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# (Signless) Laplacian Borderenergetic Graphs and the Join of Graphs*

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**Abstract**

A graph $G$ of order $n$ is *L-borderenergetic* if it has the same Laplacian energy as the complete graph $K_n$ does. Similarly, this concept can be extended to signless Laplacian energy of a graph, i.e., if a graph has the same signless Laplacian energy as the complete graph $K_n$, then it is called *Q-borderenergetic*. In this paper, we mainly survey a class of join of graphs and check that whether they are L-borderenergetic or Q-borderenergetic. Moreover, in this paper, we will show that a main result in [B. Deng, X. Li, More on L-Borderenergetic Graphs, MATCH Commun. Math. Comput. Chem. 77 (2017) 115–127.] is a directed corollary of our results.

## 1 Introduction

All graphs considered in this paper are simple and undirected. Let $G$ be a graph with its edge set $E(G)$ and vertex set $V(G)$. The complete graph of order $n$ is denoted by $K_n$. The union of two vertex-disjoint graphs $G_1$ and $G_2$ is denoted by $G_1 \cup G_2$. Let $G_1 \nabla G_2$ be the join of $G_1$ and $G_2$, obtained from the union of $G_1$ and $G_2$ by joining each vertex of $G_1$ and each vertex of $G_2$. For terminology and notation not given here, we refer to [1].

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Let $A(G)$ be an adjacency matrix of $G$ and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of the adjacency matrix $A(G)$. If $D(G)$ is the diagonal matrix of the vertex degrees of $G$, $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the Laplacian matrix and signless Laplacian matrix of $G$, respectively. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $q_1 \geq q_2 \geq \cdots \geq q_n = 0$ be the eigenvalues of $L(G)$ and $Q(G)$, respectively. The energy of a graph $G$, denoted by $\mathcal{E}(G)$, is defined as [8,9]

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| \,.$$

For additional information on graph energy and its applications in chemistry, we refer to [9–11,15].

Recently, Gong et al. [7] proposed the concept of *borderenergetic* graphs, namely graphs of order $n$ satisfying $\mathcal{E}(G) = 2(n-1)$. The corresponding results on borderenergetic graphs can be seen in [4,13,16,17,19].

For the Laplacian energy of a graph $G$ [3,12], denoted by $LE(G)$, F. Tura [20] proposed the concept of *L-borderenergetic* graphs. That is, a graph $G$ of order $n$ is $L$-borderenergetic if $LE(G) = LE(K_n)$, where $LE(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}|$ and $\overline{d}$ is the average degree of $G$. Note that $LE(K_n) = 2(n-1)$. Several classes of L-borderenergetic graphs [20] are obtained including the result that for each integer $r \geq 1$, there are $2r+1$ graphs, of order $n = 4r+4$, which are pairwise L-noncospectral and L-borderenergetic graphs. Let $S_n^1$ be the graph obtained from an $n$-order star $S_n$ by adding an edge. Obviously, $S_n^1$ is an unicyclic and threshold graph (see Figure 1). A main result on the graph $S_n^1$ in [5] is as follow(see Theorem 1). More results on L-borderenergetic graphs, we can refer to [5,6,18,20–22].

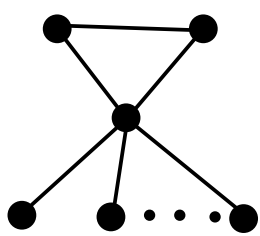

**Figure 1.** The graph $S_n^1$.

**Theorem 1.** *[5] The graph $S_n^1$ is L-borderenergetic.*

It is interesting to construct more different connected L-borderenergetic graphs. In this paper, we use the join of graphs to construct a class of L-borderenergetic graphs. That is the graph $K_1\nabla(K_t \cup pK_{t-1})$, which is shown in Figure 2. Moreover, one can check that Theorem 1 is a directed corollary of our result(Theorem 3).

