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On Some Lower Bounds of the Kirchhoff Index

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Abstract

Let G be a simple graph of order $n \geq 2$ with m edges. Denote by $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ the sequence of vertex degrees and by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ the Laplacian eigenvalues of the graph G. Lower bounds for the Kirchhoff index, $Kf(G) = n\sum_{i=1}^{n-1} \frac{1}{\mu_i}$, which depend on some of the parameters n, m, Δ (the greatest vertex degree), Δ_2 (the second greatest vertex degree) or δ (the smallest vertex degree), are obtained.

1 Introduction

Let $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, $E = \{e_1, e_2, \ldots, e_m\}$ be a simple connected graph of order $n \geq 2$ and size m. If vertices i and j are adjacent, we denote it as $i \sim j$. Denote by $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ a sequence of vertex degrees and by Δ, Δ_2 and δ the greatest, the second greatest and the smallest vertex degrees, respectively. Let $\mathbf{A}$ be the adjacency matrix of G, and $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ the diagonal matrix of its vertex degrees. Then $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is the Laplacian matrix of G. Eigenvalues of $\mathbf{L}$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ are the Laplacian eigenvalues of graph G.

The Kirchhoff index, $Kf(G)$, of a simple connected graph was defined by [6]

$$Kf(G) = \sum_{i<j} r_{ij},$$

where r_{ij} is the effective resistance between the vertices i and j.

A more appropriate formula from practical point of view, was put forward in [5]

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}.$$

This has triggered the study of this invariant and its applications in various areas, such as in spectral graph theory, molecular chemistry, computer science, etc. (see for example [1–10, 12, 15–22]).

In this paper we are concerned with the lower bounds of $Kf(G)$ which depend on some of the parameters n, m, Δ, Δ_2 or δ. Before going further, we recall some results from the literature needed for our subsequent consideration. Note that we say that two bounds belong to the same class if they depend on the same graph parameters.

2 Preliminaries

In [10] (see also [18]) the following lower bound for $Kf(G)$ was established

$$Kf(G) \geq Kf(K_n) = n - 1. \tag{1}$$

This lower bound is the best possible in its class.

Since for connected graphs hold $n(n-1) \geq 2m$, therefore the following condition have to be satisfied

$$n \geq \frac{1 + \sqrt{8m+1}}{2}.$$

Then according to (1)

$$Kf(G) \geq \frac{1}{2}\left(\sqrt{8m+1} - 1\right),$$

with equality if and only if $G \cong K_n$.

The following lower bound for $Kf(G)$ that depends on n and m was determined in [17] (see also [12])

$$Kf(G) \geq \frac{n(n-1)^2}{2m}, \tag{2}$$

with equality if and only if $G \cong K_n$.

The lower bound for $Kf(G)$ that depends on n and Δ was obtained in [15]

$$Kf(G) \geq \frac{(n-1)^2}{\Delta}, \tag{3}$$

with equality if and only if $G \cong K_n$.

Let $N(i)$ be the set of all neighborhoods of the vertex i, i.e. $N(i) = \{k \mid k \in V,\ k \sim i\}$, and $d(i,j)$ the distance between vertices i and j. Denote by Γ_d a set of all d-regular graphs, $1 \le d \le n-1$, with the properties that diameter is $D = 2$ and $|\,N(i) \cap N(j)\,| = d$ (see [16]).

In [16] and [19,22] the following bounds that depend on n, m and Δ were, respectively, obtained

$$Kf(G) \ge n - 1 + \frac{n(n-1) - 2m}{\Delta}, \tag{4}$$

and

$$Kf(G) \ge n\left(\frac{1}{1+\Delta} + \frac{(n-2)^2}{2m - \Delta - 1}\right). \tag{5}$$

Equality in (4) holds if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$. whereas in (5) if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

