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A Computer Search for the L-Borderenergetic Graphs

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Abstract

The Laplacian energy of a simple connected graph G is defined as $LE(G) = \sum_{i=1}^{n} |\lambda_i - \overline{d}|$, where λ_i are the Laplacian eigenvalues of G and $\overline{d}$ is the average degree of G. Recently, Tura proposed the concept of L-borderenergetic graphs, which means that $LE(G) = LE(K_n)$. In this paper, we first consider the extremal number of edges of non-complete L-borderenergetic graph, then use a computer search to find out all the L-borderenergetic graphs on no more than 10 vertices. The number of such graphs is 185. This could provide experience for further study on the L-borderenergetic graphs on large number of vertices.

1 Introduction

All graphs considered here are simple undirected and connected. Let G be such a graph on n vertices, the energy of G [3–5], denoted by $E(G)$, is defined as the sum of the absolute values of all eigenvalues of its adjacency matrix. If a graph G on n vertices has the same energy as the complete graph K_n, that is $E(G) = E(K_n) = 2n - 2$, then G is said to be borderenergetic, see [2]. In [2] Gong et al. determined all the non-complete borderenergetic graphs on $n = 7, 8$ and 9 vertices, In [13, 14] Li and Shao et al. use a

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computer search to find out all the borderenergetic graphs on $n = 10$ vertices. For more recent results on borderenergetic graphs, see [15,16].

The Laplacian energy of a graph G [6] is defined as $LE(G) = \sum_{i=1}^{n} |\lambda_i - \overline{d}|$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$ are the Laplacian eigenvalues of G and $\overline{d}$ is the average degree of G. For its basic properties, see [7–12]. Recently, Tura [17] proposed the concept of L-borderenergetic graphs, which means $LE(G) = LE(K_n)$, and gave several classes of L-borderenergetic graphs. In this paper, we first consider the extremal number of edges of non-complete L-borderenergetic graph, then use a computer search to find out all the non-complete L-borderenergetic graphs on no more than 10 vertices. The total number of such graphs is 185. In next section, we list all the non-complete L-borderenergetic graphs on $n \leq 10$ vertices, respectively, and we hope this could provide experience for further study for the L-borderenergetic graphs on large number of vertices.

2 Main results

Let T be a tree on n vertices. In [1], it has been proven that $LE(T) \leq LE(S_n)$ with equality holding if and only if $T \cong S_n$, where S_n is a star graph on n vertices. By simple calculation we have $LE(S_n) = 2n - 4 + \frac{4}{n} < 2n - 2$, for $n > 2$, therefore, $LE(T) < LE(K_n) = 2n - 2$ hold for all trees on more than 2 vertices. That is to say, there are no any L-borderenergetic tree. Therefore, a L-borderenergetic graph on n vertices has at least n edges.

For each $n \geq 4$, let S_n^+ denote the graph obtained from star graph by adding an edge, by simple calculation, its Laplacian spectrum is $Sp(S_n^+) = \{0, 1^{n-3}, 3, n\}$, and $\overline{d_{S_n^+}} = 2$. Hence $LE(S_n^+) = (2-0) + (2-1)(n-3) + (3-2) + (n-2) = 2n-2$. Therefore, we have

Proposition 1. *For each $n \geq 4$, the graph S_n^+ is L-borderenergetic.*

Let $G_1 \bigtriangledown G_2$ denote the join of graphs G_1 and G_2, obtained from the union of G_1 and G_2 by joining every vertex of G_1 with every vertex of G_2. For even integer n, let CP_n denote the cocktail party graph obtained from K_n by deleting a perfect matching. The following lemma is from [17].

Lemma 2. *Let G_1 and G_2 be graphs on n_1 and n_2 vertices, respectively. Let L_1 and L_2 be the Laplacian matrices for G_1 and G_2, respectively, and let L be the Laplacian matrix for $G_1 \bigtriangledown G_2$. If $0 = \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{n_1}$ and $0 = \beta_1 \leq \beta_2 \leq \ldots \leq \beta_{n_2}$ are the eigenvalues of*

L_1 and L_2, respectively. Then the eigenvalues of L are $\{0, n_2+\alpha_2, n_2+\alpha_3, \ldots, n_2+\alpha_{n_1}, n_1+\beta_2, n_1+\beta_3, \ldots, n_1+\beta_{n_2}, n_1+n_2\}$.