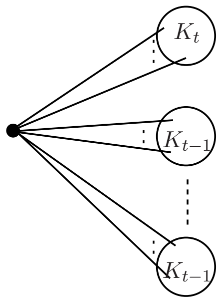


**Figure 2.** The graph $K_1\nabla(K_t \cup pK_{t-1})$.

Similarly, this concept can be extended to signless Laplacian energy of a graph $G$ [2], denoted by $QE(G)$, where $QE(G) = \sum_{i=1}^{n} |q_i - \overline{d}|$. Y. Hou et al. [14] proposed the concept of Q-borderenergetic graphs. That is, if a graph has the same signless Laplacian energy as the complete graph $K_n$, then it is called *Q-borderenergetic*. In fact, it is not hard to find a Q-borderenergetic graph. When a connected regular graph $G$ is L-borderenergetic, we have $\mathcal{E}(G) = LE(G) = QE(G) = 2(n-1)$. For example, $H_1$ and $H_2$ (see Figure 3) are two 4-regular $Q$-borderenergetic graphs with 9 vertices. And their signless Laplacian spectra are $\{2^{(3)}, 3^{(2)}, 5^{(2)}, 6, 8\}$ and $\{2^{(4)}, 5^{(4)}, 8\}$, respectively. Furthermore, in section 3 of this paper, we consider that whether this kind of graphs $K_1\nabla(K_t \cup pK_{t-1})$ are Q-borderenergetic and we will present the corresponding results.

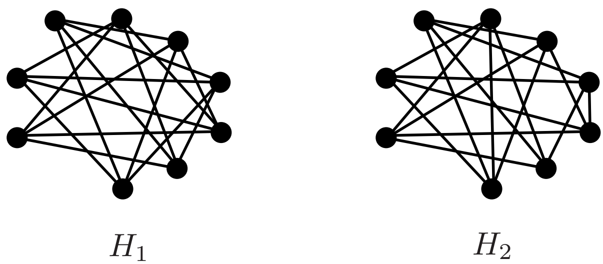


**Figure 3.** Two 4-regular $Q$-borderenergetic graphs $H_1$ and $H_2$.

## 2 Laplacian borderenergetic graphs

In this section, we show that the graph $K_1\nabla(K_t \cup pK_{t-1})$ is L-borderenergetic by using Lemma 2.

**Lemma 2.** *[20]. Let $G_1$ and $G_2$ be graphs on $n_1$ and $n_2$ vertices, respectively. Let $L_1$ and $L_2$ be the Laplacian matrices for $G_1$ and $G_2$, respectively, and let $L$ be the Laplacian matrix for $G_1\nabla G_2$. If $0 = \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n_1}$ and $0 = \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{n_2}$ are the eigenvalues of $L_1$ and $L_2$, respectively. Then the eigenvalues of $L$ are $\{0, n_2+\alpha_2, n_2+\alpha_3, \cdots, n_2+\alpha_{n_1}, n_1+\beta_2, n_1+\beta_3, \cdots, n_1+\beta_{n_2}, n_1+n_2\}$.*

**Theorem 3.** *For each integer $p \geq 1$ and $t \geq 2$, the graph $K_1\nabla(K_t \cup pK_{t-1})$ is L-borderenergetic.*

*Proof.* For $K_1$ and $K_t \cup pK_{t-1}$, their Laplacian spectra are respectively $\{0\}$ and $\{0^{(p+1)}, t^{(t-1)}, (t-1)^{(p(t-2))}\}$. By Lemma 2, the Laplacian spectrum of $K_1\nabla(K_t \cup pK_{t-1})$ is

$$\{0, (t+1)^{(t-1)}, 1^{(p)}, t^{(p(t-2))}, 1+t+p(t-1)\}.$$

Note that the average degree of $K_1\nabla(K_t \cup pK_{t-1})$ is $t$ and the vertex number of $K_1\nabla(K_t \cup pK_{t-1})$ is $t+1+p(t-1)$. So

$$\begin{aligned} LE(K_1\nabla(K_t \cup pK_{t-1})) &= t+(t-1)+p(t-1)+1+p(t-1) \\ &= 2((t+1)+p(t-1)-1). \end{aligned}$$

Hence, $K_1\nabla(K_t \cup pK_{t-1})$ is L-borderenergetic. ■

It is easy to check that $K_1\nabla(K_t \cup pK_{t-1}) \cong S_n^1$ as $t = 2$. Then we see that Theorem 1 is a directed corollary of Theorem 3.

**Corollary 4.** $K_1\nabla(K_2 \cup pK_1)(\cong S_n^1)$ *is L-borderenergetic.*

## 3 Signless Laplacian borderenergetic graphs

Moreover, in this section, we consider that whether the graph $K_1\nabla(K_t \cup pK_{t-1})$ is Q-borderenergetic. In fact, when $t = 3$, Y. Hou et al. [14] have proved that the graph $K_1\nabla(K_t \cup pK_{t-1})$ is Q-borderenergetic, which is also verified in the proof of Theorem 5. But the cases $t = 2$ and $t > 3$ for the graph $K_1\nabla(K_t \cup pK_{t-1})$ are not been discussed. Then the corresponding cases will be given below.

