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Some Inequalities for General Sum–Connectivity Index

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Abstract

Let G be a simple connected graph with n vertices and m edges. Denote by $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ and $d(e_1) \geq d(e_2) \geq \cdots \geq d(e_m) > 0$ sequences of vertex and edge degrees, respectively. Adjacency of the vertices i and j is denoted by $i \sim j$. A vertex-degree topological index, referred to as general sum-connectivity index, is defined as $\chi_\alpha = \chi_\alpha(G) = \sum_{i\sim j}(d_i + d_j)^\alpha$, where α is an arbitrary real number. Lower and upper bounds for χ_α are obtained. We also prove one generalization of discrete Kantorovich inequality.

1 Introduction

Let $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, $E = \{e_1, e_2, \ldots, e_m\}$ be a simple connected graph with n vertices and m edges. Denote by $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ and $d(e_1) \geq d(e_2) \geq \cdots \geq d(e_m) > 0$ sequences of vertex and edge degrees, respectively. If vertices i and j are adjacent, we denote it as $i \sim j$. In addition, we use the following notation: $\Delta = d_1$, $\delta = d_n$, $\Delta_e = d(e_1) + 2$, $\delta_e = d(e_m) + 2$. As usual, $L(G)$ denotes a line graph of G.

Gutman and Trinajstić [1] introduced two vertex degree topological indices, named as the first and the second Zagreb index. These are defined as

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 \qquad \text{and} \qquad M_2 = M_2(G) = \sum_{i\sim j} d_i d_j.$$

The first Zagreb index can be also expressed as (see [23])

$$M_1 = M_1(G) = \sum_{i\sim j}(d_i + d_j)\,. \tag{1}$$

Details on the mathematical theory of Zagreb indices can be found in [2–6, 21, 24].

Recently [7], a graph invariant similar to M_1 came into the focus of attention, defined as

$$F = F(G) = \sum_{i=1}^{n} d_i^3\,,$$

which for historical reasons [3] was named *forgotten* topological index. It satisfies the identities

$$F = \sum_{i\sim j}(d_i^2 + d_j^2) = \sum_{i=1}^{m}[d(e_i) + 2]^2 - 2M_2\,. \tag{2}$$

Another degree–based graph invariant was introduced in [8], and named general sum-connectivity index, χ_α. It is defined as

$$\chi_\alpha = \chi_\alpha(G) = \sum_{i\sim j}(d_i + d_j)^\alpha\,, \tag{3}$$

where α is an arbitrary real number. More on mathematical properties of this index can be found in [9–14].

In this paper we are concerned with upper and lower bounds for χ_α. Also, we present one generalization of discrete Kantorovich inequality, and show how it can be used to obtain upper bounds for M_1. The derived inequality is best possible in its class.

2 Preliminaries

In this section we recall some results for χ_α, and state a few analytical inequalities needed for our work.

In [10] (see also [9]) the following was proved:

Lemma 1. [10]. *Let G be a nontrivial connected graph with maximum degree Δ and minimum degree δ, and $\alpha \in \mathbb{R}$. Then*

$$2^{\alpha-1}\Delta^{\alpha-1}M_1 \le \chi_\alpha \le 2^{\alpha-1}\delta^{\alpha-1}M_1, \qquad \text{if } \alpha < 1, \tag{4}$$

$$2^{\alpha-1}\delta^{\alpha-1}M_1 \le \chi_\alpha \le 2^{\alpha-1}\Delta^{\alpha-1}M_1, \qquad \text{if } \alpha \ge 1\,. \tag{5}$$

The equality holds in each inequality for some $\alpha \ne 1$ if and only if G is regular.

In [8] upper and lower bounds for $\chi_\alpha(G)$ in terms of invariant M_1 and graph parameter m were obtained.