In [2] it was proved that for the Laplacian eigenvalues of G holds

$$S_\alpha(G) = \sum_{i=1}^{n} \mu_i^\alpha \ge (1+\Delta)^\alpha + \delta^\alpha + \frac{(2m - \Delta - \delta - 1)^\alpha}{(n-3)^{\alpha-1}},$$

where $\alpha \le 0$ and G is a connected graph different from K_n (i.e. $G \ne K_n$). For $\alpha = -1$ from the above inequality follows

$$Kf(G) \ge n\left(\frac{1}{1+\Delta} + \frac{1}{\delta} + \frac{(n-3)^2}{2m - \Delta - \delta - 1}\right), \tag{6}$$

whereby the equality holds if and only if $G \cong K_{1,n-1}$, or $G \cong 2K_1 \bigvee K_{n-2}$, or $G \cong (K_1 \cup K_{n-2}) \bigvee K_1$.

In [3] a lower bound for $Kf(G)$ that depends n, m, Δ, Δ_2 and δ was obtained. It was proved that

$$Kf(G) \ge \frac{n}{\Delta+1} + \frac{n}{2m - \Delta - 1}\left((n-2)^2 + \frac{(\Delta_2 - \delta)^2}{\Delta_2 \delta}\right), \tag{7}$$

with equality if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

The following lower bound for $Kf(G)$ that depends on n, m, Δ and Δ_2 was established in [1]

$$Kf(G) \ge n\left(\frac{1}{1+\Delta} + \frac{1}{\Delta_2} + \frac{(n-3)^2}{2m - \Delta - \Delta_2 - 1}\right) \tag{8}$$

In this paper we obtain lower bounds for $Kf(G)$ that are better than (2), (3) and (4) and belong to the same classes. Also, we determine new lower bounds for Kirchhoff index which are of the same class as (5), (6), (7) and (8).

3 Main results

In the following theorem we establish the lower bound for $Kf(G)$ that depends on n and m and is stronger than (2).

Theorem 1 *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n^2(n-1)-2m}{2m}, \tag{9}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$ or $G \in \Gamma_d$.

Proof. In [19] (see also [22]), the following inequality was proved

$$Kf(G) \geq -1 + (n-1)\sum_{i=1}^{n} \frac{1}{d_i}, \tag{10}$$

with equality if and only if $G \cong K_n$.

On the other hand, from the Chebyshev inequality (see [13])

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i \geq \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i, \tag{11}$$

where $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \dots, n$, are real sequences of the same monotonicity, for $p_i = d_i$, $a_i = b_i = \frac{1}{d_i}$, $i = 1, 2, \dots, n$ we obtain

$$\sum_{i=1}^{n} \frac{1}{d_i} \geq \frac{n^2}{2m}. \tag{12}$$

The inequality (9) is obtained from (10) and (12). ■

Remark 1 *Since*

$$\frac{n^2(n-1)-2m}{2m} \geq \frac{n^2(n-1)-n(n-1)}{2m} = \frac{n(n-1)^2}{2m},$$

the inequality (9) *is stronger than* (2).

Corollary 1 *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n(n-1)-\Delta}{\Delta}, \tag{13}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$ or $G \in \Gamma_d$.

Remark 2 *Since*

$$\frac{n(n-1)-\Delta}{\Delta} \geq \frac{(n-1)^2}{\Delta}$$

the inequality (13) *is stronger than* (3).

Corollary 2 *Let G be a simple connected planar graph with $n \geq 3$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n^2(n-1)}{6(n-2)} - 1, \tag{14}$$

with equality if and only if $G \cong K_3$ or $G \cong K_4$.

Remark 3 *The inequality* (14) *is stronger than*

$$Kf(G) \geq \frac{n(n-1)^2}{6(n-2)},$$

which was proved in [17].

We now give another bound for the Kirchhoff index that depends on n, m and Δ.

Theorem 2 *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + \frac{(n-1)^3}{2m-\Delta}. \tag{15}$$

Equality holds if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$ or $G \in \Gamma_d$.

