MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **77** (2017) 37-44

ISSN 0340 - 6253

L-Borderenergetic Graphs

Fernando Tura

Departamento de Matemática, UFSM, Santa Maria, RS, 97105-900, Brazil
ftura@smail.ufsm.br

(Received April 2, 2016)

Abstract

The energy of a graph is defined as the sum the absolute values of the eigenvalues of its adjacency matrix. A graph G on n vertices is said to be borderenergetic if its energy equals the energy of the complete graph K_n. In this paper, we promote this concept for the Laplacian matrix. The Laplacian energy of G, introduced by Gutman and Zhou [5], is given by $LE(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}|$, where μ_i are the Laplacian eigenvalues of G and $\overline{d}$ is the average degree of G. In this way, we say G to be L-borderenergetic if $LE(G) = LE(K_n)$. Several classes of L-borderenergetic graphs are obtained including result that for each integer $r \geq 1$, there are $2r+1$ graphs, of order $n = 4r+4$, pairwise L-noncospectral and L-bordernergetic graphs.

1 Introduction

Throughout this paper, all graphs are assumed to be finite, undirected and without loops or multiple edges. If G is a graph of order n and M is a real symmetric matrix associated with G, then the M- *energy* of G is

$$E_M(G) = \sum_{i=1}^{n} \left| \lambda_i(M) - \frac{tr(M)}{n} \right|. \tag{1}$$

The energy of a graph simply refers to using the adjacency matrix in (1). There are many results on energy [1,10–13,16] and its applications in several areas, including in chemistral see [9] for more details and the references therein.

It is well known that the complete graph K_n has $E(K_n) = 2n-2$. In this context, several authors have been presented families of graphs with same energy of the complete graph K_n. Recently, Gong, Li, Xu, Gutman and Furtula [3] introduced the concept of *bordernergetic*. A graph G on n vertices is said to be borderenergetic if its energy equals the energy of the complete graph K_n.

In [3], it was shown that there exits borderenergetic graphs on order n for each integer $n \geq 7$, and all borderenergetic graphs with $7, 8$, and 9 vertices were determined.

In [7] considered the eigenvalues and energies of threshold graphs. For each $n \geq 3$, they determined $n-1$ threshold graphs on n^2 vertices, pairwise non-cospectral and equienergetic to the complete graph K_{n^2}. Recently, Hou and Tao [6], showed that for each $n \geq 2$ and $p \geq 1$ ($p \geq 2$ if $n = 2$), there are $n-1$ threshold graphs on pn^2 vertices, pairwise non-cospectral and equienergetic with the complete graph K_{pn^2}, generalizing the results in [7].

The Laplacian energy of G, introduced by Gutman and Zhou [5], is given by

$$LE(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}| \tag{2}$$

where μ_i are the Laplacian eigenvalues of G and $\overline{d}$ is the average degree of G. Similarly for the laplacian energy, we have that $LE(K_n) = 2n - 2$.

The first purpose of this paper is to promote the concept of borderenergetic to the laplacian matrix. In this way, we say G to be L-borderenergetic if $LE(G) = LE(K_n)$. The second is to present several classes of L-borderenergetic graphs.

The paper is organized as follows. In Section 2 we describe some known results about the Laplacian spectrum of graphs. In Section 3 we present four classes of L-borderenergetic. We finalize this paper, showing that for each integer $r \geq 1$, there are $2r+1$ graphs, of order $n = 4r+4$, pairwise L-noncospectral and L-bordernergetic graphs.

2 Premilinares

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be undirected graphs without loops or multiple edges. The *union* $G_1 \cup G_2$ of graphs G_1 and G_2 is the graph $G = (V, E)$ for which $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. We denote the graph $\underbrace{G \cup G \cup \ldots \cup G}_{m}$ by mG. The *join* $G_1 \nabla G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2.

