MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **80** (2018) 467-480

ISSN 0340 - 6253

A Generalization of the Incidence Energy and the Laplacian–Energy–Like Invariant

Ezgi Kaya[a,b,*], **A. Dilek Maden**[b†]

[a]*Department of Mathematics,*
Faculty of Science and Literature, Igdir University, Igdir, Turkey
ezgi.kaya@igdir.edu.tr

[b] *Department of Mathematics, Faculty of Science,*
Selçuk University, Campus, 42075, Konya, Turkey
aysedilekmaden@selcuk.edu.tr

(Received April 11, 2018)

Abstract

For a graph G and a real number α, the graph invariant $s_\alpha(G)$ is the sum of the αth powers of the signless Laplacian eigenvalues and $\sigma_\alpha(G)$ is the sum of the αth powers of the Laplacian eigenvalues of G. In this study, for appropriate vales of alpha, we give some bounds for the generalized versions of incidence energy and of the Laplacian-energy-like invariant of graphs.

1 Introduction

Let G be a finite, simple, connected graph with n vertices and m edges. Let $V(G) = \{v_{1,}v_2, ..., v_n\}$ be the vertex set of G. $v_i \in V(G)$, the degree of the vertex v_i , denoted

*Corresponding author
†The author are partially supported by TUBITAK and the Scientific Research Project Office (BAP) of Selçuk University.

by d_i. The maximum vertex degree is denoted by Δ and the minimum vertex degree is denoted by δ.

Let $A(G)$ be the $(0,1)-$adjacency matrix of G and $D(G)$ be the diagonal matrix of vertex degrees. The matrix $L(G)=D(G)-A(G)$ (resp., $Q(G)=D(G)+A(G)$) is called the Laplacian matrix [26, 27] (resp., the signless Laplacian matrix) of G. Since $A(G)$, $L(G)$ and $Q(G)$ are all real symmetric matrices, their eigenvalues are real numbers. So, we can assume that $\lambda_1(G)\geq\lambda_2(G)\geq...\geq\lambda_n(G)$ (resp., $\mu_1(G)\geq\mu_2(G)\geq...\geq\mu_n(G)$, $q_1(G)\geq q_2(G)\geq...\geq q_n(G)$) are the adjacency (resp., Laplacian, signless Laplacian) eigenvalues of G.

The energy of G was defined by Gutman in [12, 13] as

$$E(G)=\sum_{i=1}^{n}|\lambda_i|$$

$\lambda_i,\ i=1,...,n$ are the eigenvalues of adjacency matrix of G. For survey and details on $E(G)$, see [12–15, 22, 28, 34].

Let $I(G)$ be the (vertex-edge) incidence matrix of the graph G with vertex set $\{v_1,v_2,...,v_n\}$ and edge set $\{e_1,e_2,...,e_n\}$. The (i,j) -entry of $I(G)$ is 0 if v_i is not incident with e_j and 1 if v_i is incident with e_j. Jooyandeh et al. [20] introduced the incidence energy IE of G, which is defined as the sum of the singular values of the incidence matrix of G. Gutman et al. [17] showed that

$$IE=IE(G)=\sum_{i=1}^{n}\sqrt{q_i}.$$

Some basic properties of IE may be found in [17, 18, 20].

In [23] Liu and Lu introduced a new graph invariant based on the Laplacian eigenvalues

$$LEL=LEL(G)=\sum_{i=1}^{n-1}\sqrt{\mu_i}$$

and called it Laplacian energy like invariant. At first it was considered that LEL [23] shares similar properties with Laplacian energy [16]. Then it was shown that it is much more similar to the ordinary graph energy [19]. For survey and details on LEL, see [24].

For a graph G with n vertices and a real number α, the sum of the αth powers of the non zero Laplacian eigenvalues is defined as [33]

$$\sigma_\alpha=\sigma_\alpha(G)=\sum_{i=1}^{n-1}\mu_i^\alpha.$$

For survey and details on sum of powers of the Laplacian eigenvalues of graphs, see [9]. The cases $\alpha = 0$ and $\alpha = 1$ are $\sigma_0 = n-1$ and $\sigma_1 = 2m$, where m is the number of edges of G. Note that $\sigma_{1/2}$ is equal to LEL.

Motivated by the definitions of IE, LEL and σ_α, Akbari et al. [1] introduced the sum of the αth powers of signless Laplacian eigenvalues of G as

$$s_\alpha = s_\alpha(G) = \sum_{i=1}^{n} q_i^\alpha$$

and they also gave some relations between σ_α and s_α. In this sum, the cases $\alpha = 0$ and $\alpha = 1$ are $s_0 = n$ and $s_1 = 2m$. Note that $s_{1/2}$ is equal to IE. Note further that Laplacian eigenvalues and signless Laplacian eigenvalues of bipartite graphs coincide [6, 26, 27]. Therefore, for bipartite graphs σ_α is equal to s_α [2, 25, 29, 30, 35] and and LEL is equal to IE [17].