Proof. From the inequality

$$\sum_{i=2}^{n} p_i \sum_{i=2}^{n} p_i a_i b_i \geq \sum_{i=2}^{n} p_i a_i \sum_{i=2}^{n} p_i b_i,$$

for $p_i = d_i$, $a_i = b_i = \frac{1}{d_i}$, $i = 2, 3, \dots, n$, we obtain

$$\sum_{i=1}^{n} \frac{1}{d_i} \geq \frac{1}{\Delta} + \frac{(n-1)^2}{2m-\Delta}. \tag{16}$$

The inequality (15) is obtained from (10) and (16). ■

Remark 4 *Since*

$$f(x) = \frac{1}{x} + \frac{(n-1)^2}{2m-x}$$

is a monotone increasing function, for $x \geq \frac{2m}{n}$ we have that

$$\begin{aligned} Kf(G) &\geq \frac{n-1-\Delta}{\Delta} + \frac{(n-1)^3}{2m-\Delta} \geq \frac{n^2(n-1)-2m}{2m} = n-1+\frac{n(n(n-1)-2m)}{2m} \geq \\ &\geq n-1+\frac{n(n-1)-2m}{\Delta}. \end{aligned}$$

This means that the inequality (15) is stronger than (4).

Remark 5 *Inequalities* (15) *and* (5) *are exact when* $G \cong K_n$ *or* $G \cong K_{1,n-1}$. *The inequality* (15) *is stronger than* (5) *when* $G \cong P_n$, *or* $G \cong K_{\frac{n}{2},\frac{n}{2}}$, *or when sequence of vertex degrees of a connected graph is of the form* $D = (n-1, d_2, \ldots, d_n)$. *We have performed testing on the set of connected (regular) graphs with* $n \geq 4$ *vertices to find out the case, if any, when the inequality* (5) *is stronger than* (15), *but we didn't find it. However, it is an open question whether* (15) *is always stronger than* (5).

Following the similar procedure as in the case of Theorem 2, the following can be proved:

Theorem 3 *Let* G *be a simple connected graph with* $n \geq 3$ *vertices and* m *edges. Then*

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + \frac{n-1}{\delta} + \frac{(n-1)(n-2)^2}{2m-\Delta-\delta} \tag{17}$$

and

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + \frac{n-1}{\Delta_2} + \frac{(n-1)(n-2)^2}{2m-\Delta-\Delta_2}. \tag{18}$$

Equalities hold if and only if $G \cong K_n$ *or* $G \cong K_{1,n}$, *or* $G \cong K_{\frac{n}{2},\frac{n}{2}}$ *or* $G \in \Gamma_d$.

Remark 6 *Inequalities* (17) *and* (6) *are incomparable. Thus, for example, equality in* (17) *occurs when* $G \cong K_{\frac{n}{2},\frac{n}{2}}$, *but not for* (6). *Conversely, equality in* (6) *is attained when* $G \cong 2K_1 \bigvee K_{n-2}$, *but not in the inequality* (17).

The inequality (18) *is stronger than* (8) *when* $G \cong K_{\frac{n}{2},\frac{n}{2}}$ *and* $G \cong P_n$. *However, it is an open question whether* (18) *is always better than* (8).

In the following theorem we establish lower bound for $Kf(G)$ that depends on n, m, Δ and δ.

Theorem 4 *Let* G *be a simple connected graph with* $n \geq 3$ *vertices and* m *edges. Then*

$$Kf(G) \geq \frac{n^2(n-1)-2m}{2m} + \frac{(n-1)\left(\sqrt{\frac{\Delta}{\delta}} - \sqrt{\frac{\delta}{\Delta}}\right)^2}{\Delta+\delta}. \tag{19}$$

Equality holds if and only if $G \cong K_n$, *or* $G \cong K_{\frac{n}{2},\frac{n}{2}}$ *or* $G \in \Gamma_d$.