The Laplacian spectrum of $G_1 \cup \ldots \cup G_k$ is the union of Laplacian spectra of $G_1, \ldots, G_k$, while the Laplacian spectra of the complement of n- vertex graph G consists of values $n - \mu_i$, for each Laplacian eigenvalue μ_i of G, except for a single instance of eigenvalue 0 of G.

Lemma 1 *Let G be a graph on n vertices with Laplacian matrix L. Let $0 = \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$ be the eigenvalues of L. Then the eigenvalues of $\overline{G}$ are*

$$0 \leq n - \mu_n \leq n - \mu_{n-1} \leq n - \mu_{n-2} \leq \ldots \leq n - \mu_2$$

with the same corresponding eigenvectors.

Proof: Note that the Laplacian matrix of $\overline{G}$ satisfies $L(\overline{G}) = nI + J - L$, where I is the identity matrix and J is the matrix each of whose entries is equal 1. Therefore, for $i = 2, \ldots, n$, if x is an eigenvector of L corresponding to μ_i, then $Jx = 0$. Therefore

$$L(\overline{G})x = (nI + J - L)x = nIx + Jx - Lx = (n - \mu_i)x.$$

Thus $n - \mu_i$ is an eigenvalue with x_i as a corresponding eigenvector. Finally, $e = (1, \ldots, 1)$ is an eigenvector of $L(\overline{G})$ corresponding to 0. ■

Recall that G is laplacian integral if its spectrum consists entirely of integers [8, 14]. Follows from Lemma 1 that G is laplacian integral if and only if $\overline{G}$ is laplacian integral.

Theorem 1 *Let G_1 and G_2 be graphs on n_1 and n_2 vertices, respectively. Let L_1 and L_2 be the Laplacian matrices for G_1 and G_2, respectively, and let L be the Laplacian matrix for $G_1 \nabla G_2$. If $0 = \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{n_1}$ and $0 = \beta_1 \leq \beta_2 \leq \ldots \leq \beta_{n_2}$ are the eigenvalues of L_1 and L_2, respectively. Then the eigenvalues of L are*

$$0,\ \ n_2 + \alpha_2,\ \ n_2 + \alpha_3, \ldots,\ n_2 + \alpha_{n_1}$$

$$n_1 + \beta_2,\ \ n_1 + \beta_3, \ldots,\ n_1 + \beta_{n_2},\ \ n_1 + n_2.$$

Proof: Since that the join of graphs G_1 and G_2 is given by $G_1 \nabla G_2 = \overline{\overline{G_1} \cup \overline{G_2}}$ (see [5]), the proof follows immediately from the Lemma 1. ■

3 L-Borderenergetic graphs

Recall that the L-energy of a graph G is obtained by $LE(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}|$, where μ_i are the laplacian eigenvalues of G and $\overline{d}$ is the average degree of G. It is known that the complete graph K_n has Laplacian energy $2(n-1)$. We exhibit four infinite classes $\Omega_i = \{G_1, G_2, \ldots, G_r, \ldots\}$ for $i = 1, \ldots, 4$ such that each G_r, of order $n = 4r + 4$, satisfies $LE(G_r) = LE(K_{4r+4})$.

3.1 The class Ω_1

For each integer $r \geq 1$, we define the graph $G_r \in \Omega_1$ to be the following join

$$G_r = (rK_1 \cup (K_1 \nabla (r+1)K_1)) \nabla (rK_1 \cup (K_1 \nabla (r+1)K_1)).$$

G_r has order $n = 4r + 4$. We let μ^m denote the laplacian eigenvalue μ with multiplicity equals to m.