In this paper, we give some generalizations for the Incidence energy and the Laplacian-energy-like invariant of graphs.

2 Lemmas

The following lemmas will be used for our main results.

Lemma 2.1 ([26]) *Let G be a graph on n vertices with at least one edge. Then*

$$\mu_1 \geq \Delta + 1.$$

Moreover, if G is connected, then the equality holds if and only if $\Delta = n - 1$.

Lemma 2.2 ([26]) *Let G be a graph of order n and $\overline{G}$ its complement. If $Spec(G) = \{\mu_1, \mu_2, ..., \mu_{n-1}, 0\}$, then $Spec(\overline{G}) = \{n - \mu_1, n - \mu_2, ..., n - \mu_{n-1}, 0\}$. From this, it follows that $\mu_1(G) \leq n$ with equality holding if and only if $\overline{G}$ is connected.*

Lemma 2.3 ([8]) *Let G be a connected graph with $n \geq 3$ vertices. Then $\mu_2 = \mu_3 = ... = \mu_{n-1}$ if and only if $G \cong K_n$ or $G \cong K_{\Delta,\Delta}$ or $G \cong K_{1,n-1}$.*

Lemma 2.4 ([8]) *Let G be a connected graph of order n. Then $\mu_1 = \mu_2 = ... = \mu_{n-1}$ if and only if $G \cong K_n$.*

Let $t = t(G)$ be the number of spanning trees of a graph G. Let $G_1 \times G_2$ denotes the Cartesian product of the graphs G_1 and G_2 [5]. Now we introduce the following two auxiliary quantities for a graph G as

$$\begin{aligned} t_1 &= t_1(G) = \frac{2t(G \times K_2)}{t(G)} \ , \\ t_2 &= t_2(G) = \frac{\Delta + \delta + \sqrt{(\Delta - \delta)^2 + 4\Delta}}{2} \end{aligned} \tag{1}$$

and where Δ and δ are the maximum and minimum vertex degree of G, respectively.

Lemma 2.5 ([7]) *If G is connected bipartite graph of order n, then $\prod_{i=1}^{n-1} \mu_i = \prod_{i=1}^{n-1} q_i = nt(G)$. If G is a connected non-bipartite graph of order n, then $\prod_{i=1}^{n} q_i = t_1$.*

Lemma 2.6 ([4], [32]) *Let G be a connected graph with $n \geq 3$ vertices and Δ be the maximum vertex degree of G. Then $q_1 \geq t_2 \geq \Delta + 1$ with either equalities if and only if G is a star graph $K_{1,n-1}$.*

Lemma 2.7 ([6], [26], [27]) *The spectra of $L(G)$ and $Q(G)$ coincide if and only if the graph G is bipartite.*

Lemma 2.8 ([31]) *Let G be simple connected graph with n vertices. Then $q_1 \leq 2\Delta$, with equality if and only if G is a regular graph.*

Lemma 2.9 ([21]) *Let $x_1, x_2, ..., x_N$ be non-negative numbers, and let*

$$\beta = \frac{1}{N}\sum_{i=1}^{N} x_i \quad \textit{and} \quad \gamma = \left(\prod_{i=1}^{N} x_i\right)^{1/N}$$

be their arithmetic and geometric means. Then

$$\frac{1}{N(N-1)}\sum_{i<j} \left(\sqrt{x_i} - \sqrt{x_j}\right)^2 \leq \beta - \gamma \leq \frac{1}{N}\sum_{i<j} \left(\sqrt{x_i} - \sqrt{x_j}\right)^2 .$$

Moreover, equality holds if and only if $x_1 = x_2 = ... = x_N$.

Lemma 2.10 ([11]) *For $a_1, a_2, ..., a_n \geq 0$ and $p_1, p_2, ..., p_n \geq 0$ such that $\sum_{i=1}^{n} p_i = 1$*

$$\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} a_i^{p_i} \geq n\lambda \left(\frac{1}{n}\sum_{i=1}^{n} a_i - \prod_{i=1}^{n} a_i^{1/n}\right)$$

where $\lambda = \min\{p_1, p_2, ..., p_n\}$. Moreover, equality holds if and only if $a_1 = a_2 = ... = a_n$.

3 Main Results

After all above materials, we are now ready to present our main results.

Theorem 3.1 *Let α be a real number with $0<\alpha<1$ and let G be a connected graph of order n with maximum degree Δ and t spanning trees. Then*

$$\sigma_\alpha(G) = \begin{cases} \leq & n^\alpha + \sqrt{(n-3)\left[\sigma_{2\alpha} - (\Delta+1)^{2\alpha}\right] + (n-2)\left(\frac{tn}{\Delta+1}\right)^{2\alpha/n-2}} \\ \geq & (\Delta+1)^\alpha + \sqrt{\sigma_{2\alpha} - n^{2\alpha} + (n-2)(n-3)t^{2\alpha/n-2}}. \end{cases} \tag{2}$$

Equality hold on both sides if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