Lemma 3. *([18]) Let G be a graph with n vertices and minimum degree δ. If $G \neq K_n$ then $\lambda_{n-1} \leq \delta$.*

Lemma 4. *([8]) Let G be a graph with $n \geq 3$ vertices. Then*

$$LE(G) \geq 4\overline{d} - 2\lambda_{n-1}.$$

Proposition 5. *For each even integer n, $n \geq 6$, let H_n denote the non-complete L-borderenergetic graph on n vertices has maximum edges, then $H_n \cong K_2 \bigtriangledown CP_{n-2}$.*

Proof. It is easy to get that $LE(H_n) = 2n-2$, and noting that H_n has $\frac{n(n-1)}{2} - (\frac{n}{2}-1)$ edges. Let G be any non-complete graph on n vertices and m, $m = \frac{n(n-1)}{2} - (\frac{n}{2}-1) + k$, where $1 \leq k < \frac{n}{2} - 1$, edges with minimum degree δ. We will show $LE(G) > 2n-2$, hence G can not be a L-borderenergetic graph.

If $\delta = n-2$, then G can be seen as obtained from H_n by adding k edges. In this case, we have $G \cong K_{2+2k} \bigtriangledown CP_{n-2-2k}$. By lemma 2 and direct calculation one has that the eigenvalues of G are $\{0, n-2^{(\frac{n}{2}-1-k)}, n^{(\frac{n}{2}+k)}\}$. Hence $LE(G) = 2n-2+2k-\frac{4k+4k^2}{n} > 2n-2$ holds.

If $\delta \leq n-3$, then we have $\lambda_{n-1} \leq n-3$. By lemma 4, we have

$$LE(G) \geq 4\overline{d} - 2\lambda_{n-1} \geq 4(\frac{n^2-2n+2+2k}{n}) - 2(n-3) = 2n-2+\frac{8k+8}{n} > 2n-2.$$

This completes the proof. □

Remark 1: For each odd integer $n \geq 5$, if G is a non-complete L-borderenergetic graph on n vertices and m edges, then we can show that $m < \frac{n(n-1)}{2} - \frac{n-1}{2}$. Otherwise, let δ denote the minimum degree of G, if $m \geq \frac{n(n-1)}{2} - \frac{n-1}{2}$ and $\delta = n-2$, then $G \cong K_{n_1} \bigtriangledown CP_{n_2}$ for some even integer n_2 and $n_1 + n_2 = n$, by direct calculation, the eigenvalues of G are $\{0, (n-2)^{(\frac{n_2}{2})}, n^{(n_1+\frac{n_2}{2}-1)}\}$, consequently, $LE(G) = 2n-2+\frac{n_2(n_1-2)}{n} \neq 2n-2$ for n_1 is odd.

If $\delta \leq n-3$, then $\lambda_{n-1} \leq n-3$ by Lemma 3. Hence we have

$$LE(G) \geq 4\overline{d} - 2\lambda_{n-1} \geq 4(\frac{n(n-1)-n+1}{n}) - 2(n-3) = 2n-2+\frac{4}{n} > 2n-2.$$

Remark 2: From [2] we know that, for each $n \geq 10$, even or odd, there is a $(n-3)$-regular graph $\overline{pC_4 \bigcup qC_6 \bigcup rC_3}$, where p, q, and r are non-negative integers with $p+q=2$

and $4p + 6q + 3r = n$, is borderenergetic. And it is well known that, for any regular graph, the energy and the laplacian energy are equal, hence $\overline{pC_4 \bigcup qC_6 \bigcup rC_3}$ is also L-borderenergetic, whose number of edges is $m = \frac{n(n-1)}{2} - n$, therefore, if G is a non-complete L-borderenergetic graph on n vertices with maximum edges, we have $m \geq \frac{n(n-1)}{2} - n$. Further, we should point out that non-complete L-borderenergetic graphs on n vertices with $\frac{n(n-1)}{2} - n$ edges may not be unique. For example, if $\overline{pC_4 \bigcup qC_6 \bigcup rC_3}$ is L-borderenergetic and $p \neq 0$ one can replace C_4 with S_4^+ in graph $\overline{pC_4 \bigcup qC_6 \bigcup rC_3}$, then the resulted graph is also L-borderenergetic graph with $\frac{n(n-1)}{2} - n$ edges, this can be verified by simple calculation.