Lemma 2. [8]. *Let G be a graph with $m \geq 1$ edges. If $0 < \alpha < 1$, then*

$$\chi_\alpha(G) \leq M_1^\alpha m^{1-\alpha}\,, \tag{6}$$

and if $\alpha < 0$ or $\alpha > 1$, then

$$\chi_\alpha(G) \geq M_1^\alpha m^{1-\alpha}\,. \tag{7}$$

Equality holds if and only if $d_i + d_j$ is constant, for any edge $\{i,j\} \in E$.

For the real number sequences the following result was proved in [15] (see also [16]):

Lemma 3. [15]. *Let $p = (p_i)$, and $a = (a_i)$, $i = 1, 2, \ldots, m$, be two positive real number sequences with the properties*

$$\sum_{i=1}^{m} p_i = 1 \qquad \text{and} \qquad 0 < r \leq a_i \leq R < +\infty\,.$$

Then

$$\sum_{i=1}^{m} p_i a_i + rR \sum_{i=1}^{m} \frac{p_i}{a_i} \leq r + R\,, \tag{8}$$

with equality if and only if for some k, $1 \leq k \leq m$, holds $R = a_1 = \cdots = a_k \geq a_{k+1} = \cdots = a_m = r$.

In [19] the following was proved:

Lemma 4. [19]. *Let $q = (q_i)$ be a sequence of positive real numbers, and $a = (a_i)$ and $b = (b_i)$ sequences of real numbers with the properties*

$$0 < r_1 \leq a_i \leq R_1 < +\infty \qquad \text{and} \qquad 0 < r_2 \leq b_i \leq R_2 < +\infty\,,$$

$i = 1, 2, \ldots, m$. Denote with S a subset of $I_m = \{1, 2, \ldots, m\}$ which minimizes the expression

$$\left| \sum_{i \in S} q_i - \frac{1}{2} \sum_{i=1}^{m} q_i \right|.$$

Then

$$\left| \sum_{i=1}^{m} q_i \sum_{i=1}^{m} q_i a_i b_i - \sum_{i=1}^{m} q_i a_i \sum_{i=1}^{m} q_i b_i \right| \leq (R_1 - r_1)(R_2 - r_2) \sum_{i \in S} q_i \left(\sum_{i=1}^{m} q_i - \sum_{i \in S} q_i \right). \tag{9}$$

In the following lemma we recall well-known Chebyshev inequality (see for example [16]) which will be used later.

Lemma 5. *Let $q = (q_i)$ be a sequence of positive real numbers, and $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \ldots, m$, sequences of non-negative real numbers of similar monotonicity. Then*

$$\sum_{i=1}^{m} q_i \sum_{i=1}^{m} q_i a_i b_i \geq \sum_{i=1}^{m} q_i a_i \sum_{i=1}^{m} q_i b_i \,. \tag{10}$$

If sequences $a = (a_i)$ and $b = (b_i)$ has opposite monotonicity, then the sense of (10) reverses.

3 Main result

3.1 A new inequality for real number sequences

In this section we prove a new inequality for real number sequences.

Theorem 1. *Let $p = (p_i)$ and $a = (a_i)$, $i = 1, 2, \ldots, m$, be real number sequences, with $a = (a_i)$ being monotonic and $0 < r \leq a_i \leq R < +\infty$. Let S be a subset of $I_m = \{1, 2, \ldots, m\}$ which minimizes the expression*

$$\left| \sum_{i \in S} p_i - \frac{1}{2} \sum_{i=1}^{m} p_i \right| .$$

Then

$$\sum_{i=1}^{m} p_i a_i \sum_{i=1}^{m} \frac{p_i}{a_i} \leq \left(1 + \gamma(S) \frac{(R-r)^2}{rR} \right) \left(\sum_{i=1}^{m} p_i \right)^2 , \tag{11}$$

where

$$\gamma(S) = \frac{\sum_{i \in S} p_i}{\sum_{i=1}^{m} p_i} \left(1 - \frac{\sum_{i \in S} p_i}{\sum_{i=1}^{m} p_i} \right) .$$

Equality is attained if $R = a_1 = \cdots = a_m = r$.

