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# Numerical Study of Oxygen and Carbon Substrate Concentrations in Excess Sludge Production Using Sinc–Collocation Method

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**Abstract**

A numerical procedure is developed to obtain the solution of a mathematical model that relates the concentrations of carbon substrate and oxygen within a microbial floc particle. This model can be reduced to a system of two coupled nonlinear singular differential equations. Our approach is based on sinc-collocation method. Sinc based methods are characterized by exponentially decaying errors associated with their approximations. Also, approximation by sinc methods handles singularities in the problem. Properties of the sinc functions are utilized to reduce the computation of this model to systems of algebraic equations. The method is easy to implement and the results are compared with some well-known results which show that they are accurate.

## 1 Introduction

Conventional activated sludge is one of the most widely used methods for treatment of organic wastes. Air or oxygen is introduced into the screened and primary treated sewage or industrial wastewater combined with microorganisms to convert dissolved and suspended biodegradable carbonaceous organic contaminants into biological floc (sludge), water and $CO_2$ gas. One of the challenging issues of conventional activated sludge is high sludge production. The wide application of the process and more stringent wastewater regulations

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have amplified the problem of excess sludge management and hygienic disposal. About 50% to 60% of the total operating cost in a sewage treatment plant is accounted for excess sludge storage, transport, digestion and disposal [17]. Several studies have shown that carbon substrate and oxygen concentration are two main factors influencing excess sludge production [1].

A mathematical model that relates the concentration of the carbon substrate and the concentration of oxygen within a microbial floc particle is given in [1,5,7,16] as

$$\frac{d^2u}{dx^2} + \frac{2}{x}\frac{du}{dx} = -\alpha_2 + F_1(u(x), v(x)), \tag{1}$$

$$\frac{d^2v}{dx^2} + \frac{2}{x}\frac{dv}{dx} = F_2(u(x), v(x)), \tag{2}$$

subject to boundary conditions:

$$u'(0) = 0, \; u(1) = 1, \qquad v'(0) = 0, \; v(1) = 1. \tag{3}$$

Here, $u(x)$ and $v(x)$ are the dimensionless concentrations of carbon substrate and oxygen, respectively. Also, $x$ denotes the radius of a spherical floc particle and $F_1, F_2$ are given by

$$F_1(u(x), v(x)) = \alpha_1 \frac{u(x)v(x)}{(\ell_1 + u(x))(m_1 + v(x))} + \alpha_3 \frac{u(x)v(x)}{(\ell_2 + u(x))(m_2 + v(x))}, \tag{4}$$

$$F_2(u(x), v(x)) = \alpha_4 \frac{u(x)v(x)}{(\ell_1 + u(x))(m_1 + v(x))} + \alpha_5 \frac{u(x)v(x)}{(\ell_2 + u(x))(m_2 + v(x))}, \tag{5}$$

where $\ell_1, \ell_2, m_1, m_2$ and $\alpha_i, \; i = 1, 2, .., 5$ are some constants. For a complete model's description, we refer the interested reader to [1,7].

In [7] the Adomian decomposition method is applied to derive a solution of this problem. In [5] the Adomian decomposition method combined with the Duan-Rach modified recursion scheme is applied to solve this problem. Also, the authors of [16] used the variational iteration method for the solution of this problem. Moreover, for similar problems, we refer to [3,19–21].

The main purpose of this work is to develop sinc-collocation method for numerical solution of problem (1)-(3). Our approach consists of reducing the solution of this problem to a set of algebraic equations by expanding $u(x)$ and $v(x)$ as sinc functions with unknown coefficients. The properties of sinc functions are then used to evaluate the unknown coefficients. A general review of sinc function approximation is given in [6,15]. As pointed out in [6,15], there are several advantages to approximate by sinc functions: (i) it may readily handle the singularity, (ii) sinc numerical methods are easily implemented and give good accuracy. These methods are characterized by exponentially decaying errors. In the last three decades or so, sinc methods are widely used in various problems such squeezing flow [12], boundary value problems [14, 18], fractional convection-diffusion equations [13], Bagley-Torvik equation [2], Troesch's problem [8], Volterra's population model [11], Thomas-Fermi equation [9], Falkner-Skan boundary-layer equation [10] and Schrödinger equation [4].