Next, we perform a computer-aided calculation for $n \leq 10$ vertices graphs and obtain that there are totally 185 non-complete L-borderenergetic graphs, among which there are 2 on $n = 4$ vertices, 1 on $n = 5$ vertices, 11 on $n = 6$ vertices, 4 on $n = 7$ vertices, 31 on $n = 8$ vertices, 16 on $n = 9$ vertices and 120 on $n = 10$ vertices. We list the results as follows

Proposition 6. *There are totally 18 L-borderenergetic graphs on less than 8 vertices, which are listed in Figure 1 in Appendix, their Laplacian spectra are listed in the following tabular.*

$Sp(G_1^4)=\{0,1,3,4\}$	$Sp(G_4^6)=\{0,1,3,3,5,6\}$	$Sp(G_{10}^6)=\{0,3,4,5,6,6\}$
$Sp(G_2^4)=\{0,2,4,4\}$	$Sp(G_5^6)=\{0,1,3,4,4,6\}$	$Sp(G_{11}^6)=\{0,4,4,6,6,6\}$
$Sp(G_1^5)=\{0,1,1,3,5\}$	$Sp(G_6^6)=\{0,2,2,4,4,6\}$	$Sp(G_1^7)=\{0,1,1,1,1,3,7\}$
$Sp(G_1^6)=\{0,1,1,1,3,6\}$	$Sp(G_7^6)=\{0,2,3,4,5,6\}$	$Sp(G_2^7)=\{0,2,4,4,5,6,7\}$
$Sp(G_2^6)=\{0,1,1,3,3,6\}$	$Sp(G_8^6)=\{0,3,3,4,6,6\}$	$Sp(G_3^7)=\{0,2,4,5,5,5,7\}$
$Sp(G_3^6)=\{0,1,2,3,4,6\}$	$Sp(G_9^6)=\{0,3,3,5,5,6\}$	$Sp(G_4^7)=\{0,3,3,4,5,6,7\}$

Proposition 7. *There are exactly 31 L-borderenergetic graphs on 8 vertices, which are listed in Figure 2 in Appendix, their Laplacian spectra are listed in the following tabular.*

$Sp(G_1^8)$={0,1,1,1,1,1,3,8}	$Sp(G_{17}^8)$={0,4,4,4,5,5,8,8}
$Sp(G_2^8)$={0,1,1,1,3,3,3,8}	$Sp(G_{18}^8)$={0,4,4,4,5,6,7,8}
$Sp(G_3^8)$={0,1,1,2,3,3,4,8}	$Sp(G_{19}^8)$={0,4,4,4,6,6,6,8}
$Sp(G_4^8)$={0,1,1,3,3,4,4,8}	$Sp(G_{20}^8)$={0,3,5,5,5,7,7,8}
$Sp(G_5^8)$={0,1,1,3,3,3,5,8}	$Sp(G_{21}^8)$={0,3,5,5,6,6,7,8}
$Sp(G_6^8)$={0,1,2,2,4,5,$5\pm\sqrt{3}$}	$Sp(G_{22}^8)$={0,4,4,5,5,6,8,8}
$Sp(G_7^8)$={0,1,3,3,4,4,5,8}	$Sp(G_{23}^8)$={0,4,4,5,5,7,7,8}
$Sp(G_8^8)$={0,1,4,4,4,4,7,8}	$Sp(G_{24}^8)$={0,4,4,5,6,6,7,8}
$Sp(G_9^8)$={0,1,4,4,5,5,5,8}	$Sp(G_{25}^8)$={0,4,4,6,6,6,6,8}
$Sp(G_{10}^8)$={0,2,3,4,5,5,6,7}	$Sp(G_{26}^8)$={0,4,5,5,6,6,8,8}
$Sp(G_{11}^8)$={0,3,3,3,5,5,5,8}	$Sp(G_{27}^8)$={0,4,5,5,6,7,7,8}
$Sp(G_{12}^8)$={0,2,4,4,5,7,$6\pm\sqrt{2}$}	$Sp(G_{28}^8)$={0,4,6,6,6,6,8,8}
$Sp(G_{13}^8)$={0,3,3,4,5,5,6,8}	$Sp(G_{29}^8)$={0,5,5,5,6,7,8,8}
$Sp(G_{14}^8)$={0,2,5,5,5,5,6,8}	$Sp(G_{30}^8)$={0,5,6,6,7,8,8,8}
$Sp(G_{15}^8)$={0,3,4,4,6,7,$6\pm\sqrt{2}$}	$Sp(G_{31}^8)$={0,6,6,6,8,8,8,8}
$Sp(G_{16}^8)$={0,3,4,4,5,5,7,8}	

Proposition 8. *There are exactly 16 L-borderenergetic graphs on 9 vertices, which are listed in Figure 3 in Appendix, their Laplacian spectra are listed in the following tabular.*