Proof. For $q_i = \dfrac{p_i}{\sum_{i=1}^{m} p_i}$, $a_i = a_i$, $b_i = \frac{1}{a_i}$, $R_1 = R$, $r_1 = r$, $R_2 = \frac{1}{r}$ and $r_2 = \frac{1}{R}$, $i = 1, 2, \ldots, m$, the inequality (9) becomes

$$\left| 1 - \frac{\sum_{i=1}^{m} p_i a_i \sum_{i=1}^{m} \frac{p_i}{a_i}}{\left(\sum_{i=1}^{m} p_i \right)^2} \right| \leq (R - r) \left(\frac{1}{r} - \frac{1}{R} \right) \frac{\sum_{i \in S} p_i}{\sum_{i=1}^{m} p_i} \left(1 - \frac{\sum_{i \in S} p_i}{\sum_{i=1}^{m} p_i} \right) . \tag{12}$$

For $q_i = \dfrac{p_i}{\sum_{i=1}^{m} p_i}$, $a_i = a_i$, $b_i = \frac{1}{a_i}$, $i = 1, 2, \ldots, m$, the inequality (10) transforms into

$$1 \leq \frac{\sum\limits_{i=1}^{m} p_i a_i \sum\limits_{i=1}^{m} \dfrac{p_i}{a_i}}{\left(\sum\limits_{i=1}^{m} p_i\right)^2} . \tag{13}$$

Combining (12) and (13), gives

$$\frac{\sum\limits_{i=1}^{m} p_i a_i \sum\limits_{i=1}^{m} \dfrac{p_i}{a_i}}{\left(\sum\limits_{i=1}^{m} p_i\right)^2} \leq 1 + \frac{(R-r)^2}{rR} \cdot \frac{\sum\limits_{i \in S} p_i}{\sum\limits_{i=1}^{m} p_i} \left(1 - \frac{\sum\limits_{i \in S} p_i}{\sum\limits_{i=1}^{m} p_i} \right) ,$$

wherefrom we arrive at (11). ■

Remark 1. *The inequality* (11) *is a revision of the inequality*

$$\sum_{i=1}^{m} p_i a_i \sum_{i=1}^{m} \frac{p_i}{a_i} \leq \frac{\left(\left\lfloor \frac{m}{2} \right\rfloor R + \left\lfloor \frac{m+1}{2} \right\rfloor r\right)\left(\left\lfloor \frac{m+1}{2} \right\rfloor R + \left\lfloor \frac{m}{2} \right\rfloor r\right)}{rRm^2}$$

given in [17]. *The above inequality is not always correct. It is correct when* $p_i = \frac{1}{m}$, $i = 1, 2 \ldots, m$. *However, if* $p_i \neq \frac{1}{m}$ *and* $p_1 + p_2 + \cdots + p_m = 1$, *the above inequality might be incorrect. Thus, for example for* $m = 5$, $p_1 = p_2 = \frac{1}{4}$, $p_3 = p_4 = p_5 = \frac{1}{6}$, $a_1 = a_2 = 3$, $a_3 = a_4 = a_5 = 2$, $r = 2$ *and* $R = 3$, *one obtains that* $625 \leq 624$, *which is obviously wrong.*

Since $\gamma(S) \leq \frac{1}{4}$ for each $S \subset I_m$, the following corollary of Theorem 1 is valid.

Corollary 1. *Let* $p = (p_i)$, *be a sequence of positive real numbers and* $a = (a_i)$, $i = 1, 2, \ldots, m$, *a monotone sequence of positive real numbers, with the properties*

$$p_1 + \cdots + p_m = 1, \qquad 0 < r \leq a_i \leq R < +\infty .$$

Then

$$\sum_{i=1}^{m} p_i a_i \sum_{i=1}^{m} \frac{p_i}{a_i} \leq \frac{(R+r)^2}{4rR} . \tag{14}$$

Remark 2. *The inequality* (14) (*proved in* [20]) *is a generalization of Kantorovich inequality* (*see for example* [16]).