The organization of this paper is as follows: in the next section, some preliminary results of sinc functions are given. In Section 3, we apply the sinc-collocation method on the studied model. Results and discussion of the proposed method is shown in Section 4. A brief conclusion is given in Section 5. Note that we have computed the numerical results by Maple programming.

## 2 Sinc function approximation

The books [6, 15] have provided overviews of methods based on sinc functions. We recall here the main properties of sinc functions required for our subsequent development.

The sinc function is defined on $-\infty < x < \infty$, by

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

For each $k = 0, \pm 1, \pm 2, \ldots$ and $h > 0$ the sinc basis functions are defined on the whole real line by

$$S(k,h)(x) \equiv \operatorname{sinc}\left(\frac{x-kh}{h}\right) = \begin{cases} \frac{\sin[\frac{\pi}{h}(x-kh)]}{\frac{\pi}{h}(x-kh)}, & x \neq kh, \\ 1, & x = kh. \end{cases} \tag{6}$$

The sinc functions form an interpolatory set of functions, i.e.,

$$S(k,h)(jh) = \delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

If $f$ is an analytic function defined over the whole real line, then the Whittaker cardinal expansion of $f$ is defined by

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh) \operatorname{Sinc}\left(\frac{x-kh}{h}\right),$$

whenever this series converges. The properties of the Whittaker cardinal expansion have been studied in [6]. These properties are derived in the infinite strip $D_S$ on the complex

$w$-plane, where for $d > 0$,

$$D_S = \left\{ w = t + is \ : |s| < d \leq \frac{\pi}{2} \right\}.$$

In this paper, we use approximations on the interval $(0, 1)$. Here, we consider the conformal mapping

$$w = \phi(z) = \ln \left( \frac{z}{1 - z} \right),$$

which transforms the eye-shaped domain, $D_E$, in the z-plane, onto $D_S$, where

$$D_E = \left\{ z = x + iy \ : \left| arg \left( \frac{z}{1 - z} \right) \right| < d \leq \frac{\pi}{2} \right\}.$$

The sinc basis functions over $(0, 1)$ are given by

$$S_k(x) \equiv S(k, h) \circ \phi(x) = \operatorname{sinc} \left( \frac{\phi(x) - kh}{h} \right), \tag{7}$$

where $S(k, h) \circ \phi(x)$ is defined by $S(k, h)(\phi(x))$. The inverse map of $w = \phi(z)$ is

$$z = \phi^{-1}(w) = \frac{\exp(w)}{1 + \exp(w)}.$$

Therefore, we may define the inverse images of the real line and the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ as

$$\Gamma = \{\psi(t) \in D_E \ : -\infty < t < \infty\} = (0, 1),$$

and

$$x_k = \psi(kh) = \frac{e^{kh}}{1 + e^{kh}}, \qquad k = 0, \pm 1, \pm 2, \ldots \tag{8}$$

respectively.

**Definition 2.1.** *Let $B(D_E)$ be the class of functions $F$ which are analytic in $D_E$, satisfy*

$$\int_{\psi(t+L)} |F(z)dz| \longrightarrow 0, \qquad t \longrightarrow \pm\infty,$$

*where $L = \left\{ iv \ : |v| < d \leq \frac{\pi}{2} \right\}$, and on the boundary of $D_E$, (denoted $\partial D_E$), satisfy*

$$N(F) = \int_{\partial D_E} |F(z)dz| < \infty.$$

The next theorem, whose proof can be found in [15], shows the exponential convergence of the sinc approximation for function in $B(D_E)$.

**Theorem 2.2.** *If* $\phi' F \in B(D_E)$ *then for all* $x \in \Gamma$

$$\left| F(x) - \sum_{k=-\infty}^{\infty} F(x_k) S(k,h) \circ \phi(x) \right| \leq \frac{N(F\phi')}{2\pi d \sinh(\pi d/h)}$$

$$\leq \frac{2N(F\phi')}{\pi d} e^{-\pi d/h}.$$

*Moreover, if* $|F(x)| \leq C e^{-\alpha|\phi(x)|}$, $x \in \Gamma$, *for some positive constants* $C$ *and* $\alpha$, *and if the selection* $h = \sqrt{\pi d/\alpha N} \leq 2\pi d/\ln 2$, *then*

$$\left| F(x) - \sum_{k=-N}^{N} F(x_k) S(k,h) \circ \phi(x) \right| \leq C_2 \sqrt{N} \exp(-\sqrt{\pi d \alpha N}), \quad x \in \Gamma,$$

*where* $C_2$ *depends only on* $F$, $d$ *and* $\alpha$.