$Sp(G_1^9)$={0,1,1,1,1,1,1,3,9}	$Sp(G_9^9)$={0,1,3,4,4,4,5,6,9}
$Sp(G_2^9)$={0,2,3,3,4,6,6,$6\pm\sqrt{2}$}	$Sp(G_{10}^9)$={0,2,3,3,4,5,5,6,8}
$Sp(G_3^9)$={0,2,3,3,4,5,6,6,7}	$Sp(G_{11}^9)$={0,4,5,5,5,5,7,8,9}
$Sp(G_4^9)$={0,1,3,4,5,5,5,6,7}	$Sp(G_{12}^9)$={0,4,6,6,6,7,7,9,9}
$Sp(G_5^9)$={0,2,3,3,5,5,5,6,7}	$Sp(G_{13}^9)$={0,4,6,6,6,7,8,8,9}
$Sp(G_6^9)$={0,2,3,3,5,5,6,6,6}	$Sp(G_{14}^9)$={0,5,5,6,6,7,7,9,9}
$Sp(G_7^9)$={0,1,3,4,4,4,4,7,9}	$Sp(G_{15}^9)$={0,4,6,6,6,6,8,9,9}
$Sp(G_8^9)$={0,1,3,4,4,5,6,6,7}	$Sp(G_{16}^9)$={0,6,6,7,7,7,9,9,9}

Proposition 9. *There are exactly 120 L-borderenergetic graphs on 10 vertices, which are listed in Figure 4 and Figure 5 in Appendix, their Laplacian spectra are listed in following two tables, where* λ_1,λ_2,λ_3 *are three roots of equation* $x^3 - 20x^2 + 129x - 268 = 0$.