For $p_i = 1$, $i = 1, 2, \ldots, m$, the following corollary of Theorem 1 holds:

Corollary 2. *Let $a = (a_i)$, $i = 1, 2, \ldots, m$, be a real number sequence with the property $0 < r \le a_i \le R < +\infty$. Then*

$$\sum_{i=1}^{m} a_i \sum_{i=1}^{m} \frac{1}{a_i} \le m^2 \left(1 + \alpha(m) \frac{(R-r)^2}{rR}\right), \tag{15}$$

where

$$\alpha(m) = \frac{1}{4}\left(1 - \frac{(-1)^{m+1}+1}{2m^2}\right).$$

Remark 3. *The inequality* (15) *was proved in* [17]*. Since $\alpha(m) \le \frac{1}{4}$, it is a generalization of the inequality*

$$\sum_{i=1}^{m} a_i \sum_{i=1}^{m} \frac{1}{a_i} \le \frac{m^2}{4} \cdot \frac{(R+r)^2}{rR},$$

proved in [22].

3.2 Some inequalities for general sum–connectivity index

In what follows we derive lower and upper bounds for the degree-based topological index χ_α in terms of topological indices M_1, M_2 and F and graph parameters m, Δ_e and δ_e.

Theorem 2. *Let G be a simple connected graph with n vertices and $m \ge 2$ edges. Then, for any $\alpha \ge 2$,*

$$(F + 2M_2)\delta_e^{\alpha-2} \le \chi_\alpha \le (F + 2M_2)\Delta_e^{\alpha-2}. \tag{16}$$

If $\alpha \ge 1$, then

$$M_1 \delta_e^{\alpha-1} \le \chi_\alpha \le M_1 \Delta_e^{\alpha-1}. \tag{17}$$

If $\alpha \ge 0$, then

$$m\delta_e^{\alpha} \le \chi_\alpha \le m\Delta_e^{\alpha}.$$

Equalities in the above inequalities are attained, respectively, for $\alpha = 2$, $\alpha = 1$, $\alpha = 0$, or if $L(G)$ is regular.

When $\alpha \le 2$, $\alpha \le 1$ and $\alpha \le 0$, respectively, the opposite inequalities are valid.

Proof. Let $e = \{i, j\}$ be an arbitrary edge of graph G. Then $d(e) = d_i + d_j - 2$. According to (3), topological index χ_α can be computed from the following expression

$$\chi_\alpha = \sum_{i \sim j} (d_i + d_j)^\alpha = \sum_{i=1}^{m} (d(e_i) + 2)^\alpha, \qquad \chi_0 = m. \tag{18}$$

From (3) follows

$$F + 2M_2 = \chi_2 = \sum_{i \sim j} (d_i + d_j)^2 = \sum_{i=1}^{m} (d(e_i) + 2)^2.$$

Since

$$\chi_\alpha = \sum_{i=1}^{m} (d(e_i)+2)^\alpha = \sum_{i=1}^{m} (d(e_i)+2)^2 (d(e_i)+2)^{\alpha-2}\,,$$

for $\alpha \geq 2$ holds

$$\delta_e^{\alpha-2} \sum_{i=1}^{m} (d(e_i)+2)^2 \leq \chi_\alpha \leq \Delta_e^{\alpha-2} \sum_{i=1}^{m} (d(e_i)+2)^2\,,$$

i.e.

$$(F+2M_2)\delta_e^{\alpha-2} \leq \chi_\alpha \leq (F+2M_2)\Delta_e^{\alpha-2}\,.$$

By a similar procedure, the remaining inequalities in Theorem 2 can be proved. ■

Remark 4. *Let α and β be arbitrary real numbers such that $\alpha-\beta \geq 0$. Then, according to*

$$\chi_\alpha = \sum_{i=1}^{m} (d(e_i)+2)^\alpha = \sum_{i=1}^{m} (d(e_i)+2)^\beta (d(e_i)+2)^{\alpha-\beta}$$

follows that

$$\delta_e^{\alpha-\beta}\chi_\beta \leq \chi_\alpha \leq \Delta_e^{\alpha-\beta}\chi_\beta\,, \tag{19}$$

with equality if and only if $\alpha=\beta$, or $L(G)$ is regular.