Proof. Taking $N = n-2$ and $x_i = \mu_i^{2\alpha}$, $i = 2, 3, ..., n-1$ in Lemma 2.9 and using Lemmas 2.1 and 2.2, we obtain

$$\frac{\sum\limits_{2\leq i<j\leq n-1} \left(\mu_i^\alpha - \mu_j^\alpha\right)^2}{(n-2)\,(n-3)} \leq \frac{\sigma_{2\alpha} - \mu_1^{2\alpha}}{n-2} - \left(\frac{nt}{\mu_1}\right)^{2\alpha/n-2} \leq \frac{\sum\limits_{2\leq i<j\leq n-1} \left(\mu_i^\alpha - \mu_j^\alpha\right)^2}{(n-2)}.$$

Since $\sum\limits_{i=1}^{n-1} \mu_i^{2\alpha} = \sigma_{2\alpha}$, we have

$$\begin{aligned} \sum_{2\leq i<j\leq n-1} \left(\mu_i^\alpha - \mu_j^\alpha\right)^2 &= (n-3)\sum_{i=2}^{n-1}\mu_i^{2\alpha} - 2\sum_{2\leq i<j\leq n-1} \left(\mu_i\mu_j\right)^\alpha \\ &= (n-3)\left(\sigma_{2\alpha} - \mu_1^{2\alpha}\right) - \left(\sum_{i=2}^{n-1}\mu_i^\alpha\right)^2 + \sum_{i=2}^{n-1}\mu_i^{2\alpha} \\ &= (n-2)\left(\sigma_{2\alpha} - \mu_1^{2\alpha}\right) - \left(\sigma_\alpha - \mu_1^\alpha\right)^2 . \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{(n-2)\left(\sigma_{2\alpha} - \mu_1^{2\alpha}\right) - \left(\sigma_\alpha - \mu_1^\alpha\right)^2}{(n-2)\,(n-3)} &\leq \frac{\sigma_{2\alpha} - \mu_1^{2\alpha}}{n-2} - \left(\frac{nt}{\mu_1}\right)^{2\alpha/n-2} \\ &\leq \frac{(n-2)\left(\sigma_{2\alpha} - \mu_1^{2\alpha}\right) - \left(\sigma_\alpha - \mu_1^\alpha\right)^2}{(n-2)}. \end{aligned}$$

This implies that,

$$\sigma_\alpha \leq \mu_1^\alpha + \sqrt{(n-3)\left[\sigma_{2\alpha} - \mu_1^{2\alpha}\right] + (n-2)\left(\frac{tn}{\mu_1}\right)^{2\alpha/n-2}} \tag{3}$$

and

$$\sigma_\alpha \geq \mu_1^\alpha + \sqrt{\sigma_{2\alpha} - \mu_1^{2\alpha} + (n-2)(n-3)\left(\frac{tn}{\mu_1}\right)^{2\alpha/n-2}}. \tag{4}$$

Consider the function,

$$f(x) = \sqrt{(n-3)\left[\sigma_{2\alpha} - x^{2\alpha}\right] + (n-2)\left(\frac{tn}{x}\right)^{2\alpha/n-2}}$$

.

It can be easily shown that $f(x)$ is decreasing for $x \geq \Delta + 1$, as $0 < \alpha < 1$. Thus, we get

$$f(x) \leq f(\Delta+1) = \sqrt{(n-3)\left[\sigma_{2\alpha} - (\Delta+1)^{2\alpha}\right] + (n-2)\left(\frac{tn}{\Delta+1}\right)^{2\alpha/n-2}}.$$

Considering this, (3) and Lemma 2.2, we get

$$\begin{aligned}\sigma_\alpha(G) &\leq \mu_1^\alpha + \sqrt{(n-3)\left[\sigma_{2\alpha} - (\Delta+1)^{2\alpha}\right] + (n-2)\left(\frac{tn}{\Delta+1}\right)^{2\alpha/n-2}} \\ &\leq n^\alpha + \sqrt{(n-3)\left[\sigma_{2\alpha} - (\Delta+1)^{2\alpha}\right] + (n-2)\left(\frac{tn}{\Delta+1}\right)^{2\alpha/n-2}}.\end{aligned}$$

In an analogous manner,

$$g(x) = \sqrt{\sigma_{2\alpha} - x^{2\alpha} + (n-2)(n-3)\left(\frac{tn}{x}\right)^{2\alpha/n-2}}$$

is a decreasing function for $x \leq n$, as $0 < \alpha < 1$. Therefore,

$$g(x) \geq g(n) = \sqrt{\sigma_{2\alpha} - n^{2\alpha} + (n-2)(n-3)t^{2\alpha/n-2}}$$

Considering this, (4) and Lemma 2.1, we get

$$\begin{aligned}\sigma_\alpha &\geq \mu_1^\alpha + \sqrt{\sigma_{2\alpha} - n^{2\alpha} + (n-2)(n-3)t^{2\alpha/n-2}} \\ &\geq (\Delta+1)^\alpha + \sqrt{\sigma_{2\alpha} - n^{2\alpha} + (n-2)(n-3)t^{2\alpha/n-2}}.\end{aligned}$$

By this, the first part of the proof is done.