$Sp(G_1^{10})$={0,1,1,1,1,1,1,1,3,10}	$Sp(G_{41}^{10})$={0,1,2,4,4,4,4,5,6,10}
$Sp(G_2^{10})$={0,1,1,1,1,3,3,3,3,10}	$Sp(G_{42}^{10})$={0,1,2,4,4,5,5,5,7,7}
$Sp(G_3^{10})$={0,1,1,1,2,3,3,3,4,10}	$Sp(G_{43}^{10})$={0,1,2,4,4,4,5,6,7,7}
$Sp(G_4^{10})$={0,1,1,2,2,4,4,5,5,6}	$Sp(G_{44}^{10})$={0,1,3,3,4,5,5,5,6,8}
$Sp(G_5^{10})$={0,1,2,2,2,2,5,5,5,6}	$Sp(G_{45}^{10})$={0,1,2,4,4,4,4,4,7,10}
$Sp(G_6^{10})$={0,1,2,2,2,4,5,5,5,6}	$Sp(G_{46}^{10})$={0,1,3,3,4,4,5,5,7,8}
$Sp(G_7^{10})$={0,1,2,2,2,4,5,5,$\frac{1}{2}(11\pm\sqrt{17})$}	$Sp(G_{47}^{10})$={0,1,3,3,5,5,5,5,5,8}
$Sp(G_8^{10})$={0,1,2,2,3,4,5,5,5,7}	$Sp(G_{48}^{10})$={0,3,3,5,6,7,$\frac{1}{2}(5\pm\sqrt{5})$,$\frac{1}{2}(11\pm\sqrt{5})$}
$Sp(G_9^{10})$={0,1,2,2,3,4,4,5,6,7}	$Sp(G_{49}^{10})$={0,2,2,4,4,5,5,6,6,8}
$Sp(G_{10}^{10})$={0,1,2,2,3,4,5,7,$5\pm\sqrt{2}$}	$Sp(G_{50}^{10})$={0,2,3,3,4,5,5,5,7,8}
$Sp(G_{11}^{10})$={0,1,2,2,3,4,5,5,6,6}	$Sp(G_{51}^{10})$={0,2,2,4,4,5,5,5,7,8}
$Sp(G_{12}^{10})$={0,2,2,2,2,5,5,5,5,6}	$Sp(G_{52}^{10})$={0,2,3,3,4,5,5,6,6,8}
$Sp(G_{13}^{10})$={0,1,2,3,3,5,5,5,6,6}	$Sp(G_{53}^{10})$={0,2,2,4,4,5,5,6,7,7}
$Sp(G_{14}^{10})$={0,2,2,2,3,4,5,5,6,7}	$Sp(G_{54}^{10})$={0,1,3,4,4,5,5,6,6,8}
$Sp(G_{15}^{10})$={0,1,2,3,3,5,$\frac{1}{2}(11\pm\sqrt{5})$,$\frac{1}{2}(11\pm\sqrt{5})$}	$Sp(G_{55}^{10})$={0,2,3,3,4,5,5,7,$\frac{1}{2}(13\pm\sqrt{17})$}
$Sp(G_{16}^{10})$={0,2,2,2,3,5,5,5,6,6}	$Sp(G_{56}^{10})$={0,3,4,6,6,7,$\frac{1}{2}(5\pm\sqrt{5})$,$\frac{1}{2}(11\pm\sqrt{5})$}
$Sp(G_{17}^{10})$={0,2,3,4,5,6,$2\pm\sqrt{2}$,$6\pm\sqrt{2}$}	$Sp(G_{57}^{10})$={0,2,3,4,4,5,5,6,7,8}
$Sp(G_{18}^{10})$={0,1,2,3,3,4,5,6,6,6}	$Sp(G_{58}^{10})$={0,1,4,4,4,5,6,6,6,8}
$Sp(G_{19}^{10})$={0,2,2,2,3,5,5,5,5,7}	$Sp(G_{59}^{10})$={0,2,3,4,4,5,6,6,6,8}
$Sp(G_{20}^{10})$={0,1,2,3,3,5,5,5,$6\pm\sqrt{5}$}	$Sp(G_{60}^{10})$={0,2,3,4,4,5,6,λ_1,λ_2,λ_3}
$Sp(G_{21}^{10})$={0,1,2,3,3,4,4,5,6,8}	$Sp(G_{61}^{10})$={0,2,4,4,4,5,5,6,8,8}
$Sp(G_{22}^{10})$={0,2,2,3,3,5,5,5,6,7}	$Sp(G_{62}^{10})$={0,2,4,4,4,5,6,6,6,9}
$Sp(G_{23}^{10})$={0,2,2,3,3,5,5,6,6,6}	$Sp(G_{63}^{10})$={0,2,4,4,4,5,5,7,7,8}
$Sp(G_{24}^{10})$={0,2,2,3,3,4,5,5,6,8}	$Sp(G_{64}^{10})$={0,2,4,4,4,5,6,6,7,8}
$Sp(G_{25}^{10})$={0,2,2,3,3,4,5,5,7,7}	$Sp(G_{65}^{10})$={0,3,4,4,4,5,5,7,8,8}
$Sp(G_{26}^{10})$={0,1,3,3,3,5,5,5,6,7}	$Sp(G_{66}^{10})$={0,3,4,4,4,5,$\frac{1}{2}(13\pm\sqrt{5})$,$\frac{1}{2}(15\pm\sqrt{5})$}
$Sp(G_{27}^{10})$={0,2,2,3,3,5,5,5,5,8}	$Sp(G_{67}^{10})$={0,3,4,4,5,5,7,8,$7\pm\sqrt{2}$}
$Sp(G_{28}^{10})$={0,1,3,3,3,4,4,5,7,8}	$Sp(G_{68}^{10})$={0,2,4,5,5,6,6,7,7,8}
$Sp(G_{29}^{10})$={0,2,2,2,3,4,5,5,6,6,7}	$Sp(G_{69}^{10})$={0,3,3,5,5,6,6,6,8,8}
$Sp(G_{30}^{10})$={0,2,2,2,3,4,5,5,6,$\frac{1}{2}(13\pm\sqrt{17})$}	$Sp(G_{70}^{10})$={0,3,4,4,5,6,$\frac{1}{2}(13\pm\sqrt{5})$,$\frac{1}{2}(15\pm\sqrt{5})$}
$Sp(G_{31}^{10})$={0,2,4,4,6,6,$\frac{1}{2}(5\pm\sqrt{5})$,$\frac{1}{2}(13\pm\sqrt{5})$}	$Sp(G_{71}^{10})$={0,3,4,4,5,5,6,7,8,8}
$Sp(G_{32}^{10})$={0,2,2,2,3,4,5,5,7,$6\pm\sqrt{2}$}	$Sp(G_{72}^{10})$={0,5,5,5,7,8,$3\pm\sqrt{2}$,$7\pm\sqrt{2}$}
$Sp(G_{33}^{10})$={0,2,2,2,3,4,5,5,5,7,7}	$Sp(G_{73}^{10})$={0,2,4,5,5,6,6,6,8,8}
$Sp(G_{34}^{10})$={0,1,2,4,4,4,5,6,6,8}	$Sp(G_{74}^{10})$={0,3,3,5,5,5,6,7,8,8}
$Sp(G_{35}^{10})$={0,2,2,2,3,5,5,5,5,5,8}	$Sp(G_{75}^{10})$={0,2,4,5,5,5,6,7,7,9}
$Sp(G_{36}^{10})$={0,2,2,2,3,5,5,5,5,6,7}	$Sp(G_{76}^{10})$={0,2,4,5,5,6,6,6,7,9}
$Sp(G_{37}^{10})$={0,1,3,3,5,5,5,5,6,7}	$Sp(G_{77}^{10})$={0,2,4,5,5,5,7,7,7,8}
$Sp(G_{38}^{10})$={0,1,3,3,4,4,5,6,6,8}	$Sp(G_{78}^{10})$={0,3,4,4,5,6,6,6,7,9}
$Sp(G_{39}^{10})$={0,2,3,3,3,5,5,5,6,8}	$Sp(G_{79}^{10})$={0,2,4,5,6,6,6,6,7,8}
$Sp(G_{40}^{10})$={0,2,2,3,4,4,5,6,6,8}	$Sp(G_{80}^{10})$={0,3,4,5,5,6,6,7,8,8}