**Theorem 5.** *For each integer $p \geq 1$, when $t = 2$ or $t > 3$, the graph $K_1\nabla(K_t \cup pK_{t-1})$ is not Q-borderenergetic.*

*Proof.* Through suitable labeling, $Q(K_1\nabla(K_t \cup pK_{t-1}))$ has the form as follow

$$\begin{pmatrix}
t+p(t-1) & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & & & & & & \cdots & 1 \\
1 & t & 1 & 1 & \cdots & 1 & & & & & & & & & \\
1 & 1 & t & 1 & \cdots & 1 & & & & & & & & & \\
\vdots & & & \ddots & & \vdots & & & & & & & & & \\
1 & \cdots & & & & t & & & & & & & & & \\
1 & & & & & & t-1 & 1 & \cdots & 1 & & & & & \\
1 & & & & & & 1 & t-1 & \cdots & 1 & & & & & \\
\vdots & & & & & & \vdots & & \ddots & \vdots & & & & & \\
1 & & & & & & 1 & \cdots & & t-1 & & & & & \\
\vdots & & & & & & & & & & \ddots & & & & \\
1 & & & & & & & & & & & t-1 & 1 & \cdots & 1 \\
1 & & & & & & & & & & & 1 & t-1 & \cdots & 1 \\
\vdots & & & & & & & & & & & \vdots & & \ddots & \vdots \\
1 & & & & & & & & & & & 1 & \cdots & & t-1
\end{pmatrix}$$

And its signless Laplacian characteristic polynomial is $|xI - Q(K_1\nabla(K_t \cup pK_{t-1}))|$, that is, the determinant of the following matrix.

$$\begin{pmatrix}
a_1 & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & & & & & & \cdots & -1 \\
-1 & b_1 & -1 & -1 & \cdots & -1 & & & & & & & & & \\
-1 & -1 & b_1 & -1 & \cdots & -1 & & & & & & & & & \\
\vdots & & & \ddots & & \vdots & & & & & & & & & \\
-1 & \cdots & & & & b_1 & & & & & & & & & \\
-1 & & & & & & c_1 & -1 & \cdots & -1 & & & & & \\
-1 & & & & & & -1 & c_1 & \cdots & -1 & & & & & \\
\vdots & & & & & & \vdots & & \ddots & \vdots & & & & & \\
-1 & & & & & & -1 & \cdots & & c_1 & & & & & \\
\vdots & & & & & & & & & & \ddots & & & & \\
-1 & & & & & & & & & & & c_1 & -1 & \cdots & -1 \\
-1 & & & & & & & & & & & -1 & c_1 & \cdots & -1 \\
\vdots & & & & & & & & & & & \vdots & & \ddots & \vdots \\
-1 & & & & & & & & & & & -1 & \cdots & & c_1
\end{pmatrix}$$

where $a_1 = x - (t + p(t-1))$, $b_1 = x - t$ and $c_1 = x - (t-1)$.

Let $C_i$ be the $i$-th column of the determinant of above matrix. Then we first compute $C_1 + \frac{1}{x-2t+1}(C_2 + C_3 + \cdots + C_{t+1})$ so that we get $|xI - Q(K_1\nabla(K_t \cup pK_{t-1}))|$ is equal to the determinant of below matrix.

$$\begin{pmatrix}
a_2 & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & & & & & & \cdots & -1 \\
0 & b_1 & -1 & -1 & \cdots & -1 & & & & & & & & & \\
0 & -1 & b_1 & -1 & \cdots & -1 & & & & & & & & & \\
\vdots & & & \ddots & & \vdots & & & & & & & & & \\
0 & \cdots & & & & b_1 & & & & & & & & & \\
-1 & & & & & & c_1 & -1 & \cdots & -1 & & & & & \\
-1 & & & & & & -1 & c_1 & \cdots & -1 & & & & & \\
\vdots & & & & & & \vdots & & \ddots & \vdots & & & & & \\
-1 & & & & & & -1 & \cdots & & c_1 & & & & & \\
\vdots & & & & & & & & & & \ddots & & & & \\
-1 & & & & & & & & & & & c_1 & -1 & \cdots & -1 \\
-1 & & & & & & & & & & & -1 & c_1 & \cdots & -1 \\
\vdots & & & & & & & & & & & \vdots & & \ddots & \vdots \\
-1 & & & & & & & & & & & -1 & \cdots & & c_1
\end{pmatrix}$$

where $a_2 = x - (t + p(t-1)) - \frac{t}{x-2t+1}$, $b_1 = x - t$ and $c_1 = x - (t-1)$.

