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Lower Bounds for the Energy of (Bipartite) Graphs

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Abstract

The energy of a graph is defined as the sum of absolute values of its eigenvalues. In this paper, we establish some lower bounds for the energy of (bipartite) graphs involving the number of vertices (n), the number edges (m) and the determinant of the adjacency matrix ($\det A$). Our lower bound for graphs improves the lower bound in [2] for a class of graphs.

1 Introduction

Let G be a simple graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. The adjacency matrix $A = A(G)$ of G is the $n \times n$ matrix whose (i, j)-entry is 1 if the vertices v_i and v_j are adjacent and zero otherwise. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denotes the eigenvalues of A. These eigenvalues are called the eigenvalues of G and to form its spectrum [3]. As well known, λ_1 is the spectral radius of G and

$$\det A = \prod_{i=1}^{n} \lambda_i .$$

The terminology and the notations not defined here can be found in the book by Cvetković, Doob and Sachs [3].

The energy of a graph G is defined as the following [6]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| . \tag{1}$$

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This concept was first introduced in chemistry and used to approximate the total π-electron energy of a molecule [8–14, 18]. For details on $E(G)$ including also its mathematical properties, see the book [17] and the references cited therein.

McClelland obtained the first upper bound for the energy of graphs in terms of n and m as [18]

$$E(G) \leq \sqrt{2mn} \tag{2}$$

The upper bound (2) has been well studied in the chemical literature [7, 10–12, 14]. After this upper bound, various upper and lower bounds on $E(G)$ were obtained. Some of these bounds can be found [1, 2, 4, 5, 15, 16].

We now list some of lower bounds on $E(G)$.

Caporossi et al. gave the following lower bound based on the number of edges m as [2]

$$E(G) \geq 2\sqrt{m} \tag{3}$$

with equality in (3) if and only if G consist of a complete bipartite graph $K_{a,b}$, where $ab = m$ and arbitrarily many isolated vertices.

McClelland established the following lower bound in terms of n, m and $\det A$ as [18]

$$E(G) \geq \sqrt{2m + n(n-1)\left|\det A\right|^{2/n}} \tag{4}$$

Das et al. derived the following lower bound for connected non-singular graphs involving the parameters n, m and $\det A$ as [4]

$$E(G) \geq \frac{2m}{n} + n - 1 + \ln\left|\det A\right| - \ln\frac{2m}{n} \tag{5}$$

with equality in (5) if and only if G is isomorphic to the complete graph K_n.

Das and Gutman obtained the following lower bound in terms of n, m and $\det A$ as [5]

$$E(G) \geq \sqrt{2m + n(n-1)\left|\det A\right|^{2/n} + \frac{4}{(n+1)(n-2)}\left[\sqrt{\frac{2m}{n}} - \left(\frac{2m}{n}\right)^{1/4}\right]^2} \tag{6}$$

with equality in (6) if and only if G is isomorphic to $\frac{n}{2}K_2$ (n is even) or edgeless graph $\overline{K_n}$.

In [4] Das et al. showed that the lower bound (5) is better than the lower bounds in (3) and (4) under certain conditions. Das and Gutman [5] stated that the lower bound (6) is better than lower bound in (4).

In this paper, we establish some lower bounds for the energy of (bipartite) graphs involving the same parameters, namely n, m and $\det A$, with the lower bounds (4), (5) and (6). We also showed that our lower bound for graphs improves the lower bound in (3) for a class of graphs.

2 Lower Bounds for $E(G)$

In this section, we present some lower bounds for the energy of (bipartite) graphs. At first, we state the following useful lemma.

Lemma 2.1. *[3] G has only one distinct eigenvalue if and only if G is empty graph. G has two distict eigenvalues* $\lambda_1 > \lambda_2$ *with multiplicities* m_1 *and* m_2 *if and only if G is the direct sum of* m_1 *complete graphs of order* $\lambda_1 + 1$. *In this case,* $\lambda_2 = -1$ *and* $m_2 = m_1\lambda_1$.

Theorem 2.2. *Let G be a graph of order n with m edges such that* $2m \geq n$. *Then*

$$E(G) \geq \frac{2m}{n} + (n-1)\left(\frac{n\,|\det A|}{2m}\right)^{1/(n-1)} \tag{7}$$

Moreover, equality holding in (7) if and only if G is either isomorphic to $\frac{n}{2}K_2$, K_n *or a non-complete connected strongly regular graph with two non-trivial eigenvalues both with absolute value* $\frac{E(G)-\frac{2m}{n}}{n-1}$.