Also, we need derivatives of $S(k,h) \circ \phi(x)$ at some points $x_j$. The expressions required for this paper are [6, 15]

$$\delta_{k,j}^{(0)} = [S(k,h) \circ \phi(x)]|_{x=x_j} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{9}$$

$$\delta_{k,j}^{(1)} = \frac{d}{d\phi}[S(k,h) \circ \phi(x)]|_{x=x_j} = \frac{1}{h} \begin{cases} 0, & k = j, \\ \frac{(-1)^{j-k}}{j-k}, & k \neq j. \end{cases} \tag{10}$$

$$\delta_{k,j}^{(2)} = \frac{d^2}{d\phi^2}[S(k,h) \circ \phi(x)]|_{x=x_j} = \frac{1}{h^2} \begin{cases} \frac{-\pi^2}{3}, & k = j, \\ \frac{-2(-1)^{j-k}}{(j-k)^2}. & k \neq j. \end{cases} \tag{11}$$

# 3 Discretization of problem (1)–(3)

In this section, we use sinc-collocation method to approximate solutions of problem (1)-(3). The sinc basis functions $S_k(x)$ are not differentiable when $x$ tends to 0. Thus, we modify the sinc basis functions as $\frac{S_k(x)}{\phi'(x)}$. Now the first derivative of the modified sinc basis functions is defined as $x$ approaches 0 and is equal to 0. We also add boundary basis functions that are cubic polynomials. These polynomials, which are obtained by Hermite interpolation at the nodes 0 and 1, are given by

$$\mu_1(x) = (2x+1)(1-x)^2, \quad \mu_2(x) = x^2(3-2x), \quad \mu_3(x) = x^2(x-1). \tag{12}$$

In order to discretize Eqs. (1)-(2) by using sinc-collocation method, first of all, we approximate $u(x)$ and $v(x)$ as

$$u_N(x) = U_N(x) + p(x), \quad v_N(x) = V_N(x) + q(x), \tag{13}$$

where

$$U_N(x) = \sum_{k=-N}^{N} c_k \frac{S_k(x)}{\phi'(x)} = x(1-x) \sum_{k=-N}^{N} c_k S_k(x), \tag{14}$$

and

$$V_N(x) = \sum_{k=-N}^{N} d_k \frac{S_k(x)}{\phi'(x)} = x(1-x) \sum_{k=-N}^{N} d_k S_k(x). \tag{15}$$

Also, $p(x)$ and $q(x)$ are linear combination of $\mu_i(x), i = 1, 2, 3$. Hence, the boundary terms $p(x)$ and $q(x)$ corresponding to the boundary conditions (3) are chosen in the following form, so that $u_N(x)$ and $v_N(x)$ satisfies Eq. (3):

$$p(x) = c_{-N-1}\mu_1(x) + \mu_2(x) + c_{N+1}\mu_3(x), \tag{16}$$

$$q(x) = d_{-N-1}\mu_1(x) + \mu_2(x) + d_{N+1}\mu_3(x). \tag{17}$$

In Eqs. (16) and (17), $c_{-N-1}, c_{N+1}, d_{-N-1}, d_{N+1}$ are constants to be determined. The $2N + 3$ coefficients $\{c_k\}_{k=-N-1}^{N+1}$ and the $2N + 3$ coefficients $\{d_k\}_{k=-N-1}^{N-1}$ are determined by substituting $u_N(x)$ and $v_N(x)$ into Eqs. (1), (2) and evaluating the result at the sinc points:

$$x_j = \frac{e^{jh}}{1+e^{jh}}\ , \qquad j = -N-1, \ldots, N+1. \tag{18}$$

Clearly, by using Eqs. (14), (15) and (9) we have

$$\begin{cases} U_N(x_j) = c_j/\phi'(x_j), \ V_N(x_j) = d_j/\phi'(x_j), & j = -N, ..., N, \\ \quad U_N(x_j) = V_N(x_j) = 0, & j = -N-1, N+1. \end{cases} \tag{19}$$