Suppose now that equalities hold in (2). Then all the above inequalities must become equalities. For both lower and upper bounds, by Lemma 2.9 it must be $\mu_2 = \mu_3 = \ldots = \mu_{n-1}$. Then by Lemma 2.3, either $G \cong K_n$ or $G \cong K_{1,n-1}$ or $G \cong K_{\Delta,\Delta}$. Moreover $\mu_1 = \Delta + 1 = n$ for both lower and upper bounds. Then by Lemmas 2.1 and 2.2, we conclude that $G \cong K_n$ or $G \cong K_{1,n-1}$.

For the converse, if $G \cong K_n$ or $G \cong K_{1,n-1}$ it is easy to see that equalities (2) hold.

■

Corollary 3.2 *Let α be a real number with $0 < \alpha < 1$ and let T be a tree of order n with maximum degree Δ. Then*

$$\sigma_\alpha(T) = \begin{cases} \leq n^\alpha + \sqrt{(n-3)\left[\sigma_{2\alpha} - (\Delta+1)^{2\alpha}\right] + (n-2)\left(\frac{n}{\Delta+1}\right)^{2\alpha/n-2}} \\ \geq (\Delta+1)^\alpha + \sqrt{\sigma_{2\alpha} - n^{2\alpha} + (n-2)(n-3)}. \end{cases}$$

Equality holds on both sides if and onlt if $T \cong K_{1,n-1}$.

Proof. For a tree T, $t = 1$. ■

Corollary 3.3 *Let α be a real number with $0 < \alpha < 1$ and let U be a connected unicyclic graph of order n with maximum degree Δ. Then*

$$\sigma_\alpha(U) = \begin{cases} \leq n^\alpha + \sqrt{(n-3)\left[\sigma_{2\alpha} - (\Delta+1)^{2\alpha}\right] + (n-2)\left(\frac{n^2}{\Delta+1}\right)^{2\alpha/n-2}} \\ \geq (\Delta+1)^\alpha + \sqrt{\sigma_{2\alpha} - n^{2\alpha} + (n-2)(n-3)3^{2\alpha/n-2}}. \end{cases}$$

Equality holds on both sides if and only if $U \cong K_3$.

Proof. For unicyclic graphs, $3 \leq t \leq n$ and K_3 is the only unicyclic graph for which equality in Theorem 3.1 holds. ■

For a special case, if we take $\alpha = \frac{1}{2}$, we get the same bounds as in Theorem 2.5 and Corollaries 2.6 and 2.7 of the paper [10] for the LEL given as follows:

Theorem 3.4 ([10]) *Let G be a connected graph of order n with m edges, maximum degree Δ and t spanning trees. Then*

$$LEL(G) = \begin{cases} \leq \sqrt{n} + \sqrt{(n-3)(2m-\Delta-1) + (n-2)\left(\frac{tn}{\Delta+1}\right)^{1/n-2}} \\ \geq \sqrt{\Delta+1} + \sqrt{2m - n + (n-2)(n-3)t^{1/n-2}}. \end{cases}$$

Equality hold on both sides if and onlt if $G \cong K_n$ or $G \cong K_{1,n-1}$.

Corollary 3.5 ([10]) *Let T be a tree of order n with maximum degree Δ. Then*

$$LEL(T) = \begin{cases} \leq \sqrt{n} + \sqrt{(n-3)(2n-\Delta-3) + (n-2)\left(\frac{n}{\Delta+1}\right)^{1/n-2}} \\ \geq n - 2 + \sqrt{\Delta+1}. \end{cases}$$

Equality holds on both sides if and onlt if $T \cong K_{1,n-1}$.

Corollary 3.6 ([10]) *Let U be a connected unicyclic graph of order n with maximum degree Δ. Then*

$$LEL(U) = \begin{cases} \leq \sqrt{n} + \sqrt{(n-3)(2n-\Delta-1) + (n-2)\left(\frac{n^2}{\Delta+1}\right)^{1/n-2}} \\ \geq \sqrt{\Delta+1} + \sqrt{n + (n-2)(n-3)3^{1/n-2}}. \end{cases}$$

Equality holds on both sides if and onlt if $U \cong K_3$.

By using the Lemmas 2.1-2.9, we obtain the following results.

Theorem 3.7 *Let α be a real number with $0<\alpha<1$ and let G be a connected graph with $n \geq 3$ vertices, maximum degree Δ and t spanning trees and let t_1 and t_2 be given by (1).*

(i) *If G is bipartite then*

$$\sigma_\alpha(G)=s_\alpha(G)=\begin{cases} \leq n^\alpha+\sqrt{(n-3)\left[s_{2\alpha}-t_2^{2\alpha}\right]+(n-2)\left(\frac{tn}{t_2}\right)^{2\alpha/n-2}} \\ \geq t_2^\alpha+\sqrt{s_{2\alpha}-n^{2\alpha}+(n-2)(n-3)t^{2\alpha/n-2}}. \end{cases} \tag{5}$$

Moreover, equalities hold if and only if $G \cong K_{1,n-1}$.