$Sp(G_{81}^{10})$={0,3,4,5,5,7,$\frac{1}{2}(13 \pm \sqrt{5})$,$\frac{1}{2}(15 \pm \sqrt{5})$}	$Sp(G_{101}^{10})$={0,4,5,6,6,7,8,8,8 $\pm \sqrt{3}$}
$Sp(G_{82}^{10})$={0,3,5,5,5,6,6,7,8,9}	$Sp(G_{102}^{10})$={0,4,5,6,6,7,$\frac{1}{2}(15 \pm \sqrt{5})$,$\frac{1}{2}(17 \pm \sqrt{5})$}
$Sp(G_{83}^{10})$={0,4,4,5,5,8,7 $\pm \sqrt{2}$,7 $\pm \sqrt{2}$}	$Sp(G_{103}^{10})$={0,4,5,6,6,6,7,8,8,10}
$Sp(G_{84}^{10})$={0,4,4,5,5,6,6,7,8,9}	$Sp(G_{104}^{10})$={0,4,5,6,6,6,8,9,8 $\pm \sqrt{3}$}
$Sp(G_{85}^{10})$={0,4,4,5,5,6,7,9,7 $\pm \sqrt{2}$}	$Sp(G_{105}^{10})$={0,5,5,5,6,7,7,7,8,10}
$Sp(G_{86}^{10})$={0,4,4,5,5,6,7,7,8 $\pm \sqrt{2}$}	$Sp(G_{106}^{10})$={0,4,5,6,6,7,7,8,8,9}
$Sp(G_{87}^{10})$={0,3,5,5,5,6,6,8,8,8}	$Sp(G_{107}^{10})$={0,4,5,6,6,6,6,8,9,10}
$Sp(G_{88}^{10})$={0,4,5,5,5,6,7,8,8 $\pm \sqrt{2}$}	$Sp(G_{108}^{10})$={0,5,5,6,6,7,7,8,8,10}
$Sp(G_{89}^{10})$={0,4,5,5,5,7,7,7,8 $\pm \sqrt{3}$}	$Sp(G_{109}^{10})$={0,5,5,6,6,8,8 $\pm \sqrt{2}$,8 $\pm \sqrt{2}$}
$Sp(G_{90}^{10})$={0,4,5,5,5,7,$\frac{1}{2}(15 \pm \sqrt{5})$,$\frac{1}{2}(15 \pm \sqrt{13})$}	$Sp(G_{110}^{10})$={0,5,6,6,6,7,7,8,9,10}
$Sp(G_{91}^{10})$={0,4,5,5,5,6,7,8,8 $\pm \sqrt{3}$}	$Sp(G_{111}^{10})$={0,6,6,6,6,7,7,9,9,10}
$Sp(G_{92}^{10})$={0,4,5,5,5,6,6,8,8,9}	$Sp(G_{112}^{10})$={0,6,6,7,7,7,8,9,10,10}
$Sp(G_{93}^{10})$={0,4,5,5,5,6,6,9,8 $\pm \sqrt{3}$}	$Sp(G_{113}^{10})$={0,6,6,7,7,8,8,9,9,10}
$Sp(G_{94}^{10})$={0,5,5,5,5,6,6,8,8,10}	$Sp(G_{114}^{10})$={0,6,6,7,7,7,9,9,9,10}
$Sp(G_{95}^{10})$={0,5,5,5,5,6,7,7,8,10}	$Sp(G_{115}^{10})$={0,5,7,7,7,7,8,9,10,10}
$Sp(G_{96}^{10})$={0,5,5,5,6,6,8,8,8,9}	$Sp(G_{116}^{10})$={0,5,7,7,7,8,8,8,10,10}
$Sp(G_{97}^{10})$={0,4,5,6,6,7,7,7,8,10}	$Sp(G_{117}^{10})$={0,6,7,7,7,8,8,9,10,10}
$Sp(G_{98}^{10})$={0,5,5,5,6,6,7,8,8,10}	$Sp(G_{118}^{10})$={0,7,7,7,7,8,8,10,10,10}
$Sp(G_{99}^{10})$={0,4,5,6,6,6,7,8,9,9}	$Sp(G_{119}^{10})$={0,7,8,8,8,9,10,10,10,10}
$Sp(G_{100}^{10})$={0,5,5,5,6,6,6,8,9,10}	$Sp(G_{120}^{10})$={0,8,8,8,8,10,10,10,10,10}

