\varepsilon_i(G)(i-1) + \sum_{i=3}^{2n-4} i^2 \varepsilon_i(G) \\
&= 3M_1(G) - 8m + \sum_{i=3}^{2n-4} (i-1)(i-2)\varepsilon_i(G),
\end{aligned}$$

proving the lemma. ■

Corollary 2.8. *Let G be a graph with n vertices and m edges.*

1. *If $\varepsilon_0(G) = 0$, then $EM_1(G) \geq 3M_1(G) - 8m$. The equality holds if and only if $\varepsilon_i(G) = 0$ for $3 \leq i \leq 2n - 4$.*
2. *If $\varepsilon_0(G) = 0$ and $n_0(G) = 0$, then $EM_1(G) \geq 10m - 6n$. The equality holds if and only if $\varepsilon_i(G) = 0$ and $n_i(G) = 0$ for $3 \leq i \leq 2n - 4$.*

Proof. Theorem 2.6 and Lemma 2.7, give us the results. ■

Corollary 2.9. *Let G be a graph with n vertices and m edges.*

1. *If $\varepsilon_0(G) = 0$, then $M_1^{\mathcal{E}}(G) \geq 4M_1(G) - 8m$. The equality holds if and only if $\varepsilon_i(G) = 0$ for $3 \leq i \leq 2n - 4$.*
2. *If $\varepsilon_0(G) = 0$ and $n_0(G) = 0$, then $M_1^{\mathcal{E}}(G) \geq 16m - 8n$. The equality holds if and only if $\varepsilon_i(G) = n_i(G) = 0$, for $3 \leq i \leq 2n - 4$.*

Proof. Theorems 2.1(1), 2.6, and Corollary 2.8, give us the results. ■

The following result is a direct consequence of Corollary 2.9.

Corollary 2.10. *Let G be a connected graph with n vertices and m edges.*

1. *If G is a tree, then $EM_1(G) \geq 4n - 10$ and $M_1^{\mathcal{E}}(G) \geq 8n - 16$, with equality if and only if $G \cong P_n$.*
2. *$EM_1(G) \geq 10m - 6n \geq 4n$ and $M_1^{\mathcal{E}}(G) \geq 16m - 8n \geq 8n$, the equalities holds if and only if $G \cong C_n$.*

Let n be a positive integer number. Define $\alpha(n) = n\lceil \frac{n}{2} \rceil(1 - \frac{1}{n}\lceil \frac{n}{2}\rceil)$.

Theorem 2.11. (*See [4]*) *Suppose a_i and b_i, $1 \leq i \leq n$, are positive real numbers. Then*

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \alpha(n)(A - a)(B - b),$$

where a, b, A, and B are real constants, that for each i, $1 \leq i \leq n$, $a \leq a_i \leq A$, and $b \leq b_i \leq B$.

Theorem 2.12. *Let G be a nontrivial graph with n vertices and m edges. Then*

$$\begin{aligned} M_1^{\mathcal{E}}(G) \;\leq\; & \frac{4\alpha(m)(\Delta-\delta)^2}{m} + \frac{\alpha(n)(\Delta-\delta)^4}{mn^2} + \frac{8\alpha(n)(\Delta-\delta)^2 m + 16m^3}{n^2} \\ - \; & \frac{3\alpha(n)(\Delta-\delta)^2 - 12m^2}{n} + 4m, \end{aligned}$$

with equality if and only if G is a regular graph.