Also, employing Eqs. (9), (10) and (14) we obtain

$$\begin{aligned} U_N'(x_j) &= \sum_{k=-N}^{N} c_k \left[\frac{d}{dx}\left(\frac{S_k(x)}{\phi'(x)}\right)\right]_{x=x_j} \\ &= \sum_{k=-N}^{N} c_k \left[\left(\frac{-\phi''(x)}{\phi'(x)^2}\right) S_k(x) + \frac{d}{d\phi} S_k(x)\right]_{x=x_j} \\ &= \sum_{k=-N}^{N} c_k \left\{\left(\frac{-\phi''(x_j)}{\phi'(x_j)^2}\right) \delta_{kj}^{(0)} + \delta_{kj}^{(1)}\right\}. \end{aligned} \tag{20}$$

In a similar way, we get

$$V_N'(x_j) = \sum_{k=-N}^{N} d_k \left\{\left(\frac{-\phi''(x_j)}{\phi'(x_j)^2}\right) \delta_{kj}^{(0)} + \delta_{kj}^{(1)}\right\}. \tag{21}$$

Moreover, by taking the second derivative from $\frac{S_k(x)}{\phi'(x)}$ and using Eqs. (9)-(11), (14) and (15) we obtain

$$U_N''(x_j) = \sum_{k=-N}^{N} c_k \left\{\left(\frac{2\phi''(x_j)^2 - \phi'''(x_j)\phi'(x_j)}{\phi'(x_j)^3}\right) \delta_{kj}^{(0)} - \left(\frac{\phi''(x_j)}{\phi'(x_j)}\right) \delta_{kj}^{(1)} + \phi'(x_j)\delta_{kj}^{(2)}\right\},$$

(22)

$$V_N''(x_j) = \sum_{k=-N}^{N} d_k \left\{ \left( \frac{2\phi''(x_j)^2 - \phi'''(x_j)\phi'(x_j)}{\phi'(x_j)^3} \right) \delta_{kj}^{(0)} - \left( \frac{\phi''(x_j)}{\phi'(x_j)} \right) \delta_{kj}^{(1)} + \phi'(x_j)\delta_{kj}^{(2)} \right\}. \tag{23}$$

We are now ready to solve problem (1)-(3). Substituting Eq. (13) in Eq. (1) and evaluating the results at $x_j, j = -N-1, ..., N+1$, given in Eq. (18), we obtain

$$U_N''(x_j) + p''(x_j) + \frac{2}{x_j}\left(U_N'(x_j) + p'(x_j)\right) = -\alpha_2 + F_1\left(U_N(x_j) + p(x_j), V_N(x_j) + q(x_j)\right). \tag{24}$$

Similarly for Eq. (2) we get

$$V_N''(x_j) + q''(x_j) + \frac{2}{x_j}\left(V_N'(x_j) + q'(x_j)\right) = F_2\left(U_N(x_j) + p(x_j), V_N(x_j) + q(x_j)\right). \tag{25}$$

Eqs. (24) and (25) gives $4N+6$ nonlinear algebraic equations which can be solved for the unknown coefficients $c_k$ and $d_k, (k = -N-1, ..., N+1)$ by applying an iterative method, like the well known Newton's method. Consequently $u_N(x)$ and $v_N(x)$ given in Eq. (13) can be calculated. Throughout this paper, we use the Maple's `fsolve` command to find unknown coefficients $c_k$ and $d_k$ from the nonlinear system (24)-(25).

# 4 Numerical results and discussion

Here, we report the results of our numerical calculations using sinc-collocation method for solving problem (1)-(3). All the results presented in this study are obtained using $\alpha = 1$ and $d = \pi/2$ which leads, according to Theorem 1, to $h = \pi/\sqrt{2N}$. Also, we assign $m_1 = \ell_1 = m_2 = \ell_2 = 0.0001$ as in [5, 7, 16].

