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Lower Bounds for the Laplacian Resolvent Energy via Majorization

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Abstract

Using majorization we find two general lower bounds for the Laplacian Resolvent Energy of a graph, one in terms of the degrees of the vertices, the other in terms of the number of edges, and some particular lower bounds for c-cyclic graphs, $0 \leq c \leq 6$.

1 Introduction

Let $G = (V, E)$ be a finite simple connected graph with vertex set $V = \{1, 2, \ldots, n\}$, edge set E and degrees $d_1 \geq d_2 \geq \cdots \geq d_n$. We consider A to be the adjacency matrix of G, D the diagonal matrix whose diagonal elements are the degrees of G and $L = D - A$ the Laplacian matrix of G, with eigenvalues $\lambda_1 \geq ... \geq \lambda_{n-1} \geq \lambda_n = 0$ (For all graph theoretical terms the reader is referred to reference [10]). The Laplacian Resolvent Energy of a graph, proposed by Cafure et al. in [4] as an alternative to the Resolvent Energy (see [5]) is defined as

$$RL(G) = \sum_{i=1}^{n} \frac{1}{n + 1 - \lambda_i}. \tag{1}$$

In this note we use the majorization technique in order to prove two general lower bounds for the Laplacian Resolvent Energy, one in terms of the degrees of the graph that is attained by the n-star graph, the other in terms of the number of edges. We also find particular lower bounds for c-cyclic graphs, $0 \leq c \leq 6$.

Majorization has been applied extensively to find bounds and extremal values for a variety of descriptors. We point out the book chapters [1] and [3] and the articles [7], [11] and [9] for a sample of the variety of scenarios covered with this approach.

Here is a brief summary of majorization: given two n-tuples $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ with $x_1 \geq x_2 \geq \ldots \geq x_n$ and $y_1 \geq y_2 \geq \ldots \geq y_n$, we say that $\mathbf{x}$ majorizes $\mathbf{y}$ and write $\mathbf{x} \succ \mathbf{y}$ in case

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i, \tag{2}$$

for $1 \leq k \leq n-1$ and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{3}$$

A Schur-convex function $\Phi : \mathbb{R} \to \mathbb{R}$ keeps the majorization inequality, that is, if Φ is Schur-convex then $\mathbf{x} \succ \mathbf{y}$ implies $\Phi(\mathbf{x}) \geq \Phi(\mathbf{y})$. Likewise, a Schur-concave function reverses the inequality: for this type of function $\mathbf{x} \succ \mathbf{y}$ implies $\Phi(\mathbf{x}) \leq \Phi(\mathbf{y})$. A simple way to construct a Schur-convex (resp. Schur-concave) function is to consider

$$\Phi(\mathbf{x}) = \sum_{i=1}^{n} f(x_i),$$

where $f : \mathbb{R} \to \mathbb{R}$ is a convex (resp. concave) one-dimensional real function. For more details on majorization the reader is referred to [8].

2 Lower bounds for the Laplacian resolvent energy

The following lemma can be found in [8]:

Lemma 1 *Let Σ_a be the set of real n-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$ and $\sum_{i=1}^{n} x_i = a$. Then the minimal element of Σ_a, that is the element $\mathbf{x}_* \in \Sigma_a$ such that $\mathbf{x} \succ \mathbf{x}_*$ for all $\mathbf{x} \in \Sigma_a$, is given by*

$$\left(\frac{a}{n}, \frac{a}{n}, \ldots \frac{a}{n}\right).$$

The next result is due to Grone (see [6]):

Lemma 2 *For any graph G we have $(\lambda_1, \lambda_2, \ldots, \lambda_n) \succ (d_1 + 1, d_2, \ldots, d_{n-1}, d_n - 1)$.*

With these tools we can prove now the following

Proposition 1 *For any graph G we have*

$$RL(G) \geq \frac{1}{n-d_1} + \sum_{i=2}^{n-1} \frac{1}{n+1-d_i} + \frac{1}{n+2-d_n} \geq \frac{n^2}{n(n+1)-2|E|}.$$

The left inequality is attained by the n-star graph.