(ii) *If G is non-bipartite, then*

$$s_\alpha(G)=\begin{cases} <(2\Delta)^\alpha+\sqrt{(n-2)\left[s_{2\alpha}-t_2^{2\alpha}\right]+(n-1)\left(\frac{t_1}{t_2}\right)^{2\alpha/n-1}} \\ >t_2^\alpha+\sqrt{s_{2\alpha}-(2\Delta)^{2\alpha}+(n-1)(n-2)\left(\frac{t_1}{2\Delta}\right)^{2\alpha/n-1}}. \end{cases} \tag{6}$$

Proof. *(i)* Taking $N=n-2$ and $x_i=q_i^{2\alpha}$, $i=2,3,...,n-1$ in Lemma 2.9 and using Lemmas 2.1 and 2.2, we obtain

$$\frac{\sum\limits_{2\leq i<j\leq n-1}\left(q_i^\alpha-q_j^\alpha\right)^2}{(n-2)(n-3)} \leq \frac{s_{2\alpha}-q_1^{2\alpha}}{n-2}-\left(\frac{tn}{q_1}\right)^{2\alpha/n-2} \leq \frac{\sum\limits_{2\leq i<j\leq n-1}\left(q_i^\alpha-q_j^\alpha\right)^2}{(n-2)}.$$

Since $\sum\limits_{i=1}^{n-1} q_i^{2\alpha}=s_{2\alpha}$, we have

$$\begin{aligned} \sum_{2\leq i<j\leq n-1}\left(q_i^\alpha-q_j^\alpha\right)^2 &= (n-3)\sum_{i=2}^{n-1} q_i^{2\alpha}-2\sum_{2\leq i<j\leq n-1}(q_iq_j)^\alpha \\ &= (n-3)\left(s_{2\alpha}-q_1^{2\alpha}\right)-\left(\sum_{i=2}^{n-1} q_i^\alpha\right)^2+\sum_{i=2}^{n-1} q_i^{2\alpha} \\ &= (n-2)\left(s_{2\alpha}-q_1^{2\alpha}\right)-\left(s_\alpha-q_1^\alpha\right)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{(n-2)\left(s_{2\alpha}-q_1^{2\alpha}\right)-\left(s_\alpha-q_1^\alpha\right)^2}{(n-2)(n-3)} &\leq \frac{s_{2\alpha}-q_1^{2\alpha}}{n-2}-\left(\frac{nt}{q_1}\right)^{2\alpha/n-2} \\ &\leq \frac{(n-2)\left(s_{2\alpha}-q_1^{2\alpha}\right)-\left(s_\alpha-q_1^\alpha\right)^2}{(n-2)}. \end{aligned}$$

This implies that,

$$s_\alpha \leq q_1^\alpha+\sqrt{(n-3)\left[s_{2\alpha}-q_1^{2\alpha}\right]+(n-2)\left(\frac{tn}{q_1}\right)^{2\alpha/n-2}} \tag{7}$$

and

$$s_\alpha \geq q_1^\alpha + \sqrt{s_{2\alpha} - q_1^{2\alpha} + (n-2)(n-3)\left(\frac{tn}{q_1}\right)^{2\alpha/n-2}}. \tag{8}$$

Consider the function,

$$f(x) = \sqrt{(n-3)\left[s_{2\alpha} - x^{2\alpha}\right] + (n-2)\left(\tfrac{tn}{x}\right)^{2\alpha/n-2}}.$$

It can be easily shown that $f(x)$ is a decreasing function for $x \geq \Delta + 1$, as $0 < \alpha < 1$. By Lemma 2.6, we have $q_1 \geq t_2 \geq \Delta + 1$.Therefore,

$$f(x) \leq f(t_2) = \sqrt{(n-3)\left[s_{2\alpha} - t_2^{2\alpha}\right] + (n-2)\left(\frac{tn}{t_2}\right)^{2\alpha/n-2}}$$

considering this, (7) and Lemma 2.2, we get

$$\begin{aligned} s_\alpha &\leq q_1^\alpha + \sqrt{(n-3)\left[s_{2\alpha} - t_2^{2\alpha}\right] + (n-2)\left(\frac{tn}{t_2}\right)^{2\alpha/n-2}} \\ &\leq n^\alpha + \sqrt{(n-3)\left[s_{2\alpha} - t_2^{2\alpha}\right] + (n-2)\left(\frac{tn}{t_2}\right)^{2\alpha/n-2}}. \end{aligned}$$

In an analogous manner,

$$g(x) = \sqrt{s_{2\alpha} - x^{2\alpha} + (n-2)(n-3)\left(\tfrac{tn}{x}\right)^{2\alpha/n-2}}$$

is a decreasing function for $x \leq n$, as $0 < \alpha < 1$. Therefore,

$$g(x) \geq g(n) = \sqrt{s_{2\alpha} - n^{2\alpha} + (n-2)(n-3)t^{2\alpha/n-2}}$$

considering this, (8) and Lemma 2.6, we get

$$\begin{aligned} s_\alpha &\geq q_1^\alpha + \sqrt{s_{2\alpha} - n^{2\alpha} + (n-2)(n-3)t^{2\alpha/n-2}} \\ &\geq t_2^\alpha + \sqrt{s_{2\alpha} - n^{2\alpha} + (n-2)(n-3)t^{2\alpha/n-2}}. \end{aligned}$$

By this, the first part of the proof is done.