Proof. Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. For each i, $1 \le i \le n$, $\delta \le deg_G(v_i) \le \Delta$ and for each i, $1 \le i \le m$, $2\delta - 2 \le deg_G(e_i) \le 2\Delta - 2$. Therefore, by Theorem 2.11,

$$\left| n \sum_{i=1}^{n} deg_G(v_i)^2 - \Big(\sum_{i=1}^{n} deg_G(v_i) \Big)^2 \right| \le \alpha(n)(\Delta - \delta)^2,$$
$$\left| m \sum_{i=1}^{m} deg_G(e_i)^2 - \Big(\sum_{i=1}^{m} deg_G(e_i) \Big)^2 \right| \le 4\alpha(m)(\Delta - \delta)^2.$$

By Cauchy-Schwarz inequality, $n \sum_{i=1}^{n} deg_G(v_i)^2 \ge \Big(\sum_{i=1}^{n} deg_G(v_i) \Big)^2$ and $m \sum_{i=1}^{m} deg_G(e_i)^2 \ge \Big(\sum_{i=1}^{m} deg_G(e_i) \Big)^2$. Hence, $M_1(G) \le \frac{4m^2 + \alpha(n)(\Delta-\delta)^2}{n}$ and we have:

$$\begin{aligned} EM_1(G) &\le \frac{(M_1(G) - 2m)^2 + 4\alpha(m)(\Delta - \delta)^2}{m} \\ &\le \frac{1}{mn^2}\Big[\alpha(n)(\Delta - \delta)^4 + (8m^2 - 4mn)\alpha(n)(\Delta - \delta)^2 \\ &+ 4\alpha(m)(\Delta - \delta)^2 n^2 + 4(2m - n)^2 m^2\Big]. \end{aligned}$$

The equalities holds if and only if G is a regular graph. Finally, by Theorem 2.1(1), $M_1^{\mathcal{E}}(G) \le \frac{4\alpha(m)(\Delta-\delta)^2}{m} + \frac{\alpha(n)(\Delta-\delta)^4}{mn^2} + \frac{8\alpha(n)(\Delta-\delta)^2 m + 16m^3}{n^2} - \frac{3\alpha(n)(\Delta-\delta)^2 - 12m^2}{n} + 4m$ and the equality holds if and only if G is a regular graph. ■

Theorem 2.13. *(See [5]) Suppose a_i and b_i, $1 \le i \le n$ are positive real numbers, then*

$$\sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i \le (r + R) \sum_{i=1}^{n} a_i b_i,$$

where r, and R are real constants, that for each i, $1 \le i \le n$, $ra_i \le b_i \le Ra_i$.

Theorem 2.14. *Let G be a nontrivial graph with n vertices and m edges. Then*

$$\begin{aligned} M_1^{\mathcal{E}}(G) &\le 2(\delta + \Delta - 2)\Big(2m(\delta + \Delta - 1) - \delta\Delta n\Big) \\ &- 2m\Big(2\delta\Delta - 3(\delta + \Delta) + 2\Big) - \delta\Delta n, \end{aligned}$$

the equality holds if and only if G is a regular graph.

Proof. Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. For each i, $1 \le i \le n$, $\delta \cdot 1 \le deg_G(v_i) \le \Delta \cdot 1$ and for each i, $1 \le i \le m$, $2(\delta - 1) \cdot 1 \le deg_G(e_i) \le 2(\Delta - 1) \cdot 1$. Therefore, by Theorem 2.13, $\sum_{i=1}^{n} deg_G(v_i)^2 + \delta\Delta \sum_{i=1}^{n} 1 \le (\delta + \Delta) \sum_{i=1}^{n} 1 \cdot deg_G(v_i)$ and $\sum_{i=1}^{m} deg_G(e_i)^2 + 4(\delta - 1)(\Delta - 1) \sum_{i=1}^{m} 1 \le 2(\delta + \Delta - 2) \sum_{i=1}^{m} 1 \cdot deg_G(e_i)$. So, $M_1(G) \le (\delta + \Delta)2m - \delta\Delta n$, $EM_1(G) \le 2(\delta + \Delta - 2)(M_1(G) - 2m) - 4m(\delta - 1)(\Delta - 1) \le$

$2(\delta+\Delta-2)\Big(2m(\delta+\Delta-1)-\delta\Delta n\Big) - 4m(\delta-1)(\Delta-1)$. The equalities hold if and only if G is a regular graph. Finally, by Theorem 2.1(1), $M_1^{\mathcal{E}}(G) \leq 2(\delta+\Delta-2)\Big(2m(\delta+\Delta-1)-\delta\Delta n\Big)$ $- 2m\Big(2\delta\Delta - 3(\delta+\Delta)+2\Big) - \delta\Delta n$, with equality hold if and only if G is a regular graph.