Proof. The first inequality follows from lemma 2 and the fact that the Laplacian Resolvent Energy is a Schur-convex function on account of the real function $f(x) = \frac{1}{n+1-x}$ being convex for $x < n+1$. The second inequality follows from considering in the first lemma the set $\Sigma_{2|E|}$ to which $(\lambda_1, \ldots, \lambda_n)$ and $\bar{\mathbf{d}} = (d_1+1, \ldots, d_n-1)$ belong and whose mimimal element is $\mathbf{x}_* = (\frac{2|E|}{n}, \ldots, \frac{2|E|}{n})$. Then with some abuse of notation, if we denote by $RL(\mathbf{x})$ the Laplacian Resolvent Energy of a graph evaluated, not at the n-tuple $(\lambda_1, \ldots, \lambda_n)$, but at an arbitrary n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$, with $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$ and $\sum_i x_i = 2|E|$, we have

$$RL(G) \geq RL(\bar{\mathbf{d}}) \geq RL(\mathbf{x}_*) = \frac{n^2}{n(n+1) - 2|E|} \, .$$

Now for the n-star graph S_n its eigenvalues are $n, 1$ with multiplicity $n-2$ and 0 and therefore $RL(S_n) = 1 + \dfrac{n-2}{n} + \dfrac{1}{n+1}$, which coincides with the lower bound when $d_1 = n-1$ and $d_i = 1$ for $2 \leq i \leq n$ ■

We are usually interested in connected graphs, but if we consider the n-vertex graph without edges, our rightmost bound recovers the lower bound $\dfrac{n}{n+1}$ shown in [4].

Now we will use the following result found in [2]:

Lemma 3 *For c-cyclic graphs, the minimal degree sequences with respect to the majorization order are given by* $(\mathbf{2^{n-2}, 1, 1})$ *, in case* $c = 0$ *and* $n > 2$*,* $(\mathbf{2^n})$ *, in case* $c = 1$*, and* $n > 2$*,* $(\mathbf{3^{2c-2}, 2^{n-2c+2}})$ *, in case* $2 \leq c \leq 6$ *and* $n > 2c-2$*.*

Then we can prove the following

Proposition 2 *If* T *is a tree and* $n > 2$ *we have* $RL(T) \geq \frac{1}{n-2} + \frac{n-3}{n-1} + \frac{1}{n} + \frac{1}{n+1}$ *. If* G *is a unicyclic graph and* $n > 2$ *we have* $RL(G) \geq \frac{1}{n-2} + \frac{n-2}{n-1} + \frac{1}{n}$ *. If* G *is a c-cyclic graph,* $2 \leq c \leq 6$*, and* $n > 2c-2$ *we have* $RL(G) \geq \frac{1}{n-3} + \frac{2c-3}{n-2} + \frac{n+1-2c}{n-1} + \frac{1}{n}$ *.*

Proof. Assume T is a tree, $(\lambda_1, \ldots, \lambda_n)$ its n-tuple of eigenvalues and $(d_1, \ldots, d_n)$ its degree sequence. Then by lemma 2 we have $(\lambda_1, \ldots, \lambda_n) \succ (d_1+1, d_2, \ldots, d_{n-1}, d_n - 1) = \bar{\mathbf{d}}$, and by lemma 3 $(d_1, d_2, \ldots, d_n) \succ (2, 2, \ldots, 2, 1, 1)$. Now it is clear that $\bar{\mathbf{d}} \succ (3, 2, \ldots, 2, 1, 0) = \mathbf{d}_*$, and therefore

$$RL(G) \geq R(\mathbf{d}_*) = \frac{1}{n-2} + \frac{n-3}{n-1} + \frac{1}{n} + \frac{1}{n+1} \, .$$

The other statements follow similarly ■

Remarks. The lower bounds in Proposition 2 in general are not tight. In the case of trees, it was shown in [4] that $RL(T) \geq RL(P_n)$, and $RL(P_n)$ is strictly larger than our bound $\frac{1}{n-2}+\frac{n-3}{n-1}+\frac{1}{n}+\frac{1}{n+1}$ except for the case $n = 3$. For other c-cyclic graphs, there are no lower bounds for $RL(G)$ that we know of, and we conjecture that the minimum value is attained precisely at one of those graphs with minimal degree sequence.

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