Proof. Starting with the arithmetic-geometric mean inequality, we have

$$E(G) = \lambda_1 + \sum_{i=2}^{n} |\lambda_i| \geq \lambda_1 + (n-1)\left(\prod_{i=2}^{n} |\lambda_i|\right)^{1/(n-1)} = \lambda_1 + (n-1)\left(\frac{|\det A|}{\lambda_1}\right)^{1/(n-1)}$$

Now we consider the function

$$f(x) = x + (n-1)\left(\frac{|\det A|}{x}\right)^{1/(n-1)}.$$

Note that f is increasing for $x \geq (|\det A|)^{1/n}$. As well known from [3],

$$\lambda_1 \geq \frac{2m}{n}$$

Moreover, from (2) and the arithmetic geometric mean inequality, we have

$$\lambda_1 \geq \frac{2m}{n} \geq \sqrt{\frac{2m}{n}} \geq \frac{E(G)}{n} = \frac{\sum_{i=1}^{n} |\lambda_i|}{n} \geq (|\det A|)^{1/n}.$$

Therefore

$$E(G) \geq \frac{2m}{n} + (n-1)\left(\frac{n\,|\det A|}{2m}\right)^{1/(n-1)}$$

Hence the inequality (7) holds. Now we assume that the equality holds in (7). Then all inequalities in the above proof must be equalities. Then, from $\lambda_1 = \frac{2m}{n}$ we have that G is $\frac{2m}{n}$-regular graph [3]. By arithmetic-geometric mean inequality, we have $|\lambda_i| = \frac{E(G)-\lambda_1}{n-1}$ for $2 \leq i \leq n$. Thus we arrive at three possibilities. We note that the similar way in Theorem 1 of [15] will be followed in each possibilities.

If G has two eigenvalues with same absolute value, then $\lambda_1 = |\lambda_i| = \frac{E(G)-\lambda_1}{n-1}$ $(2 \le i \le n)$. Then by Lemma 2.1, we conclude that $|\lambda_i| = \frac{E(G)-\lambda_1}{n-1} = 1$ and $E(G) = 2m = n$ which implies that $G \cong \frac{n}{2}K_2$. If G has two eigenvalues with different absolute value, then by Lemma 2.1, $\lambda_i = -1$ $(2 \le i \le n)$. In this case, $G \cong K_n$. If G has three eigenvalues with different absolute values equal to $\frac{2m}{n}$ and $\frac{E(G)-\frac{2m}{n}}{n-1}$, then from [3], we conclude that G is non-complete connected strongly regular graph. This completes the proof. ■

Let Γ be the class of graphs of order n with the following condition:

$$|\det A| \ge \frac{2m}{n} \tag{8}$$

Note that K_n, $K_{n,n-1} \in \Gamma$.

Remark 2.3. *From Eqs. (7) and (8) and the proof of Theorem 3 in [4], we have*

$$\frac{2m}{n} + (n-1)\left(\frac{n\,|\det A|}{2m}\right)^{1/(n-1)} \ge \frac{2m}{n} + (n-1) \ge \sqrt{\left(\frac{2m}{n} - 1\right)^2 + n^2 + 4m - 2n} \ge 2\sqrt{m}$$

Then the lower bound (7) is better than (3) for any graph of Γ. Further note that non-complete connected strongly regular graph is an extremal graph for the lower bound (7). Hence the lower bound (7) is better than the lower bounds (5) and (6) for these type of graphs, when $2m \ge n$.

Theorem 2.4. *Let G be a bipartite graph of order $n > 2$ with m edges such that $2m \ge n$. Then*

$$E(G) \ge \frac{4m}{n} + (n-2)\left(\frac{n^2\,|\det A|}{4m^2}\right)^{1/(n-2)} \tag{9}$$

Moreover, equality holding in (9) if and only if G is either isomorphic to $\frac{n}{2}K_2$, $K_{\frac{n}{2},\frac{n}{2}}$ $(n = 2\sqrt{m})$ or incidence graph of symmetric 2-(v,k,λ)-design with $n = 2v$, $k = \frac{2m}{n}$ and $\lambda = \frac{k(k-1)}{v-1}$.