Suppose now that equalities hold in (5). Then all the above inequalities must become equalities. For both lower and upper bounds, by Lemma 2.9 it must be $q_2 = q_3 = ... = q_{n-1}$. Then by Lemmas 2.3 and 2.7, either $G \cong K_n$ or $G \cong K_{1,n-1}$ or $G \cong K_{\Delta,\Delta}$. Moreover $q_1 = t_2 = n$ for both lower and upper bounds. Then by Lemmas 2.1, 2.2 and 2.6 , we conclude that $G \cong K_{1,n-1}$.

Conversely, equalities hold on both sides of (5) for $G \cong K_{1,n-1}$.

(ii) Taking $N = n - 1$ and $x_i = q_i^{2\alpha}$, $i = 2, 3, ..., n$ in Lemma 2.9 and using Lemma 2.5, we obtain

$$\frac{\sum\limits_{2\le i<j\le n}\left(q_i^\alpha - q_j^\alpha\right)^2}{(n-1)(n-2)} \le \frac{s_{2\alpha}-q_1^{2\alpha}}{n-1} - \left(\frac{t_1}{q_1}\right)^{2\alpha/n-1} \le \frac{\sum\limits_{2\le i<j\le n}\left(q_i^\alpha - q_j^\alpha\right)^2}{n-1}.$$

Since $\sum\limits_{i=1}^{n} q_i^{2\alpha} = s_{2\alpha}$, we have

$$\begin{aligned}
\sum_{2\le i<j\le n}\left(q_i^\alpha - q_j^\alpha\right)^2 &= (n-2)\sum_{i=2}^{n} q_i^{2\alpha} - 2\sum_{2\le i<j\le n}(q_i q_j)^\alpha \\
&= (n-2)\left(s_{2\alpha} - q_1^{2\alpha}\right) - \left(\sum_{i=2}^{n} q_i^\alpha\right)^2 + \sum_{i=2}^{n} q_i^{2\alpha} \\
&= (n-1)\left(s_{2\alpha} - q_1^{2\alpha}\right) - \left(s_\alpha - q_1^\alpha\right)^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{(n-1)\left(s_{2\alpha} - q_1^{2\alpha}\right) - \left(s_\alpha - q_1^\alpha\right)^2}{(n-1)(n-2)} &\le \frac{s_{2\alpha}-q_1^{2\alpha}}{n-1} - \left(\frac{t_1}{q_1}\right)^{2\alpha/n-1} \\
&\le \frac{(n-1)\left(s_{2\alpha} - q_1^{2\alpha}\right) - \left(s_\alpha - q_1^\alpha\right)^2}{n-1}.
\end{aligned}$$

This implies that,

$$s_\alpha \le q_1^\alpha + \sqrt{(n-2)\left[s_{2\alpha} - q_1^{2\alpha}\right] + (n-1)\left(\frac{t_1}{q_1}\right)^{2\alpha/n-1}} \tag{9}$$

and

$$s_\alpha \ge q_1^\alpha + \sqrt{s_{2\alpha} - q_1^{2\alpha} + (n-1)(n-2)\left(\frac{t_1}{q_1}\right)^{2\alpha/n-1}}. \tag{10}$$

Consider the function,

$$f(x) = \sqrt{(n-2)\left[s_{2\alpha} - x^{2\alpha}\right] + (n-1)\left(\frac{t_1}{x}\right)^{2\alpha/n-1}}.$$

It can be easily shown that $f(x)$ is a decreasing function for $x \ge \Delta + 1$, as $0 < \alpha < 1$. By Lemma 2.6, we have $q_1 \ge t_2 \ge \Delta + 1$.Therefore,

$$f(x) \le f(t_2) = \sqrt{(n-2)\left[s_{2\alpha} - t_2^{2\alpha}\right] + (n-1)\left(\tfrac{t_1}{t_2}\right)^{2\alpha/n-1}}.$$

Considering this, (9) and Lemma 2.8, we get

$$\begin{aligned}
s_\alpha &\le q_1^\alpha + \sqrt{(n-2)\left[s_{2\alpha} - t_2^{2\alpha}\right] + (n-1)\left(\tfrac{t_1}{t_2}\right)^{2\alpha/n-1}} \\
&\le (2\Delta)^\alpha + \sqrt{(n-2)\left[s_{2\alpha} - t_2^{2\alpha}\right] + (n-1)\left(\tfrac{t_1}{t_2}\right)^{2\alpha/n-1}}.
\end{aligned}$$