If $\alpha-\beta \leq 0$, the opposite inequality is valid.

The question is for which values of parameter β the inequality (19) *has practical importance. For $\beta=0$, $\beta=1$ and $\beta=2$ it was considered in Theorem 2. Since*

$$\sum_{i=1}^{m} (d(e_i)+2)^3 = EF + 6F + 12M_2 - 12M_1 + 8m\,,$$

for $\alpha \geq 3$ holds

$$\delta_e^{\alpha-3}(EF+6F+12M_2-12M_1+8m) \leq \chi_\alpha \leq \Delta_e^{\alpha-3}(EF+6F+12M_2-12M_1+8m)\,,$$

where EF is the reformulated forgotten topological index. When $\alpha \leq 3$, the opposite inequality is valid. Obviously, these inequalities depend on a large number of graph invariants.

Another question is how would (19) *look like if $\beta \geq 4$ and its practical usability. For $\alpha = -\frac{1}{2}$ and $\beta=-1$, the inequality* (19) *gives a connection between harmonic and sum-connectivity indices.*

Remark 5. *Since*

$$2\delta \leq \delta_e \leq \Delta_e \leq 2\Delta\,,$$

then for $\alpha \geq 1$ and $\alpha \leq 1$, from (17) *the inequalities* (4) *and* (5) *are obtained. Hence, the inequality* (17) *is stronger than these inequalities.*

Corollary 3. *Let G be a simple connected graph with n vertices and $m \geq 2$ edges. Then, for any $\alpha \geq 2$*

$$4M_2\delta_e^{\alpha-2} \leq \chi_\alpha \leq 2F\Delta_e^{\alpha-2}\,.$$

Equality is attained if G is regular.

Proof. The required inequality is obtained based on (16) and

$$4M_2 \leq F + 2M_2 \leq 2F\,.$$

■

Corollary 4. *Let G be a simple connected graph with n vertices and $m \geq 2$ edges. Then, for any $\alpha \leq 1$,*

$$m\delta_e\Delta_e^{\alpha-1} \leq \chi_\alpha \leq m\Delta_e\delta_e^{\alpha-1}\,,$$

with equality if and only if $L(G)$ is regular.

In the next Theorem we establish a lower bound for χ_α in terms of M_1, M_2 and F.

Theorem 3. *Let G be a simple connected graph with n vertices and m edges. Then, for any real α, $\alpha \leq 1$ or $\alpha \geq 2$,*

$$\chi_\alpha \geq \frac{(F+2M_2)^{\alpha-1}}{M_1^{\alpha-2}}\,. \tag{20}$$

If $1 \leq \alpha \leq 2$, the opposite inequality is valid. Equality is attained if and only if $\alpha = 1$, or $\alpha = 2$, or $L(G)$ is regular.

Proof. Let $p = (p_i)$ and $a = (a_i)$, $i = 1, 2, \dots, m$, be positive real number sequences, where $p_1 + p_2 + \cdots + p_m = 1$. Then, for any real t, $t \leq 0$ or $t \geq 1$, Jensen's inequality holds (see [16, 18])

$$\sum_{i=1}^{m} p_i a_i^t \geq \left(\sum_{i=1}^{m} p_i a_i\right)^t\,. \tag{21}$$

If $0 \leq t \leq 1$ the opposite inequality is valid in (21).