In Figure 1, we plot the curves of the approximate solutions $u_N(x)$ and $v_N(x)$ for $\alpha_1 = 5, \alpha_2 = 1, \alpha_3 = 0.1, \alpha_4 = 0.1$ and $\alpha_5 = 0.05$ with $N = 8$. Also, for such values for the parameters, the following expressions are obtained by Wazwaz et al. [16].

$$u(x) = 0.3170218446 + 0.682978155\ x^2,$$

$$v(x) = 0.9750104464 + 0.02498955359\ x^2.$$

For the purpose of comparison in Table 1 the results of the sinc-collocation method with $N = 8$ are compared with the variational iteration method (VIM) [16]. We observe from Table 1 that the results obtained with the present method are in good agreement with the results of [16].

**Table 1.** Results for $u(x)$ and $v(x)$

| | $u(x)$ | | $v(x)$ | |
|---|---|---|---|---|
| $x$ | VIM [16] | sinc-collocation $N=8$ | VIM [16] | sinc-collocation $N=8$ |
| 0.1 | 0.32385162 | 0.32384926 | 0.97526035 | 0.97526027 |
| 0.2 | 0.34434097 | 0.34433211 | 0.97601003 | 0.97600977 |
| 0.3 | 0.37848987 | 0.37847159 | 0.97725951 | 0.97725898 |
| 0.4 | 0.42629835 | 0.42627014 | 0.97900878 | 0.97900793 |
| 0.5 | 0.48776638 | 0.48772986 | 0.98125784 | 0.98125679 |
| 0.6 | 0.56289398 | 0.56285247 | 0.98400669 | 0.98400546 |
| 0.7 | 0.65168114 | 0.65163952 | 0.98725533 | 0.98725411 |
| 0.8 | 0.75412786 | 0.75409236 | 0.99100376 | 0.99100272 |
| 0.9 | 0.87023415 | 0.87021223 | 0.99525199 | 0.99525134 |

Now, we choose $N = 20$ to examine the effects of the parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$ to the solutions. The effect of $\alpha_1$ on $u_N(x)$ when $\alpha_2 = 1, \alpha_3 = 0.1, \alpha_4 = 0.1$ and $\alpha_5 = 0.05$ is shown in Figure 2. Moreover, the effect of $\alpha_2$ on $u_N(x)$ when $\alpha_1 = 5, \alpha_3 = 0.1, \alpha_4 = 0.1$ and $\alpha_5 = 0.05$ is shown in Figure 3. Figure 4 illustrate the the effect of $\alpha_3$ on $u_N(x)$ when $\alpha_1 = 5, \alpha_2 = 1, \alpha_4 = 0.1$ and $\alpha_5 = 0.05$. It is found that in Figures 2-4, by increasing $\alpha_1$ or $\alpha_3$, the approximate solution $u_N(x)$ decreases, but $u_N(x)$ increases with the increasing of $\alpha_1$. Also, we checked that the effects of $\alpha_4, \alpha_5$ on the approximate solution $u_N(x)$ and the effects of $\alpha_1, \alpha_2, \alpha_3$ on the approximate solution $v_N(x)$ are very weak.

The effect of $\alpha_4$ on $v_N(x)$ when $\alpha_1 = 5, \alpha_2 = 0.1, \alpha_3 = 0.1$ and $\alpha_5 = 0.05$ is presented in Figure 5. Also, Figure 5 illustrate the the effect of $\alpha_5$ on $v_N(x)$ when $\alpha_1 = 5, \alpha_2 = 0.1, \alpha_3 = 0.1$ and $\alpha_4 = 0.01$. According to Figure 5, we find that by increasing of $\alpha_4$ or $\alpha_5$, the approximate solution $v_N(x)$ decreases.

In addition, to check the effect of the parameter $\ell_1$ on the solutions, we assign $\alpha_1 = 5, \alpha_2 = 1, \alpha_3 = 0.1, \alpha_4 = 0.1$ and $\alpha_5 = 0.05$. The results are illustrated graphically in Figure 6. From this figure, we can see that when $\ell_1$ increases, both $u_N(x)$ and $v_N(x)$ increases. Also, we checked that the increasing of parameters $m_1, m_2$ and $\ell_2$ leads to increasing of $u_N(x)$ and $v_N(x)$.