Proof. Let $\overline{I}$ and $\overline{J}$ be two finite non empty disjoint index sets, $a = (a_i)$ and $b = (b_i)$, $i \in \overline{I} \cup \overline{J}$, sequences of non negative real numbers of the similar monotonicity, and $p = (p_i)$, $i \in \overline{I} \cup \overline{J}$ sequence of real numbers. Denote with $T(a, b, p; \overline{I})$

$$T(a,b,p;\overline{I}) = \sum_{i \in \overline{I}} p_i a_i b_i - \frac{\sum_{i \in \overline{I}} p_i a_i \sum_{i \in \overline{I}} p_i b_i}{\sum_{i \in \overline{I}} p_i}.$$

In [11] (see also [14]) the following inequality was proved

$$T(a,b,p\,;\overline{I}\cup\overline{J}) \geq T(a,b,p\,;\overline{I}) + T(a,b,p\,;\overline{J}). \tag{20}$$

Let $I=\{1,2,\ldots,n\}$, $J_k=\{i_1,i_2,\ldots,i_k\}$, $J_k\subset I$, $1<i_1<\cdots<i_k<n$, $0\leq k\leq n-2$, $I_{n-k}=I-J_k$, $I_n=I$, $I_2=\{1,n\}$ and $I_1=\{1\}$ are index sets. For $\overline{I}=I_{n-k}$ and $\overline{J}=J_k$, $I_{n-k}\cap J_k=\emptyset$, from (20) follows

$$T(a,b,p\,;I_n) \geq T(a,b,p\,;I_{n-k}) + T(a,b,p\,;J_k).$$

For $k=1$, $T(a,b,p\,;J_1)=0$, from the above follows

$$T(a,b,p\,;I_n) \geq T(a,b,p\,;I_{n-1}).$$

After iterating the above inequality we obtain

$$T(a,b,p\,;I_n) \geq T(a,b,p\,;I_{n-1}) \geq \cdots \geq T(a,b,p\,;I_2) \geq 0. \tag{21}$$

From

$$T(a,b,p\,;I_n) \geq T(a,b,p\,;I_2)$$

for $p_i=d_i$, $a_i=b_i=\frac{1}{d_i}$, $i=1,2,\ldots,n$ we obtain

$$\sum_{i=1}^{n} d_i \sum_{i=1}^{n}\frac{1}{d_i} - n^2 \geq \frac{(d_1-d_n)^2}{d_1 d_n(d_1+d_n)}\sum_{i=1}^{n} d_i,$$

i.e.

$$\sum_{i=1}^{n}\frac{1}{d_i} \geq \frac{n^2}{2m} + \frac{\left(\sqrt{\frac{\Delta}{\delta}}-\sqrt{\frac{\delta}{\Delta}}\right)^2}{\Delta+\delta} \tag{22}$$

Now the inequality (19) is obtained from (10) and (22). Equality in (22) holds if and only if $d_1=d_2=\cdots=d_n$. Equality in (10) holds if and only if $G\cong K_n$ or $G\cong K_{\frac{n}{2},\frac{n}{2}}$, so equality in (19) holds if and only if $G\cong K_n$ or $G\cong K_{\frac{n}{2},\frac{n}{2}}$. Note that equality in (19) holds also for $G\in\Gamma_d$. ∎

Remark 7 *Inequalities* (19) *and* (6) *are not comparable. Thus, for example, if* $G\cong K_{\frac{n}{2},\frac{n}{2}}$, *the inequality* (19) *is stronger than* (6). *But, if* $G\cong K_{1,n-1}$, *the inequality* (6) *is stronger than* (19).

In the following corollary of Theorem 4, a lower bound for $Kf(G)$ that depends on n, Δ and δ is determined.

Corollary 3 *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq (n-1)\left(1+\frac{\left(\sqrt{\frac{\Delta}{\delta}}-\sqrt{\frac{\delta}{\Delta}}\right)^2}{\Delta+\delta}\right). \tag{23}$$

Equality holds if and only if $G \cong K_n$.

Remark 8 *Inequalities* (23) *and* (13) *are not comparable. For $G \cong K_{\frac{n}{2},\frac{n}{2}}$, the inequality* (13) *is stronger than* (23). *On the other hand, for $G \cong K_{1,n-1}$, the inequality* (23) *is stronger than* (13).