For $t = \alpha - 1$, $p_i = \dfrac{d(e_i)+2}{\sum_{i=1}^{m}(d(e_i)+2)}$, $a_i = d(e_i) + 2$, $i = 1, \dots, m$, the inequality (21) becomes

$$\frac{\sum\limits_{i=1}^{m}(d(e_i)+2)^\alpha}{\sum\limits_{i=1}^{m}(d(e_i)+2)} \geq \left(\frac{\sum\limits_{i=1}^{m}(d(e_i)+2)^2}{\sum\limits_{i=1}^{m}(d(e_i)+2)}\right)^{\alpha-1}\,.$$

According to (18), for $\alpha = 1$ and $\alpha = 2$, this inequality transforms into

$$\frac{\chi_\alpha}{M_1} \geq \frac{(F + 2M_2)^{\alpha-1}}{M_1^{\alpha-1}},$$

wherefrom the inequality (20) is obtained.

Equality in (21) holds if and only if $t = 0$, or $t = 1$, or $a_1 = a_2 = \cdots = a_m$. Therefore equality in (20) holds if and only if $\alpha = 1$, or $\alpha = 2$, or $d(e_1) + 2 = \cdots = d(e_m) + 2$, i.e. if $L(G)$ is regular. ■

Corollary 5. *Let G be a simple connected graph with n vertices and m edges. Then, for any real $\alpha \geq 1$,*

$$\chi_\alpha \geq \frac{4^{\alpha-1} M_2^{\alpha-1}}{M_1^{\alpha-2}},$$

with equality if $\alpha = 1$, or G is regular.

Remark 6. *For $q_i = a_i = d(e_i) + 2$ and $b_i = \frac{1}{d(e_i)+2}$, $i = 1, 2, \ldots, m$, the inequality* (10) *becomes*

$$m(F + 2M_2) \geq M_1^2.$$

Then, for any $\alpha \geq 1$

$$\frac{(F + 2M_2)^{\alpha-1}}{M_1^{\alpha-2}} \geq \frac{M_1^\alpha}{m^{\alpha-1}}.$$

Therefore (20) *is stronger than* (7).

In the next Theorem we establish a connection between χ_α, $\chi_{\alpha-1}$ and $\chi_{\alpha-2}$.

Theorem 4. *Let G be a simple connected graph with n vertices and $m \geq 2$ edges. Then*

$$\chi_\alpha - (\Delta_e + \delta_e)\chi_{\alpha-1} + \Delta_e \delta_e \chi_{\alpha-2} \leq 0, \tag{22}$$

with equality if and only if for some k, $1 \leq k \leq m$, $\Delta_e = d(e_1) + 2 = \cdots = d(e_k) + 2 \geq d(e_{k+1}) + 2 = \cdots = d(e_m) + 2 = \delta_e$.

Proof. For $p_i = \dfrac{(d(e_i)+2)^{\alpha-1}}{\sum_{i=1}^m (d(e_i)+2)^{\alpha-1}}$, $a_i = d(e_i) + 2$, $i = 1, 2, \ldots, m$, $r = \delta_e = d(e_m) + 2$ and $R = \Delta_e = d(e_1) + 2$, the inequality (8) transforms into

$$\frac{\sum\limits_{i=1}^m (d(e_i)+2)^\alpha}{\sum\limits_{i=1}^m (d(e_i)+2)^{\alpha-1}} + \Delta_e \delta_e \frac{\sum\limits_{i=1}^m (d(e_i)+2)^{\alpha-2}}{\sum\limits_{i=1}^m (d(e_i)+2)^{\alpha-1}} \leq \Delta_e + \delta_e.$$

From the above and (18) we get

$$\frac{\chi_\alpha}{\chi_{\alpha-1}} + \Delta_e \delta_e \frac{\chi_{\alpha-2}}{\chi_{\alpha-1}} \leq \Delta_e + \delta_e \,,$$

wherefrom (22) is obtained. ■

Corollary 6. *Let G be a simple connected graph with n vertices and $m \geq 2$ edges. Then*

$$\chi_\alpha \leq \frac{\chi_{\alpha-1}^2}{4\chi_{\alpha-2}} \left(\sqrt{\frac{\Delta_e}{\delta_e}} + \sqrt{\frac{\delta_e}{\Delta_e}} \right)^2 , \tag{23}$$

with equality if $L(G)$ is regular.