It is worthy to mention here that, the pictures in Figures 1-6 are almost the same as the ones obtained in [5].

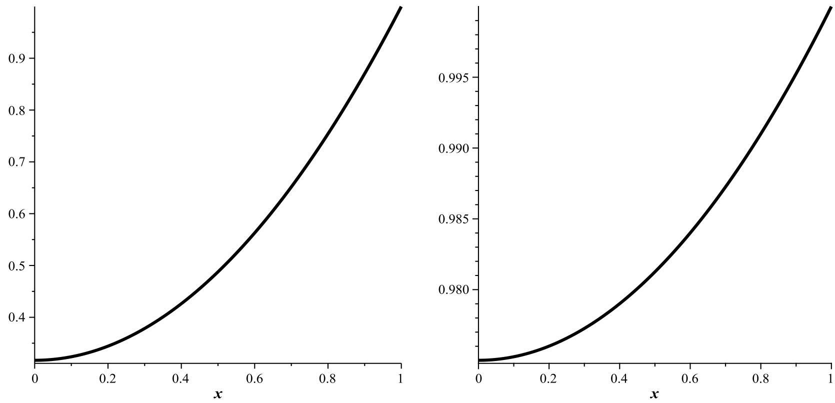


**Figure 1.** Plot of the approximate solutions $u_8(x)$ (left) and $v_8(x)$ (right), for $\alpha_1 = 5, \alpha_2 = 1, \alpha_3 = 0.1, \alpha_4 = 0.1$ and $\alpha_5 = 0.05$.