Next we go on performing $C_1 + \frac{1}{x-2t+3}(C_{t+2} + \cdots + C_{2t})$, $C_1 + \frac{1}{x-2t+3}(C_{2t+1} + \cdots + C_{3t-1})$, $\cdots$, $C_1 + \frac{1}{x-2t+3}(C_{3+p(t-1)} + \cdots + C_{1+t+p(t-1)})$. Then directly expanding along the first column, it arrives at

$$\begin{aligned}
& |xI - Q(K_1 \nabla (K_t \cup pK_{t-1}))| \\
= \ & [x - (t + p(t-1)) - \frac{t}{x-2t+1} - \frac{(t-1)p}{x-2t+3}] \\
& (x-2t+1)(x-t+1)^{t-1}(x-2t+3)^p(x-t+2)^{p(t-2)} \\
= \ & [x^3 - (pt + 5t - p - 4)x^2 + (4pt^2 + 8t^2 - 9pt - 13t + 5p + 3)x \\
& -2(t-1)(2pt^2 + 2t^2 - 5pt - 3t + 2p)] \\
& (x-t+1)^{t-1}(x-2t+3)^{p-1}(x-t+2)^{p(t-2)}.
\end{aligned}$$

We can see that the cubic polynomial $x^3 - (pt+5t-p-4)x^2 + (4pt^2+8t^2-9pt-13t+5p+3)x - 2(t-1)(2pt^2+2t^2-5pt-3t+2p)$ is as a factor of above signless Laplacian characteristic polynomial. Set $x_1, x_2$ and $x_3$ are its eigenvalues, respectively. Since a signless Laplacian matrix is positive semidefinite, all its eigenvalues are nonnegative real numbers. Then we have $x_i \geq 0$ $(i = 1, 2, 3)$. By Vieta theorem, it attains $x_1 + x_2 + x_3 = pt + 5t - p - 4$. From the signless Laplacian characteristic polynomial of $K_1 \nabla (K_t \cup pK_{t-1})$, its signless Laplacian spectrum is

$$\{x_1,\ x_2,\ x_3,\ (t-1)^{(t-1)},\ (2t-3)^{(p-1)},\ (t-2)^{(p(t-2))}\}.$$

By the definition of signless Laplacian energy, we get

$$\begin{aligned} QE(K_1\nabla(K_t \cup pK_{t-1})) &= |x_1 - t| + |x_2 - t| + |x_3 - t| + \\ &\quad |t-1| + |(t-3)(p-1)| + |2p(t-2)|. \end{aligned} \tag{1}$$

Then for the integer $t$, we discuss the following cases.

Case 1. $t = 2$. Then we have

$$\begin{aligned} & |xI - Q(K_1\nabla(K_2 \cup pK_1))| \\ = \; & [x^3 - (p+6)x^2 + (3p+9)x - 4](x-1)^p. \end{aligned} \tag{2}$$

If $K_1\nabla(K_2 \cup pK_1)$ is Q-borderenergetic, we have

$$QE(K_1\nabla(K_2 \cup pK_1)) = 2(2+p). \tag{3}$$

Suppose $x_1 \geq x_2 \geq x_3$ and note that $x_1 > 2$ and $x_1 \leq p+2$. By (2), we see that $x_i \neq 0$, $(i = 1, 2, 3)$ and $x_1 + x_2 + x_3 = p + 6$. Based on the relations between $x_2, x_3$ and the average degree of $K_1\nabla(K_2 \cup pK_1)$, we distinguish the following cases.

Subcase 1.1. $x_2 \geq 2$ and $x_3 \geq 2$. Then $QE(K_1\nabla(K_2 \cup pK_1)) = x_1 + x_2 + x_3 - 6 + p = 2p < 2(2+p)$.

Subcase 1.2. $x_2 \geq 2$ and $x_3 < 2$. Then $QE(K_1\nabla(K_2 \cup pK_1)) = x_1 + x_2 - x_3 - 2 + p < x_1 + x_2 + x_3 - 2 + p = 2(2+p)$.

Subcase 1.3. $x_2 < 2$ and $x_3 < 2$. Then $QE(K_1\nabla(K_2 \cup pK_1)) = x_1 - x_2 - x_3 + 2 + p = 2x_1 - p - 6 + 2 + p = 2x_1 - 4 < 2(2+p)$.

Thus, all above subcases make a contradiction with (3), which means that $K_1\nabla(K_t \cup pK_{t-1})$ is not Q-borderenergetic as $t = 2$.

Case 2. $t = 3$. It is easy to check that its signless Laplacian spectrum is

$$\{p + \frac{9}{2} \pm \frac{1}{2}\sqrt{4p^2 - 4p + 9},\ 1^{(p)},\ 2^{(3)},\ 3^{(p-1)}\},$$

and $QE(K_1\nabla(K_3 \cup pK_2)) = 4p + 6 = 2((2p+4) - 1)$. Thus, $K_1\nabla(K_t \cup pK_{t-1})$ is Q-borderenergetic as $t = 3$.