■

Theorem 2.15. *Let G be a connected graph with n vertices and m edges. Then*

1. $M_2^{\mathcal{E}}(G) \geq \frac{4}{n}(\delta-1)^2(2m^2 - mn) + m(5\delta^2 - 4\delta)$,
2. $M_2^{\mathcal{E}}(G) \leq 2(\Delta-1)^2(n\Delta^2 - 2m) + m(5\Delta^2 - 4\Delta)$.

The equalities holds if and only if G is a regular graph.

Proof. (1) By definition,

$$\begin{aligned}
M_2^{\mathcal{E}}(G) &= \sum_{xy\in E(G)} deg_G(x)deg_G(y) + \sum_{x,y\in E(G),x\sim y} deg_G(x)deg_G(y)\\
&+ \sum_{x\in V(G)}\sum_{xy\in E(G)} deg_G(x)deg_G(xy)\\
&\geq \sum_{xy\in E(G)} \delta^2 + \sum_{x,y\in E(G),x\sim y} (2\delta-2)^2\\
&+ \sum_{x\in V(G)}\sum_{xy\in E(G)} \delta(2\delta-2)\\
&= m\delta^2 + (2\delta-2)^2 \sum_{x\in V(G)} \binom{deg_G(x)}{2}\\
&+ \sum_{x\in V(G)} deg_G(x)\delta(2\delta-2)\\
&= m\delta^2 + 2(\delta-1)^2(M_1(G)-2m) + 4m\delta(\delta-1).
\end{aligned}$$

By inequality (2.1),

$$M_2^{\mathcal{E}}(G) \geq \frac{4}{n}(\delta-1)^2(2m^2 - mn) + m(5\delta^2 - 4\delta),$$

and the equality holds if and only if G is a regular graph.

(2) By a similar argument as (1),

$$M_2^{\mathcal{E}}(G) \leq m\Delta^2 + 2(\Delta-1)^2(M_1(G)-2m) + 4m\Delta(\Delta-1).$$

Since, $M_1(G) \leq n\Delta^2$, we have

$$M_2^{\mathcal{E}}(G) \leq 2(\Delta-1)^2(n\Delta^2 - 2m) + m(5\Delta^2 - 4\Delta),$$

and the equality holds if and only if G is a regular graph. ■

For positive integers $x_1, \ldots, x_m$, and $y_1, \ldots, y_m$, let $T(x_1^{y_1}, \ldots, x_m^{y_m})$ be the class of all trees in which the vertex y_i has degree x_i, $i = 1, \ldots, m$.

Theorem 2.16. *(See [3]) Suppose $n \geq 12$. Choose $T_1, \cdots, T_{11}$ in such a way that $T_1 := P_n$, $T_2 \in T(3^1, 2^{n-4}, 1^3)$, $T_3 \in T(3^2, 2^{n-6}, 1^4)$, $T_4 \in T(4^1, 2^{n-5}, 1^4)$, $T_5 \in T(3^3, 2^{n-8}, 1^5)$, $T_6 \in T(4^1, 3^1, 2^{n-7}, 1^5)$, $T_7 \in T(3^4, 2^{n-10}, 1^6)$, $T_9 \in T(4^1, 3^2, 2^{n-9}, 1^6)$, $T_{11} \in T(3^5, 2^{n-12}, 1^7)$ and $T \in \tau(n) \setminus \{T_1, T_2, \cdots, T_7, T_9, T_{11}\}$. Then $M_1(T_1) < M_1(T_2) < M_1(T_3) < M_1(T_4) = M_1(T_5) < M_1(T_6) = M_1(T_7) < M_1(T_9) = M_1(T_{11}) < M_1(T)$.*