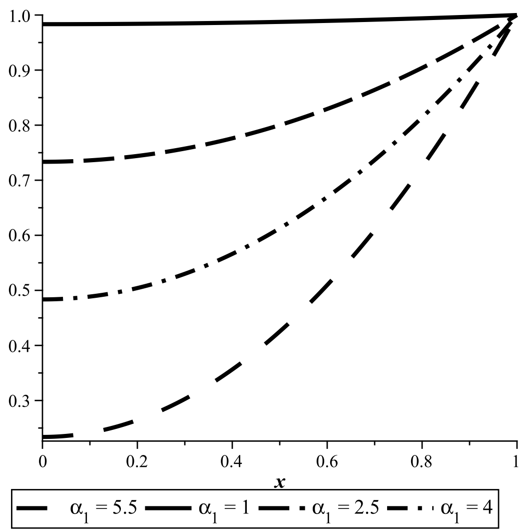


**Figure 2.** Effect of $\alpha_1$ on $u_N(x)$ for $\alpha_2 = 1, \alpha_3 = 0.1, \alpha_4 = 0.1$ and $\alpha_5 = 0.05$.

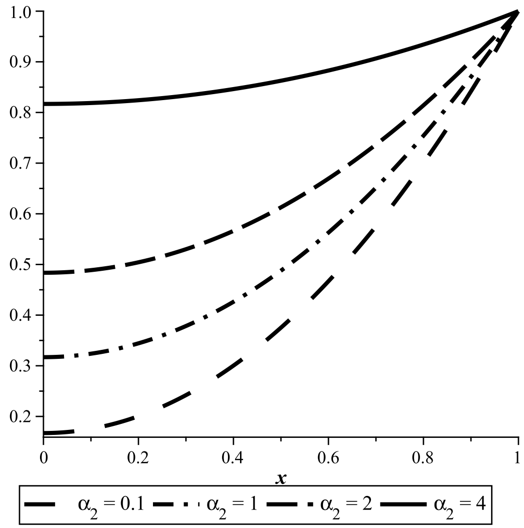


**Figure 3.** Effect of $\alpha_2$ on $u_N(x)$ for $\alpha_1 = 5, \alpha_3 = 0.1, \alpha_4 = 0.1$ and $\alpha_5 = 0.05$.

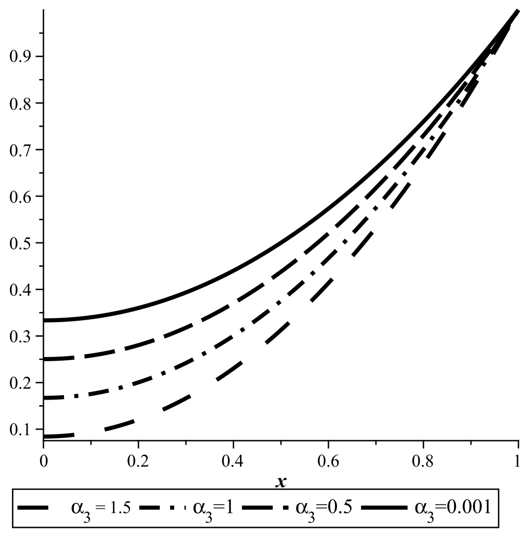


**Figure 4.** Effect of $\alpha_3$ on $u_N(x)$ for $\alpha_1 = 5, \alpha_2 = 1, \alpha_4 = 0.1$ and $\alpha_5 = 0.05$.

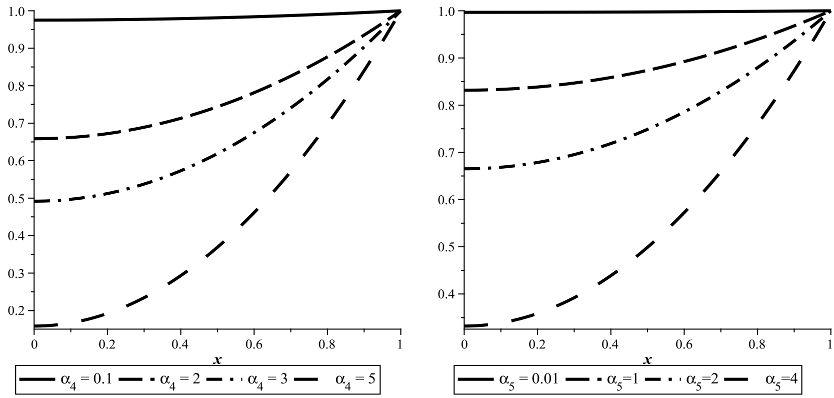


**Figure 5.** Effects of $\alpha_4$ (left) and $\alpha_5$ (right) on $v_N(x)$.

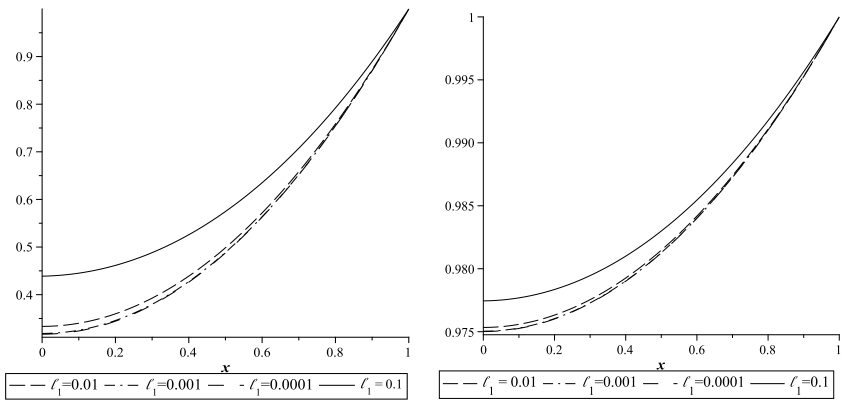


**Figure 6.** Effect of $\ell_1$ on $u_N(x)$ (left) and $v_N(x)$(right) when $m_1 = m_2 = \ell_2 = 0.0001$.

# 5 Conclusion

In this paper a sinc-collocation method with modified sinc basis functions is successfully used to solve the system of two coupled nonlinear singular differential equations, that governs the concentrations of oxygen and the carbon substrate. This approach reduced the computation of this problem to some algebraic equations. Moreover, the effects of the various values of parameters $\alpha_i,\ i = 1, 2, .., 5$ and $\ell_1, \ell_2, m_1, m_2$ on the concentrations of oxygen and the carbon substrate are discussed. These results will be beneficial in excess sludge minimization by increase of oxygen concentration in activated sludge flocs. Our results are in excellent agreement with those obtained by Adomian decomposition method [5] and variational iteration method [16].

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