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L-Borderenergetic Graphs and Normalized Laplacian Energy

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Abstract

The Laplacian and normalized Laplacian energy of G are given by expressions $E_L(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}|, E_{\mathcal{L}}(G) = \sum_{i=1}^{n} |\lambda_i - 1|$, respectively, where μ_i and λ_i are the eigenvalues of Laplacian matrix L and normalized Laplacian matrix $\mathcal{L}$ of G. An interesting problem in *spectral graph theory* is to find graphs $\{L, \mathcal{L}\}-$noncospectral with the same $E_{\{L,\mathcal{L}\}}(G)$. In this paper, we present graphs of order n, which are L-borderenergetic (in short, $E_L(G) = 2n - 2$) and graphs $\mathcal{L}$-noncospectral with the same normalized Laplacian energy.

1 Introduction

Throughout this paper, all graphs are assumed to be finite, undirected and without loops or multiple edges. If G is a graph of order n and M is a real symmetric matrix associated with G, then the M- *energy* of G is

$$E_M(G) = \sum_{i=1}^{n} \left| \lambda_i(M) - \frac{tr(M)}{n} \right|. \tag{1}$$

The energy $E(G)$ of a graph G simply refers to using the adjacency matrix in (1). There are many results on energy and its applications in several areas, including in chemistral see [10] for more details and the references [2, 11–14, 16].

Recently, a new concept as *borderenergetic* graphs [5] was proposed, namely graphs of order n satisfying $E(G) = 2n - 2$. In this way, several authors have been presented families of borderenergetic graphs [2, 7, 8, 11, 16].

An analogous concept as borderenergetic graphs, called L-borderenergetic graphs was proposed in [18]. That is, a graph G of order n is L-borderenergetic if $E_L(G) = 2n - 2$, where $E_L(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}|$, and $\overline{d}$ is the avarage degree of G. Some classes of L-borderenergetic of order $n = 4r + 4$ $(r \geq 1)$ are obtained in [18]. In [3], a kind of threshold graphs were found to be L-borderenergetic, and all the connected non-complete and pairwise non-isomorphic L-borderenergetic graphs of small order n depicted for n with $4 \leq n \leq 9$.

Since that finding noncospectral graphs with the same energy is an interesting problem in spectral graph theory, in this paper we continue this investigation presenting some new graphs which are L- borderenergetic and finish it showing two classes of graphs that are $\mathcal{L}$-noncospectral with the same normalized Laplacian energy.

The paper is organized as follows. In Section 2 we describe some known results about the Laplacian and normalized Laplacain spectrum of graphs. In Section 3 we present two classes of L-borderenergetic graphs. We finalize this paper, showing graphs with the same normalized Laplacian energy.

2 Premilinares

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be undirected graphs without loops or multiple edges. The *union* $G_1 \cup G_2$ of graphs G_1 and G_2 is the graph $G = (V, E)$ for which $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. We denote the graph $\underbrace{G \cup G \cup \ldots \cup G}_{m}$ by mG. The *join* $G_1 \nabla G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2.

The Laplacian spectrum of $G_1 \cup \ldots \cup G_k$ is the union of Laplacian spectra of $G_1, \ldots, G_k$, while the Laplacian spectra of the complement of n- vertex graph G consists of values $n - \mu_i$, for each Laplacian eigenvalue μ_i of G, except for a single instance of eigenvalue 0 of G.

Theorem 1 *([9]) Let G_1 and G_2 be graphs on n_1 and n_2 vertices, respectively. Let L_1 and L_2 be the Laplacian matrices for G_1 and G_2, respectively, and let L be the Laplacian matrix for $G_1 \nabla G_2$. If $0 = \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{n_1}$ and $0 = \beta_1 \leq \beta_2 \leq \ldots \leq \beta_{n_2}$ are the eigenvalues of L_1 and L_2, respectively. Then the eigenvalues of L are*

$$0, \;\; n_2 + \alpha_2, \;\; n_2 + \alpha_3, \ldots, \; n_2 + \alpha_{n_1}$$

$$n_1 + \beta_2, \;\; n_1 + \beta_3, \ldots, \; n_1 + \beta_{n_2}, \;\; n_1 + n_2.$$

The following result is due to Butler([1], Theorem 12).