In the following theorem we prove the inequality that sets up lower bound for $Kf(G)$ that depends on n, m, Δ, Δ_2 and δ.

Theorem 5 *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta}+\frac{(n-1)}{2m-\Delta}\left((n-1)^2+\left(\sqrt{\frac{\Delta_2}{\delta}}-\sqrt{\frac{\delta}{\Delta_2}}\right)^2\right). \tag{24}$$

Equality holds if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$ or $G \in \Gamma_d$.

Proof. Based on the inequality (21) we have that

$$\sum_{i=2}^{n} p_i a_i b_i - \frac{\sum\limits_{i=2}^{n} p_i a_i \sum\limits_{i=2}^{n} p_i b_i}{\sum\limits_{i=2}^{n} p_i} \geq \frac{p_2 p_n (a_2-a_n)(b_2-b_n)}{p_2+p_n}.$$

For $p_i = d_i$, $a_i = b_i = \frac{1}{d_i}$, $i = 2, \ldots, n$ the above inequality transforms into

$$\sum_{i=2}^{n}\frac{1}{d_i} - \frac{(n-1)^2}{\sum\limits_{i=2}^{n} d_i} \geq \frac{(\Delta_2-\delta)^2}{\Delta_2\delta(\Delta_2+\delta)},$$

i.e.

$$\sum_{i=1}^{n}\frac{1}{d_i} \geq \frac{1}{\Delta}+\frac{(n-1)^2}{2m-\Delta}+\frac{\left(\sqrt{\frac{\Delta_2}{\delta}}-\sqrt{\frac{\delta}{\Delta_2}}\right)^2}{\Delta_2+\delta} \tag{25}$$

On the other hand, since

$$\Delta+\Delta_2+\delta \leq 2m,$$

we have that

$$\Delta_2+\delta \leq 2m-\Delta. \tag{26}$$

The inequality (24) is obtained from (10), (25) and (26). ■

Remark 9 *Inequalities* (24) *and* (7) *are not comparable. Thus, for example, for* $G \cong K_{\frac{n}{2},\frac{n}{2}}$ *the inequality* (24) *is stronger than* (7)*, but for* $G \cong P_n$*, the inequality* (7) *is stronger than* (24).

In the following corollary of Theorem 5, a lower bound for $Kf(G)$ in terms of n, Δ, Δ_2 and δ is determined.

Corollary 4 *Let* G *be a simple connected graph with* $n \geq 3$ *vertices and* m *edges. Then*

$$Kf(G) \geq \frac{1}{\Delta}\left(n(n-1) - \Delta + \left(\sqrt{\frac{\Delta_2}{\delta}} - \sqrt{\frac{\delta}{\Delta_2}}\right)^2\right).$$

Equality holds if and only if $G \cong K_n$*, or* $G \cong K_{\frac{n}{2},\frac{n}{2}}$ *or* $G \in \Gamma_d$.

Theorem 6 *Let* G *be a simple connected graph with* $n \geq 4$ *vertices and* m *edges. Then*

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + (n-1)\left(\frac{1}{\Delta_2} + \frac{1}{\delta} + \frac{(n-3)^2}{2m - \Delta - \Delta_2 - \delta}\right). \tag{27}$$

Equality holds if $G \cong K_n$*, or* $G \cong K_{1,n-1}$*, or* $G \cong K_{\frac{n}{2},\frac{n}{2}}$ *or* $G \in \Gamma_d$.