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Appendix: Figures

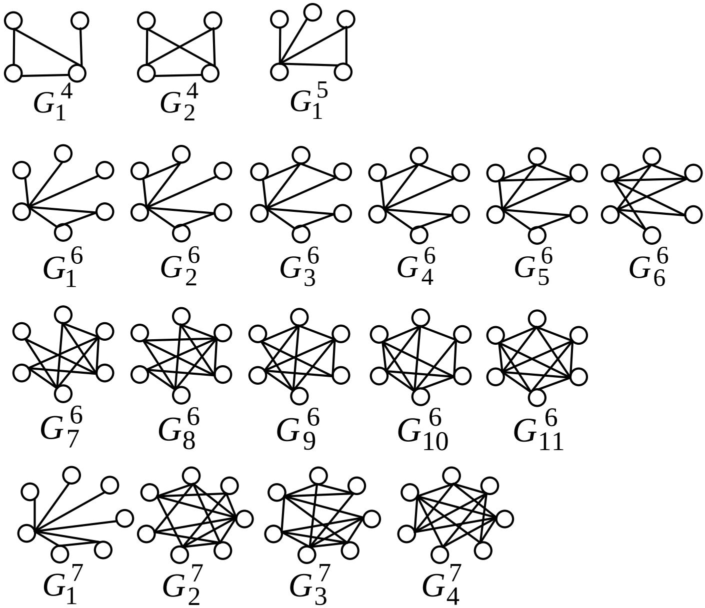

Figure 1. L-borderenergetic graphs on less than 8 vertices.

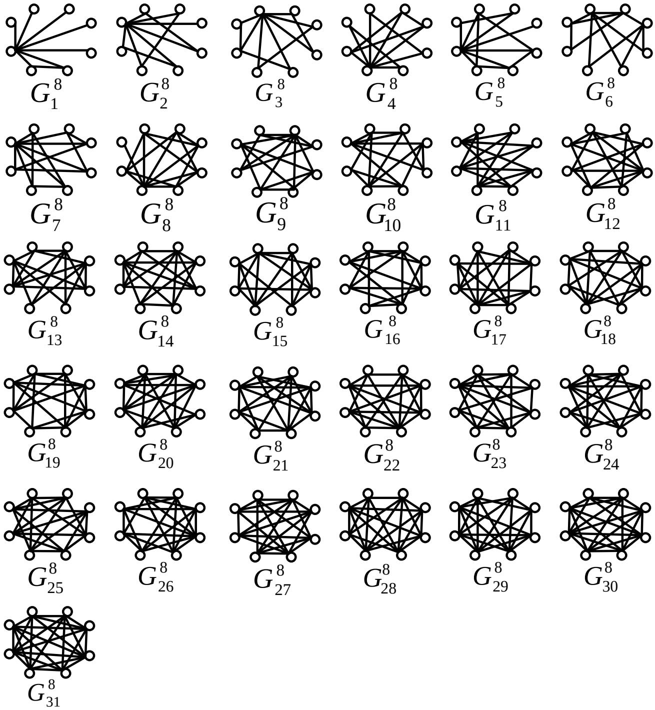

Figure 2. L-borderenergetic graphs on 8 vertices.

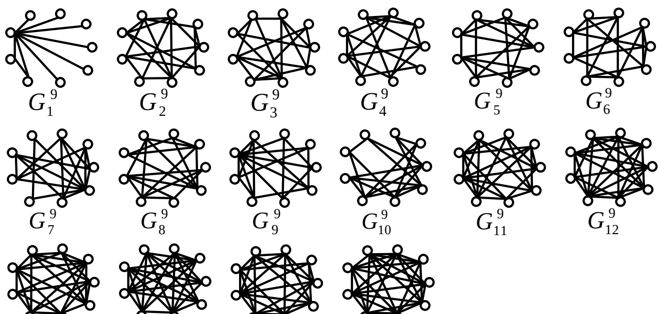

Figure 3. L-borderenergetic graphs on 9 vertices.