Theorem 2.17. *(See [7]) If $n \geq 11$, the sets of trees $T_i \in A_i$, $i = 1, 2, \ldots 14$, are defined in Table 1 and $T \in \tau(n) \diagdown \{T_1, T_2, ..., T_{14}\}$, then $EM_1(T_1) < EM_1(T_2) < EM_1(T_3) < EM_1(T_4) < EM_1(T_5) < EM_1(T_6) < EM_1(T_7) = EM_1(T_8) < EM_1(T_9) = EM_1(T_{10}) < EM_1(T_{11}) = EM_1(T_{12}) < EM_1(T_{13}) = EM_1(T_{14}) < EM_1(T)$.*

Corollary 2.18. *If $n \geq 11$, $T_1 \in A_1$, $T_2 \in A_2$, $T_3 \in A_3$, $T_4 \in A_4$, $T_5 \in A_5$, $T_6 \in A_6$, $T_7 \in A_7$, $T_8 \in A_8$, $T_9 \in A_9$, $T_{10} \in A_{10}$, $T_{11} \in A_{11}$, $T_{12} \in A_{12}$, $T_{13} \in A_{13}$, $T_{14} \in A_{14}$ and $T \in \tau(n) \diagdown \{T_1, T_2, ..., T_{14}\}$, then $M_1^{\mathcal{E}}(T_1) < M_1^{\mathcal{E}}(T_2) < M_1^{\mathcal{E}}(T_3) < M_1^{\mathcal{E}}(T_4) < M_1^{\mathcal{E}}(T_5) < M_1^{\mathcal{E}}(T_6) < M_1^{\mathcal{E}}(T_7) = M_1^{\mathcal{E}}(T_8) < M_1^{\mathcal{E}}(T_9) = M_1^{\mathcal{E}}(T_{10}) < M_1^{\mathcal{E}}(T_{11}) = M_1^{\mathcal{E}}(T_{12}) < M_1^{\mathcal{E}}(T_{13}) < M_1^{\mathcal{E}}(T_{14}) < M_1^{\mathcal{E}}(T)$.*

Proof. The proof follows from Theorems 2.1(1), 2.16, and 2.17. ■

Corollary 2.19. *Suppose $T_2 \in A_2$, $T_3 \in A_3$, $T_4 \in A_4$, and $T_5 \in A_5$.*

1. *Let G be a connected graph with $n \geq 6$ vertices and $G \notin \{T_2, T_3, P_n, C_n\}$. Then $EM_1(P_n) < EM_1(T_2) < EM_1(T_3) = EM_1(C_n) < EM_1(G)$.*

2. *Let G be a connected graph with $n \geq 7$ vertices and $G \notin \{T_2, T_3, T_4, P_n, C_n\}$. Then $M_1^{\mathcal{E}}(P_n) < M_1^{\mathcal{E}}(T_2) < M_1^{\mathcal{E}}(T_3) < M_1^{\mathcal{E}}(T_4) < M_1^{\mathcal{E}}(C_n) < M_1^{\mathcal{E}}(G)$.*

Proof. Since $EM_1(T_4) = 4n + 2$ and $M_1^{\mathcal{E}}(T_5) = 8n + 2$, the proof follows from Corollary 2.10 and Theorems 2.17, 2.18. ■

Table 1. Trees in Theorems 2.17 and 2.18.