Proof. According to the arithmetic-geometric mean inequality [16], we have that

$$2\sqrt{\Delta_e \delta_e \chi_{\alpha-2} \chi_\alpha} \leq \chi_\alpha + \Delta_e \delta_e \chi_{\alpha-2} \leq (\Delta_e + \delta_e)\chi_{\alpha-1} \,,$$

wherefrom (23) is obtained. ■

Remark 7. *For $\alpha = 2$ and $\alpha = 1$ from (23) we obtain*

$$F \leq \frac{M_1^2}{4m} \left(\sqrt{\frac{\Delta_e}{\delta_e}} + \sqrt{\frac{\delta_e}{\Delta_e}} \right)^2 - 2M_2 \,, \tag{24}$$

and

$$M_1 \leq \frac{m^2}{2H} \left(\sqrt{\frac{\Delta_e}{\delta_e}} + \sqrt{\frac{\delta_e}{\Delta_e}} \right)^2 , \tag{25}$$

where $H = 2\chi_{-1}$ is a harmonic index.

According to Theorem 1, i.e. inequality (11), the following is valid:

Theorem 5. *Let G be a simple connected graph with n vertices and $m \geq 2$ edges. Denote by S a subset of $I_m = \{1, 2, \ldots, m\}$ which minimizes the expression*

$$\left| \sum_{i \in S} (d(e_i) + 2)^{\alpha-1} - \frac{1}{2}\chi_{\alpha-1} \right| .$$

Then

$$\chi_\alpha \leq \frac{\chi_{\alpha-1}^2}{\chi_{\alpha-2}} \left(1 + \beta(S) \left(\sqrt{\frac{\Delta_e}{\delta_e}} - \sqrt{\frac{\delta_e}{\Delta_e}} \right)^2 \right) , \tag{26}$$

where

$$\beta(S) = \frac{\sum_{i \in S} (d(e_i) + 2)^{\alpha-1}}{\chi_{\alpha-1}} \left(1 - \frac{\sum_{i \in S} (d(e_i) + 2)^{\alpha-1}}{\chi_{\alpha-1}} \right) .$$

Equality is attained if $L(G)$ is regular.

Proof. The inequality (26) is obtained from (11) for $p_i = (d(e_i)+2)^{\alpha-1}$, $a_i = d(e_i)+2$, $i = 1, 2, \ldots, m$, $R = \Delta_e = d(e_1)+2$, and $r = \delta_e = d(e_m)+2$. ∎

Remark 8. *Since $\beta(S) \leq \frac{1}{4}$, the inequality (26) is stronger than (23). Thus, for example, for $\alpha = 1$, from (26) we obtain*

$$M_1 \leq \frac{2m^2}{H}\left(1 + \alpha(m)\left(\sqrt{\frac{\Delta_e}{\delta_e}} - \sqrt{\frac{\delta_e}{\Delta_e}}\right)^2\right), \tag{27}$$

where

$$\alpha(m) = \frac{1}{4}\left(1 - \frac{(-1)^{m+1}+1}{2m^2}\right).$$

The above inequality is stronger than (25) when m is odd.

Theorem 6. *Let G be a simple connected graph with n vertices and $m \geq 2$ edges. Then*

$$\chi_\alpha \chi_{-\alpha} \leq m^2 \left(1 + \alpha(m)\left(\sqrt{\frac{\Delta_e^\alpha}{\delta_e^\alpha}} - \sqrt{\frac{\delta_e^\alpha}{\Delta_e^\alpha}}\right)^2\right) \tag{28}$$

with equality if $L(G)$ is regular.

Proof. For $p_i = 1$, $a_i = (d(e_i)+2)^\alpha$, $i = 1, \ldots, m$, $R = \Delta_e^\alpha = (d(e_1)+2)^\alpha$, and $r = \delta_e^\alpha = (d(e_m)+2)^\alpha$, according to Theorem 1 we obtain (28). ∎

Remark 9. *For $\alpha = 1$ the inequality (28) reduces to (27).*

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