Theorem 2 *Let $G_1 = (V_1, E_1)$ be an r- regular graph on n vertices and $G_2 = (V_2, E_2)$ be an s- regular graph on m vertices. Suppose*

$$0 = \lambda_1 \leq \ldots \leq \lambda_n \leq 2$$

are the $\mathcal{L}$- eigenvalues of G_1 and

$$0 = \mu_1 \leq \ldots \leq \mu_m \leq 2$$

are the $\mathcal{L}$- eigenvalues of G_2. Then the $\mathcal{L}$-eigenvalues of $G_1 \nabla G_2$ are

$$0, \frac{m + r\lambda_2}{m + r}, \ldots, \frac{m + r\lambda_n}{m + r}, \frac{n + s\mu_2}{n + s}, \ldots, \frac{n + s\mu_m}{n + s}, \frac{m}{m + r} + \frac{n}{n + s}.$$

3 Constructing new L-borderenergetic graphs

Recall that the *Laplacian energy* $E_L(G)$ of G is defined to be $\sum_{i=1}^{n} |\mu_i - \overline{d}|$, where $0 = \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$ are the Laplacian eigenvalues of G and $\overline{d}$ is the average degree of G. It is known that the complete graph K_n has Laplacian energy $2(n-1)$. We exhibit two infinite classes which are L-noncospectral and L-borderenergetic graphs.

3.1 The class $K_{n-1} \odot K_n$

For each integer $n \geq 3$, we define the graph G in $K_{n-1} \odot K_n$ to be the following join

$$G = (K_{n-1} \cup K_{n-2}) \nabla K_1$$

of order $2n-2$. Let μ^m denote the laplacian eigenvalue μ with multiplicity equals to m. The Figure 1 shows the graph $K_4 \odot K_5$.

Lemma 1 *Let $G = K_{n-1} \odot K_n$ be a graph of order $2n-2$. Then the Laplacian spectrum of G is given by*

$$0;\ \ 1;\ \ (n-1)^{n-3};\ \ n^{n-2};\ \ 2n-2.$$

Proof: Let G be a graph in $K_{n-1} \odot K_n$. Since that K_{n-1} and K_{n-2} have Laplacian spectrum equal to $\{(n-1)^{n-2}, 0\}$ and $\{(n-2)^{n-3}, 0\}$, respectively. Taking $G_1 = K_{n-1} \cup K_{n-2}$ and $G_2 = K_1$, according by Theorem 1, follows that the Laplacian spectrum of G is equal to

$$0;\ \ 1;\ \ (n-1)^{n-3};\ \ n^{n-2};\ \ 2n-2.$$

Theorem 3 *For each $n \geq 3$, $G = K_{n-1} \odot K_{n-2}$ is L-borderenergetic and L-noncospectral graph with K_{2n-2}.*

Proof: Clearly G and K_{2n-2} are L-noncospectral. Let $\overline{d}$ be the average degree of G. Since that $\overline{d}$ is equal to average of Laplacian eigenvalues of G then $\overline{d} = \frac{2n-2+n(n-2)+(n-1)(n-3)+1}{2n-2} = n-1$. Using Lemma 1, $E_L(G) = |2n-2-(n-1)| + (n-2)|n-(n-1)| + (n-3)|(n-1)-(n-1)| + |1-(n-1)| + |0-(n-1)| = 4n-6 = E_L(K_{2n-2})$.

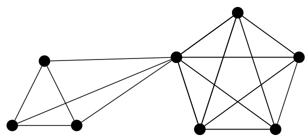

Figure 1. Graph $K_4 \odot K_5$

3.2 Another Construction

For each integer $n \geq 3$, we define the graph G of order $2n$, obtained from two copies of the complete graph by adding n edges between one vertex of a copy of K_n and n vertices of the other copy.

Remark: This construction was first introduced by Stevanović in [17], where it is studied other spectral properties of this graph, as Laplacian energy. Let's denote this class by $K_n \cdot K_n$. The Figure 2 shows the graph $K_3 \cdot K_3$.