Case 3. $t \geq 4$. If $t = 4$ and $p = 1$, we can find that $QE(K_1\nabla(K_4 \cup K_3)) \approx 14.74 \neq 2(8-1)$, i.e., $K_1\nabla(K_4 \cup K_3)$ is not Q-borderenergetic. Otherwise, we suppose that $K_1\nabla(K_t \cup pK_{t-1})$ is Q-borderenergetic. So by (1), we get

$$QE(K_1\nabla(K_t \cup pK_{t-1})) = 2((1 + t + p(t-1)) - 1)$$

$$\geq \quad |x_1 + x_2 + x_3 - 3t + (t-1) + (t-3)(p-1) + 2p(t-2)|$$
$$= \quad 2(2pt + t - 4p - 1). \tag{4}$$

Hence, we obtain $3 + \frac{1}{p} \geq t$ from (4), which is a contradiction. Thus, $K_1 \nabla (K_t \cup pK_{t-1})$ is not Q-borderenergetic in the case of $t \geq 4$. ■

# References


[1] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, New York, 2008.

[2] D. Cvetković, S. Simić, Towards a spectral theory of graphs based on the signless Laplacian, *Publ. Inst. Math. Belgrade* **85** (2009) 19–33.

[3] K. C. Das, S. A. Mojallal, I. Gutman, On Laplacian energy in terms of graph invariants, *Appl. Math. Comput.* **259** (2015) 470–479.

[4] B. Deng, X. Li, I. Gutman, More on borderenergetic graphs, *Lin. Algebra Appl.* **497** (2016) 199–208.

[5] B. Deng, X. Li, More on L-borderenergetic graphs, *MATCH Commun. Math. Comput. Chem.* **77** (2017) 115–127.

[6] B. Deng, X. Li, J. Wang, Further results on L-borderenergetic graphs, *MATCH Commun. Math. Comput. Chem.* **77** (2017) 607–616.

[7] S. C. Gong, X. Li, G. H. Xu, I. Gutman, B. Furtula, Borderenergetic graphs, *MATCH Commun. Math. Comput. Chem.* **74** (2015) 321–332.

[8] I. Gutman, Acyclic systems with extremal Hückel $\pi$-electron energy, *Theor. Chim. Acta.* **45** (1977) 79–87.

[9] I. Gutman, The energy of a graph, *Ber. Math. Stat. Sekt. Forschungsz. Graz* **103** (1978) 1–22.

[10] I. Gutman, X. Li, J. Zhang, Graph energy, in: M. Dehmer, F. Emmert–Streib (Eds.), *Analysis of Complex Networks. From Biology to Linguistics*, Wiley–VCH, Weinheim, 2009, pp. 145–174.

[11] I. Gutman, O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer, Berlin, 1986.

[12] I. Gutman, B. Zhou, Laplacian energy of a graph, *Lin. Algebra Appl.* **414** (2006) 29–37.

[13] Y. Hou, Q. Tao, Borderenergetic threshold graphs, *MATCH Commun. Math. Comput. Chem.* **75** (2016) 253–262.

[14] Q. Tao, Y. Hou, Q-borderenergetic graphs, *AKCE Int. J. Graphs Comb.*, submitted.

[15] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[16] X. Li, M. Wei, S. Gong, A computer search for the borderenergetic graphs of order 10, *MATCH Commun. Math. Comput. Chem.* **74** (2015) 333–342.

[17] X. Li, M. Wei, X. Zhu, Borderenergetic graphs with small maximum or large minimum degrees, *MATCH Commun. Math. Comput. Chem.* **77** (2016) 25–36.

[18] L. Lu, Q. Huang, On the existence of non-complete L-borderenergetic graphs, *MATCH Commun. Math. Comput. Chem.* **77** (2017) 625–634.

[19] Z. Shao, F. Deng, Correcting the number of borderenergetic graphs of order 10, *MATCH Commun. Math. Comput. Chem.* **75** (2016) 263–266.

[20] F. Tura, L-borderenergetic graphs, *MATCH Commun. Math. Comput. Chem.* **77** (2017) 37–44.

[21] F. Tura, L-borderenergetic graphs and normalized Laplacian energy, *MATCH Commun. Math. Comput. Chem.* **77** (2017) 617–624.

[22] Q. Tao, Y. Hou, A computer search for the L-borderenergetic graphs, *MATCH Commun. Math. Comput. Chem.* **77** (2017) 595–606.