In an analogous manner,

$$g(x)=\sqrt{s_{2\alpha}-x^{2\alpha}+(n-1)(n-2)\left(\frac{t_1}{x}\right)^{2\alpha/n-1}}$$

is a decreasing function for $x \leq 2\Delta$, as $0<\alpha<1$. Therefore,

$$g(x)\geq g(2\Delta)=\sqrt{s_{2\alpha}-(2\Delta)^{2\alpha}+(n-1)(n-2)\left(\frac{t_1}{2\Delta}\right)^{2\alpha/n-1}}$$

considering this, (10) and Lemma 2.6, we get

$$\begin{aligned} s_\alpha &\geq q_1^\alpha+\sqrt{s_{2\alpha}-(2\Delta)^{2\alpha}+(n-1)(n-2)\left(\frac{t_1}{2\Delta}\right)^{2\alpha/n-1}} \\ &\geq t_2^\alpha+\sqrt{s_{2\alpha}-(2\Delta)^{2\alpha}+(n-1)(n-2)\left(\frac{t_1}{2\Delta}\right)^{2\alpha/n-1}}. \end{aligned}$$

Hence the inequalities (9) and (10) hold. Either equalities in (9) and (10) hold if and only if $q_1=t_2=2\Delta$ and $q_2=q_3=...=q_n$. From the conditions $q_1=2\Delta$ and $q_2=q_3=...=q_n$, we conclude that $G\cong K_n$. However

$$q_1\left(K_n\right)=2(n-1)\neq n-1+\sqrt{n-1}=t_2\left(K_n\right)$$

Thus, (9) and (10) cannot become equalities. ■

Corollary 3.8 *Let α be a real number with $0<\alpha<1$ and let T be a tree with $n\geq 3$ vertices, maximum degree Δ and let t_2 be given by (1). Then,*

$$\sigma_\alpha(T)=s_\alpha(T)=\begin{cases} \leq & n^\alpha+\sqrt{(n-3)\left[s_{2\alpha}-t_2^{2\alpha}\right]+(n-2)\left(\frac{n}{t_2}\right)^{2\alpha/n-2}} \\ \geq & t_2^\alpha+\sqrt{s_{2\alpha}-n^{2\alpha}+(n-2)(n-3)}. \end{cases}$$

Equality holds on both sides if and only if $G\cong K_{1,n-1}$.

Proof. Every tree is a bipartite. For a tree T, $t=1$. ■

Note that, if we take $\alpha=\frac{1}{2}$, we get the same bounds as in the Theorem 3.1 of the paper [3] for the IE given as follows:

Theorem 3.9 ([3]) *Let G be a connected graph with $n\geq 3$ vertices, m edges, maximum degree Δ and t spanning trees and let $t_1$and t_2 be given by (1).*

(i) If G is bipartite then

$$LEL(G)=IE(G)=\begin{cases} \leq & \sqrt{n}+\sqrt{(n-3)\left(2m-t_2\right)+(n-2)\left(\frac{tn}{t_2T}\right)^{1/n-2}} \\ \geq & \sqrt{t_2}+\sqrt{2m-n+(n-2)(n-3)t^{1/n-2}}. \end{cases}$$

Moreover, equalities hold if and only if $G \cong K_{1,n-1}$.

(ii) If G is non-bipartite, then

$$IE(G) = \begin{cases} < \sqrt{2\Delta} + \sqrt{(n-2)\,(2m - t_2) + (n-1)\left(\frac{t_1}{t_2}\right)^{1/n-1}} \\ > \sqrt{t_2} + \sqrt{2m - 2\Delta + (n-1)(n-2)\left(\frac{t_1}{2\Delta}\right)^{1/n-1}}. \end{cases}$$

For a special case, if we take $\alpha = \frac{1}{2}$ in Corollary 3.8, we get the following result.

Corollary 3.10 *Let* T *be a tree with* $n \geq 3$ *vertices, maximum degree* Δ *and let* t_2 *be given by (1). Then,*

$$LEL(T) = IE(T) = \begin{cases} \leq & \sqrt{n} + \sqrt{(n-3)\,(2n-2-t_2) + (n-2)\left(\frac{n}{t_2}\right)^{1/n-2}} \\ \geq & \sqrt{t_2} + (n-2). \end{cases}$$

Equality holds on both sides if and onlt if $T \cong K_{1,n-1}$.

References

[1] S. Akbari, E. Ghorbani, J. H. Koolen, M. R. Oboudi, On sum of powers of the Laplacian and signless Laplacian eigenvalues of graphs, *El. J. Comb.* **17** (2010) #R115.

[2] Ş. B. Bozkurt, I. Gutman, Estimating the incidence energy, *MATCH Commun. Math. Comput. Chem.* **70** (2013) 143–156.

[3] Ş. B. Bozkurt, D. Bozkurt, On incidence energy, *MATCH Commun. Math. Comput. Chem.* **72** (2014) 215–225.