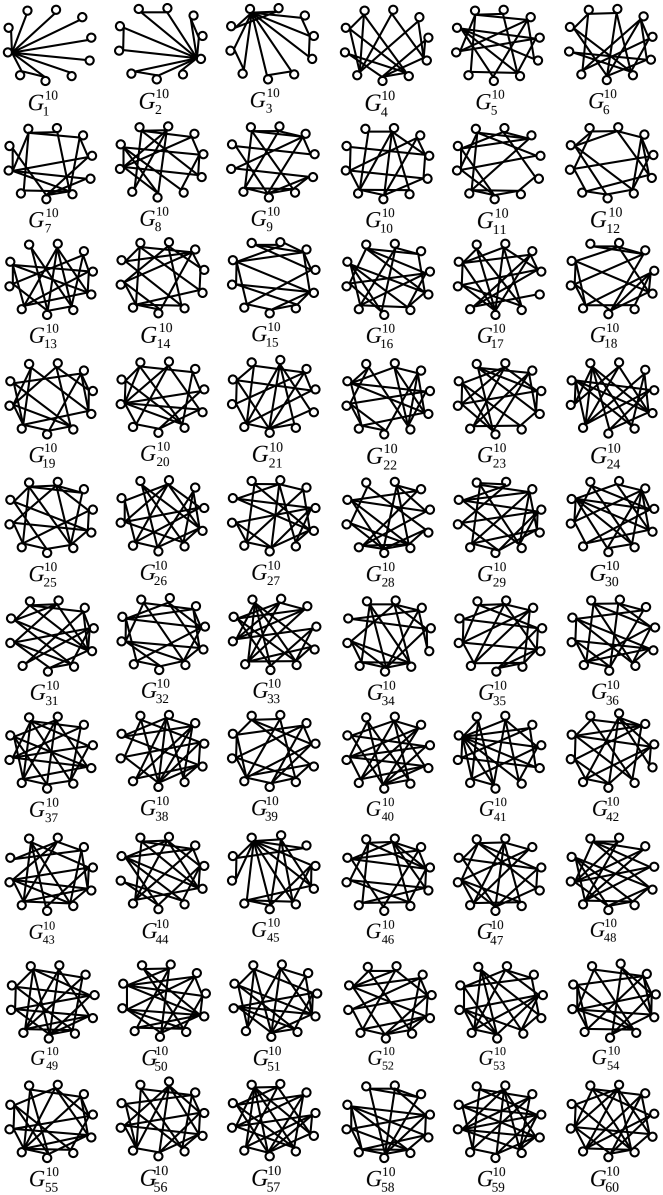

Figure 4. *L*-borderenergetic graphs on 10 vertices.

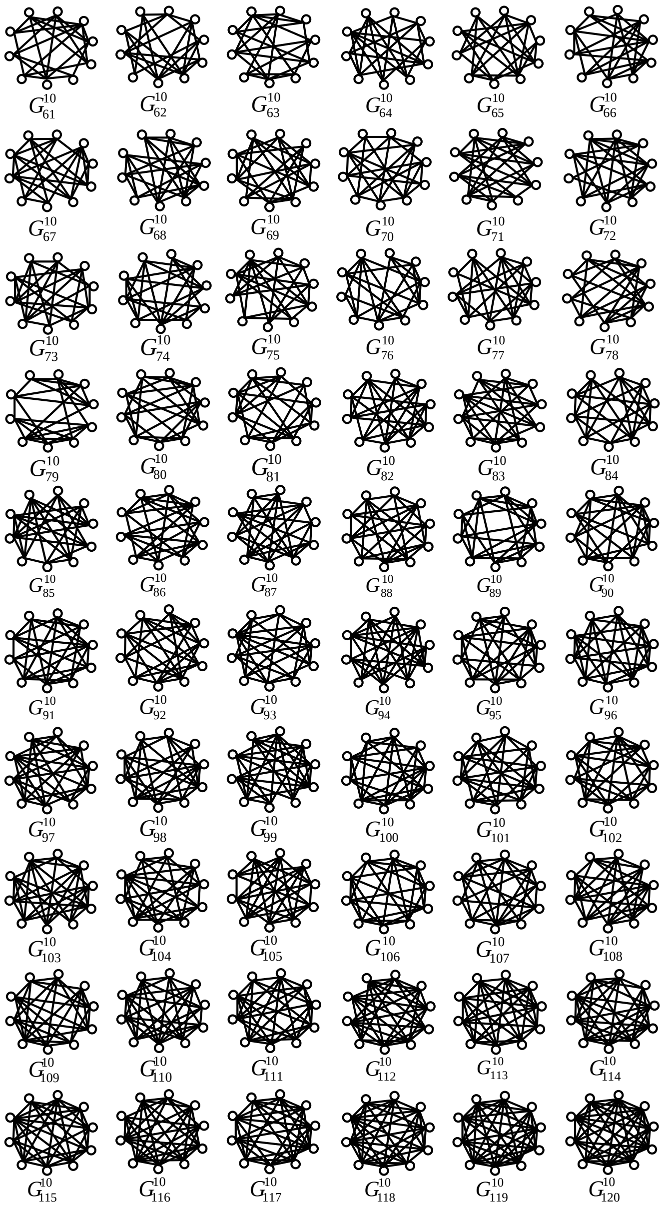

Figure 5. *L*-borderenergetic graphs on 10 vertices (continued).