Lemma 2 *Let $G = K_n \cdot K_n$ be a graph of order $2n$. Then the Laplacian spectrum of G is given by*

$$0;\ \ 1;\ \ n^{n-2};\ \ (n+1)^{n-1};\ \ 2n.$$

Proof: Let $G = K_n \cdot K_n$ be a graph of order $2n$. Since that G can be viewed as the join $(K_n \cup K_{n-1})\nabla K_1$, the proof is similar to Lemma 1.

Theorem 4 *For each $n \geq 3$, $G = K_n \cdot K_n$ is L-borderenergetic and L-noncospectral graph with K_{2n}.*

Proof: Clearly $G = K_n \cdot K_n$ and K_{2n} are L-noncospectral. Let $\overline{d}$ be the average degree of G. Since that $\overline{d} = n$ then $E_L(G) = |2n-n| + (n-1)|n+1-n| + (n-2)|n-n| + |1-n| + |0-n| = 4n-2 = E_L(K_{2n})$.

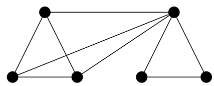

Figure 2. Graph $K_3 \cdot K_3$

4 Graphs with same normalized Laplacian energy

In this section we present graphs which have the same normalized Laplacian energy $E_{\mathcal{L}}(G)$. Let K_n be the complete graph on n vertices. For positive integer $b \geq 2$ we define the following classes of graphs:

- The class of graphs $2K_2 \nabla K_b$.
- The class of graphs $2K_2 \nabla bK_1$.

We let λ_i^m denote the m- multiplicity of normalized laplacian eigenvalue λ_i.

Lemma 3 *Let be an integer positive $b \geq 2$ and $G = 2K_2 \nabla K_b$ a graph of order $n = b + 4$. Then*

$$0, \frac{b}{b+1}, \left(\frac{b+2}{b+1}\right)^2, \left(\frac{b+4}{b+3}\right)^{b-1}, \frac{b}{b+1} + \frac{4}{b+3}.$$

are the $\mathcal{L}$- eigenvalues of G.

Proof: Let $G_1 = 2K_2$ and $G_2 = K_b$ be the graphs of order $n = 4$ and $m = b$, respectively. Since that G_1 is an 1-regular graph and G_2 is an $(b-1)$-regular graph, and the $\mathcal{L}$-eigenvalues of G_1 and G_2 are given by $\{0^2, 2^2\}$ and $\{0, (\frac{b}{b-1})^{b-1}\}$, respectively. Taking $r = 1$ and $s = b - 1$, then the result follows by Theorem (2).

Lemma 4 *Let be an integer positive $b \geq 2$ and $G' = 2K_2 \nabla bK_1$ a graph of order $n = b+4$. Then*

$$0, \frac{b}{b+1}, \left(\frac{b+2}{b+1}\right)^2, (1)^{b-1}, \frac{b}{b+1} + 1.$$

are the $\mathcal{L}$- eigenvalues of G'.

Proof: Similar to Lemma 3.

Theorem 5 *Let be an integer positive $b \geq 2$, $G = 2K_2 \nabla K_b$ and $G' = 2K_2 \nabla bK_1$ graphs of order $n = b + 4$. Then G and G' are $\mathcal{L}$-noncospectral and have the same normalized Laplacian energy. Furthermore*

$$E_{\mathcal{L}}(G) = E_{\mathcal{L}}(G') = \frac{2b+4}{b+1}.$$

Proof: Let $G = 2K_2 \nabla K_b$ and $G' = 2K_2 \nabla bK_1$ be graphs of order $n = b+4$. Clearly G and G' are $\mathcal{L}$-noncospectral. Using that $E_{\mathcal{L}}(G) = \sum_{i=1}^{n} |\lambda_i - 1|$ and Lemma 3, follows

$$E_{\mathcal{L}}(G) = |0-1| + \left|\frac{b}{b+1} - 1\right| + 2\left|\frac{b+2}{b+1} - 1\right| + (b-1)\left|\frac{b+4}{b+3} - 1\right| + \left|\frac{b}{b+1} + \frac{4}{b+3} - 1\right|$$

$$E_{\mathcal{L}}(G) = \frac{2b^2+10b+12}{(b+1)(b+3)} = \frac{2(b+2)(b+3)}{(b+1)(b+3)} = \frac{2b+4}{b+1}.$$