Proof. From the inequality

$$\sum_{i=3}^{n-1} p_i \sum_{i=3}^{n-1} p_i a_i b_i \geq \sum_{i=3}^{n-1} p_i a_i \sum_{i=3}^{n-1} p_i b_i,$$

for $p_i = d_i$, $a_i = b_i = \frac{1}{d_i}$, $i = 3, \ldots, n-1$, we get

$$\sum_{i=1}^{n} \frac{1}{d_i} \geq \frac{1}{\Delta} + \frac{1}{\Delta_2} + \frac{1}{\delta} + \frac{(n-3)^2}{2m - \Delta - \Delta_2 - \delta}. \tag{28}$$

The inequality (27) is obtained from (10) and (28). Since equality in (28) holds if and only if $d_3 = d_4 = \cdots = d_{n-1}$, therefore equality in (27) holds if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$. Equality in (27) holds also when $G \in \Gamma_d$. ■

Remark 10 *The inequality* (27) *is stronger than* (7) *when* $G \cong K_{\frac{n}{2},\frac{n}{2}}$, $G \cong P_n$ *and when* G *is regular. It is an open question whether* (27) *is always better than* (7).

Remark 11 *Interestingly, for d-regular connected graphs,* $1 \leq d \leq n-1$*, from* (9), (13), (15), (17), (18), (19), (24) *and* (27) *follows*

$$Kf(G) \geq \frac{n(n-1) - d}{d}, \tag{29}$$

with equality if and only if $G \cong K_n$ *or* $G \in \Gamma_d$. *The inequality* (29) *is stronger than*

$$Kf(G) \geq \frac{(n-1)^2}{d}$$

which was proved in [15]. However, it is still open question whether the lower bound for $Kf(G)$ *determined by* (29) *is the best possible when bounds depend on parameters* n *and* d *only. Let us note that inequality* (29) *was proved in [16].*

Remark 12 *Let* k_1 *and* k_2 *be non negative integers with the property* $0 \leq k_1, k_2 \leq n-2$. *Assume that*

$$\sum_{i=1}^{0} \frac{1}{d_i} = \sum_{i=n+1}^{n} \frac{1}{d_i} = \sum_{i=0}^{-1} \frac{1}{d_i} = 0.$$

Then, according to

$$\sum_{i=k_1}^{n-k_2} p_i \sum_{i=k_1}^{n-k_2} p_i a_i b_i \geq \sum_{i=k_1}^{n-k_2} p_i a_i \sum_{i=k_1}^{n-k_2} p_i b_i,$$

for $p_i = d_i$, $a_i = b_i = \frac{1}{d_i}$, $i = k_1, k_1+1, \ldots, n-k_2$, *and* (10) *we obtain*

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + (n-1)\left(\sum_{i=2}^{k_2-1} \frac{1}{d_i} + \sum_{i=n-k_2+1}^{n} \frac{1}{d_i} + \frac{(n-k_1-k_2-1)^2}{2m - \sum\limits_{i=1}^{k_1-1} \frac{1}{d_i} - \sum\limits_{i=n-k_2+1}^{n} \frac{1}{d_i}}\right).$$

In the special case the following inequalities are valid

$$\begin{aligned}
Kf(G) &\geq \frac{n-1-\Delta}{\Delta} + (n-1)\left(\frac{1}{\Delta_2} + \cdots + \frac{1}{d_{k_1}} + \frac{(n-k_1)^2}{2m - d_1 - \cdots - d_{k_1}}\right) \geq \\
&\geq \frac{n-1-\Delta}{\Delta} + (n-1)\left(\frac{1}{\Delta_2} + \cdots + \frac{1}{d_{k_1-1}} + \frac{(n-k_1+1)^2}{2m - d_1 - \cdots - d_{k_1-1}}\right) \geq \cdots \geq \\
&\geq \frac{n-1-\Delta}{\Delta} + (n-1)\left(\frac{1}{\Delta_2} + \frac{(n-2)^2}{2m-\Delta-\Delta_2}\right) \geq \\
&\geq \frac{n-1-\Delta}{\Delta} + \frac{(n-1)^3}{2m-\Delta} \geq \frac{n^2(n-1)-2m}{2m} \geq \frac{n(n-1)-\Delta}{\Delta} \geq \\
&\geq n-1 \geq \frac{1}{2}\left(\sqrt{8m+1}-1\right).
\end{aligned}$$

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