Proof. Since $\lambda_1 = -\lambda_n$, by the arithmetic-geometric mean inequality, we have

$$E(G) = 2\lambda_1 + \sum_{i=2}^{n-1} |\lambda_i| \ge 2\lambda_1 + (n-2)\left(\prod_{i=2}^{n-1} |\lambda_i|\right)^{1/(n-2)} = 2\lambda_1 + (n-2)\left(\frac{|\det A|}{\lambda_1^2}\right)^{1/(n-2)}$$

Now we consider the function

$$f(x) = 2x + (n-2)\left(\frac{|\det A|}{x^2}\right)^{1/(n-2)}.$$

Note that f is increasing for $x \ge (|\det A|)^{1/n}$. Moreover, from the proof of Theorem 2.2, we have

$$\lambda_1 \ge \frac{2m}{n} \ge \sqrt{\frac{2m}{n}} \ge \frac{E(G)}{n} = \frac{\sum\limits_{i=1}^{n} |\lambda_i|}{n} \ge (|\det A|)^{1/n}.$$

Therefore

$$E(G) \geq \frac{4m}{n} + (n-2)\left(\frac{n^2 |\det A|}{4m^2}\right)^{1/(n-2)}.$$

Hence we get the inequality (9). Now we suppose that the equality holds in (9). Then, from $\lambda_1 = \frac{2m}{n}$ we have that G is $\frac{2m}{n}$-regular graph [3]. By arithmetic-geometric mean inequality, we have $|\lambda_i| = \frac{E(G)-2\lambda_1}{n-2}$ for $2 \leq i \leq n-1$. Then we have the following possibilities. Note that the similar idea in Theorem 1 of [16] will be considered in each possibilities.

If G has two eigenvalues with same absolute value. Then $\lambda_1 = -\lambda_n = |\lambda_i| = \frac{E(G)-2\lambda_1}{n-2}$ $(2 \leq i \leq n-1)$. Therefore by Lemma 2.1, $\lambda_n = -\frac{E(G)-2\lambda_1}{n-2} = -1$, i.e. $E(G) = 2m = n$ which implies that $G \cong \frac{n}{2}K_2$. If G has three distinct eigenvalues, then we have $\lambda_1 = -\lambda_n = \frac{2m}{n}$ and $|\lambda_i| = \frac{E(G)-2\lambda_1}{n-2} = 0$ for $2 \leq i \leq n-1$. This implies that $E(G) = 2\lambda_1 = \frac{4m}{n}$, i.e., $G \cong K_{\frac{n}{2},\frac{n}{2}}$, where $n = 2\sqrt{m}$. If G has four distinct eigenvalues, then G is connected and $\frac{2m}{n} > \frac{E(G)-2\lambda_1}{n-2}$. Thus G is the incidence graph of a symmetric 2-$\left(v, \frac{2m}{n}, \lambda\right)$-design [3]. ■

Remark 2.5. *We finally point out that considering a lower bound sharper than* $\lambda_1 \geq \frac{2m}{n}$ *in the proof of Theorem 2.2 and Theorem 2.4, we may improve and generalize Theorem 2.2 and Theorem 2.4 as in Theorem 2.6 and Theorem 2.7, respectively.*

Theorem 2.6. *Let* G *be a graph of order* n *with* m *edges such that* $2m \geq n$. *If* λ_1 *has any lower bound such that* $\lambda_1 \geq \xi \geq \frac{2m}{n}$, *then*

$$E(G) \geq \xi + (n-1)\left(\frac{|\det A|}{\xi}\right)^{1/(n-1)} \tag{10}$$

Equality holding in (10) if and only if $\lambda_1 = \xi$ *and* $|\lambda_2| = |\lambda_3| = \cdots = |\lambda_n| = \frac{E(G)-\xi}{n-1}$.

Theorem 2.7. *Let* G *be a bipartite graph of order* $n > 2$ *with* m *edges such that* $2m \geq n$. *If* λ_1 *has any lower bound such that* $\lambda_1 \geq \xi \geq \frac{2m}{n}$, *then*

$$E(G) \geq 2\xi + (n-2)\left(\frac{|\det A|}{\xi^2}\right)^{1/(n-2)} \tag{11}$$

Equality holding in (11) if and only if $\lambda_1 = -\lambda_n = \xi$ *and* $|\lambda_2| = |\lambda_3| = \cdots = |\lambda_{n-1}| = \frac{E(G)-2\xi}{n-2}$.

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