If $G' = 2K_2 \nabla bK_1$, by Lemma (4), we have that

$$E_{\mathcal{L}}(G') = |0-1| + \left|\frac{b}{b+1} - 1\right| + 2\left|\frac{b+2}{b+1} - 1\right| + (b-1)|1-1| + \left|\frac{b}{b+1} + 1 - 1\right|$$

$$E_{\mathcal{L}}(G') = 1 + \frac{3}{b+1} + \frac{b}{b+1} = \frac{2b+4}{b+1},$$

and then the result follows.

Now we present the general case. Let K_n be the complete graph on n vertices. For positive integers $a, b \geq 2$ we define the following classes of graphs:

- The class of graphs $aK_2 \nabla K_b$.
- The class of graphs $aK_2 \nabla bK_1$.

Lemma 5 *Let be the integers positive $a, b \geq 2$ and $G = aK_2 \nabla K_b$ a graph of order $n = 2a+b$. Then*

$$0, \left(\frac{b}{b+1}\right)^{a-1}, \left(\frac{b+2}{b+1}\right)^{a}, \left(\frac{2a+b}{2a+b-1}\right)^{b-1}, \frac{b}{b+1} + \frac{2a}{2a+b-1}.$$

are the $\mathcal{L}$- eigenvalues of G.

Lemma 6 *Let be the integers positive $a, b \geq 2$ and $G' = aK_2 \nabla bK_1$ a graph of order $n = 2a+b$. Then*

$$0, \left(\frac{b}{b+1}\right)^{a-1}, \left(\frac{b+2}{b+1}\right)^{a}, (1)^{b-1}, \frac{b}{b+1} + 1.$$

are the $\mathcal{L}$- eigenvalues of G'.

Theorem 6 *Let be the integers positive $a, b \geq 2$, $G = aK_2 \nabla K_b$ and $G' = aK_2 \nabla bK_1$ graphs of order $n = 2a+b$. Then G and G' are $\mathcal{L}$-noncospectral and have the same normalized laplacian energy. Furthermore*

$$E_{\mathcal{L}}(G) = E_{\mathcal{L}}(G') = \frac{2a+2b}{b+1}.$$

Proof: Let $G = aK_2 \nabla K_b$ and $G' = aK_2 \nabla bK_1$ be graphs of order $n = 2a + b$. Clearly G and G' are $\mathcal{L}$-noncospectral. Using that $E_{\mathcal{L}}(G) = \sum_{i=1}^{n} |\lambda_i - 1|$ and Lemma 5, follows

$$E_{\mathcal{L}}(G) = |0-1| + (a-1)\left|\frac{b}{b+1} - 1\right| + a\left|\frac{b+2}{b+1} - 1\right| + (b-1)\left|\frac{2a+b}{2a+b-1} - 1\right| + \left|\frac{b}{b+1} + \frac{2a}{2a+b-1} - 1\right|$$

$$E_{\mathcal{L}}(G) = \frac{4a^2 + 6ab + 2b^2 - 2a - 2b}{(b+1)(2a+b-1)} = \frac{(2a+2b)(2a+b-1)}{(b+1)(2a+b-1)} = \frac{2a+2b}{b+1}.$$

If $G' = aK_2 \nabla bK_1$, by Lemma (6), we have that

$$E_{\mathcal{L}}(G') = |0-1| + (a-1)\left|\frac{b}{b+1} - 1\right| + a\left|\frac{b+2}{b+1} - 1\right| + (b-1)|1-1| + \left|\frac{b}{b+1} + 1 - 1\right|$$

$$E_{\mathcal{L}}(G') = 1 + \frac{2a-1}{b+1} + \frac{b}{b+1} = \frac{2a+2b}{b+1},$$

and then the result follows.

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