Notation	DD	$m_{3,3}$	$m_{2,3}$	$m_{1,2}$	$m_{1,3}$	$m_{2,2}$
A_1	$T(2^{n-2}, 1^2)$	0	0	2	0	n-3
A_2	$T(3^1, 2^{n-4}, 1^3)$	0	1	1	2	n-5
A_3	$T(3^1, 2^{n-4}, 1^3)$	0	2	2	1	n-6
A_4	$T(3^1, 2^{n-4}, 1^3)$	0	3	3	0	n-7
A_5	$T(3^2, 2^{n-6}, 1^4)$	0	2	0	4	n-7
A_6	$T(3^2, 2^{n-6}, 1^4)$	0	3	1	3	n-8
A_7	$T(3^2, 2^{n-6}, 1^4)$	0	4	2	2	n-9
A_8	$T(3^2, 2^{n-6}, 1^4)$	1	1	1	3	n-7
A_9	$T(3^2, 2^{n-6}, 1^4)$	0	5	3	1	n-10
A_{10}	$T(3^2, 2^{n-6}, 1^4)$	1	2	2	2	n-8
A_{11}	$T(3^2, 2^{n-6}, 1^4)$	0	6	4	0	n-11
A_{12}	$T(3^2, 2^{n-6}, 1^4)$	1	3	3	1	n-9
A_{13}	$T(3^2, 2^{n-6}, 1^4)$	1	4	4	0	n-10
A_{14}	$T(3^3, 2^{n-8}, 1^5)$	0	4	0	5	n-10

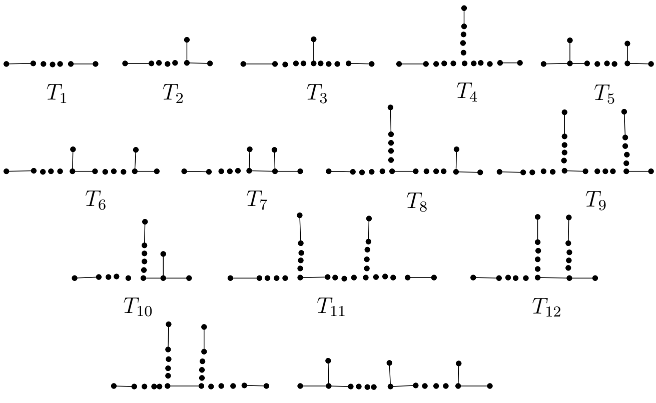

Figure 1. The Trees in Theorem 2.17 and Corollary 2.18.

Acknowledgments: The authors are indebted to the referees for their comments and helpful remarks. The research of the author is partially supported by the University of Kashan under grant number 785149/4.

References

[1] A. Ali, I. Gutman, E. Milovanović, I. Milovanović, Sum of powers of the degrees of graphs: Extremal results and bounds, *MATCH Commun. Math. Comput. Chem.* **80** (2018) 5–84..

[2] A. Alwardi, A. Alqesmah, R. Rangarajan, I. N. Cangul, Entire Zagreb indices of graphs, *Discr. Math. Alg. Appl.* **10** (2018) #1850037 (16 pages).

[3] M. Eliasi, A. Ghalavand, Extremal trees with respect to some versions of Zagreb indices via majorization, *Iranian J. Math. Chem. 8* (2017) 391–401.

[4] M. Biernacki, H. Pidek, C. Ryll–Nardzewsk, Sur une inégalité entre des intégrales définies, *Univ. Marie Curie–Sklodowska* **A4** (1950) 1–4.

[5] J. B. Diaz, F. T. Metcalf, Stronger forms of a class of inequalities of G. Pólya–G. Szegö, and L. V. Kantorovich, *Bull. Amer. Math. Soc.* **69** (1963) 415–418.

[6] B. Furtula, I. Gutman, A forgotten topological index, *J. Math. Chem.* **53** (2015) 1184–1190.

[7] A. Ghalavand, A. R. Ashrafi, Extremal trees with respect to the first and second reformulated Zagreb index, *Malaya J. Math.* **5** (2017) 524–530.

[8] I. Gutman, K. C. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.* **50** (2004) 83–92.

[9] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals, Total $\pi-$electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.

[10] A. Milićević, S. Nikolić, N. Trinajstić, On reformulated Zagreb indices, *Mol. Divers* **8** (2004) 393–399.

[11] Y. S. Yoon, J. K. Kim, A relationship between bounds on the sum of squares of degrees of a graph, *J. Appl. Math. Comput.* **21** (2006) 233–238.