[4] Y. Chen, L. Wang, Sharp upper bounds for the largest eigenvalue of the signless Laplacian of a graph, *Lin. Algebra Appl.* **433** (2010) 908–913.

[5] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application*, Academic Press, New York, 1980.

[6] D. Cvetković, P. Rowlinson, S. K. Simić, Signless Laplacian of finite graphs, *Lin. Algebra Appl.* **423** (2007) 155–171.

[7] D. Cvetković, S. K. Simić, Towards a spectral theory of graphs based on the signless Laplacian. II, *Lin. Algebra Appl.* **432** (2010) 2257–2272.

[8] K. C. Das, A sharp upper bound for the number of spanning trees of a graph, *Graphs Comb.* **23** (2007) 625–632.

[9] K. C. Das, K. Xu, M, Liu, On sum of powers of the Laplacian eigenvalues of graphs, *Lin. Algebra Appl.* **439** (2013) 3561-3575.

[10] K. C. Das, I. Gutman, A. S. Çevik, On the Laplacian–energy–like invariant, *Lin. Algebra Appl.* **442** (2014) 58–68.

[11] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Ineq.* **5** (2011) 21–31.

[12] I. Gutman, Bounds for total $\pi-$electron energy, *Chem. Phys. Lett.* **24** (1974) 283–285.

[13] I. Gutman, The energy of a graph, *Ber. Math. Statist. Sekt. Forschungsz. Graz* **103** (1978) 1–22.

[14] I Gutman, Total $\pi-$electron energy of benezoid hydrocarbons, *Topics Curr. Chem.* **162** (1992) 26–63.

[15] I. Gutman, The energy of a graph: Old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), *Algebraic Combinatorics and Applications*, Springer, Berlin, 2001, pp. 196–211.

[16] I. Gutman, B. Zhou, Laplacian energy of a graph, *Lin. Algebra Appl.* **414** (2006) 29–37.

[17] I. Gutman, D. Kiani, M. Mirzakhah, On incidence energy of graphs, *MATCH Commun. Math. Comput. Chem.* **62** (2009) 573–580.

[18] I. Gutman, D. Kiani, M. Mirzakhah, B. Zhou, On incidence energy of a graph, *Lin. Algebra Appl.* **431** (2009) 1223–1233.

[19] I. Gutman, B. Zhou, B. Furtula, The Laplacian–energy like invariant is an energy like invariant, *MATCH Commun. Math. Comput. Chem.* **64** (2010) 85–96.

[20] M. Jooyandeh, D. Kiani, M. Mirzakhah, Incidence energy of a graph, *MATCH Commun. Math. Comput. Chem.* **62** (2009) 561–572.

[21] H. Kober, On the arithmetic and geometric means and the Hölder inequality, *Proc. Amer. Math. Soc.* **59** (1958) 452–459.

[22] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[23] J. Liu, B. Liu, A Laplacian–energy like invariant of a graph, *MATCH Commun. Math. Comput. Chem.* **59** (2008) 355–372.

[24] B. Liu, Y. Huang, Z. You, A survey on the Laplacian–energy like invariant, *MATCH Commun. Math. Comput. Chem.* **66** (2011) 713–730.

[25] M. Liu, B. Liu On sum of powers of the signless Laplacian eigenvalues of graphs, *Hacettepe J. Math. Stat.* **41** (2012) 527–536.

[26] R. Merris, Laplacian matrices of graphs: a survey, *Lin. Algebra Appl.* **197/198** (1994) 143–176.

[27] R. Merris, A survey of graph Laplacians, *Lin. Multilin. Algebra* **68** (1995) 19–31.

[28] V. Nikiforov, The energy of graphs and matrices, *J. Math. Anal. Appl.* **326** (2007) 1472–1475.

[29] G. X. Tian, T. Z. Huang, B. Zhou, A note on sum of powers of the Laplacian eigenvalues of bipartite graphs, *Lin. Algebra Appl.* **430** (2009) 2503–2510.

[30] L. You, J. Yang, Notes on the sum of powers of the signless Laplacian eigenvalues of graphs, *Ars Comb.* **117** (2014) 85–94.

[31] G. Yu, Y. Wu, J. Shu, Sharp bounds on the signless Laplacian spectral radii of graphs, *Lin. Algebra Appl.* **434** (2011) 683–687.

[32] Y. Zhang, X. Liu, B. Zhang, X. Yong, The lollipop graphs is determined by its Q-spectrum, *Discr. Math.* **309** (2009) 3364–3369.

[33] B. Zhou, On sum of powers of the Laplacian eigenvalues of graphs, *Lin. Algebra Appl.* **429** (2008) 2239–2246.

[34] B. Zhou, Energy of a graph, *MATCH Commun. Math. Comput. Chem.* **51** (2004) 111–118.

[35] B. Zhou, A. Ilić, On the sum of powers of Laplacian eigenvalues of bipartite graphs, *Czech. Math. J.* **60** (2010) 1161–1169.