Lemma 2 *Let $G_r \in \Omega_1$ be a graph of order $n = 4r + 4$. Then the Laplacian spectrum of G_r is given by*

$$0;\ (2r+2)^{2r};\ (2r+3)^{2r};\ (3r+4)^2;\ 4r+4.$$

Proof: Let $G_r \in \Omega_1$. Let's denote $H = rK_1 \cup (K_1 \nabla (r+1)K_1)$. By definition we have that $G_r = H \nabla H$. According by Theorem 1, we just need to determine the Laplacian spectrum of the H and add its order. By direct calculus follows that the Laplacian spectrum of H is equal to

$$0^r;\ 1^r;\ r+2.$$

Since H has order $2r+2$, by Theorem 1 the result follows. ■

Theorem 2 *For each $r \geq 1$, G_r is L-borderenergetic and L-noncospectral graph with K_{4r+4}.*

Proof: Clearly G_r and K_{4r+4} are L-noncospectral. Let $\overline{d}$ be the average degree of G_r. Since that $\overline{d}$ is equal to average of Laplacian eigenvalues of G_r then $\overline{d} = \frac{2r(4r+5)+2(3r+4)+4r+4}{4r+4}$ $= 2r+3$. Using Lemma 2, $LE(G_r) = 4r+4-(2r+3)+2(3r+4-2r-3)+2r(2r+3-2r-3)+2r(2r+3-2r-2)+2r+3 = 8r+6 = LE(K_{4r+4})$. ■

3.2 The class Ω_2

For each integer $r \geq 1$, we define the graph $G_r \in \Omega_2$ to be the following join

$$G_r = (r+1)K_2 \nabla (r+1)K_2\ .$$

G_r has order $n = 4r + 4$. We let μ^m denote the laplacian eigenvalue μ with multiplicity equals to m.

Lemma 3 *Let $G_r \in \Omega_2$ be a graph of order $n = 4r+4$. Then the Laplacian spectrum of G_r is given by*

$$0;\ (2r+2)^{2r};\ (2r+4)^{2r+2};\ 4r+4.$$

Proof: Let $G_r \in \Omega_2$. Let's denote $H = (r+1)K_2$. By definition we have that $G_r = H\nabla H$. According by Theorem 1, we just need to determine the Laplacian spectrum of the H and add its order. By direct calculus follows that the Laplacian spectrum of H is equal to

$$0^{r+1};\ 2^{r+1}.$$

Since H has order $2r+2$, by Theorem 1 the result follows. ■

Theorem 3 *For each $r \geq 1$, G_r is L-borderenergetic and L-noncospectral graph with K_{4r+4}.*

Proof: Clearly G_r and K_{4r+4} are L-noncospectral. Let $\overline{d}$ be the average degree of G_r. Since that $\overline{d}$ is equal to average of Laplacian eigenvalues of G_r then $\overline{d} = \frac{(2r+2)(4r+4)+4r+4}{4r+4} = 2r+3$. Using Lemma 2, $LE(G_r) = 4r+4-(2r+3)+(2r+2)(2r+4-2r-3)+2r(2r+3-2r-2)+2r+3 = 8r+6 = LE(K_{4r+4})$. ■

3.3 The classes Ω_3 and Ω_4

For each integer $r \geq 1$, we define the following two graphs $G_r \in \Omega_3$ and $G'_r \in \Omega_4$:

$$G_r = (K_2 \cup (2r+1)K_1)\nabla(2r+1)K_1,$$

$$G'_r = ((2r+1)K_1)\nabla(2r+2)K_1\nabla K_1,$$

where G_r and G'_r have order $n = 4r+4$.

The proof of following results are similar to others above, then we will omite them.

Lemma 4 *Let $G_r \in \Omega_3$ and $G'_r \in \Omega_4$ be graphs of order $n = 4r+4$. Then the Laplacian spectrum of G_r and G'_r are given by*

$$0;\ (2r+1)^{2r+1};\ (2r+3)^{2r+1};\ 4r+4,$$

$$0;\ (2r+2)^{2r+1};\ (2r+3)^{2r};\ (4r+4)^2,$$

respectively.

Theorem 4 *For each $r \geq 1$, G_r and G'_r are L-borderenergetic and L-noncospectral graphs with K_{4r+4}.*

4 More L-Borderenergetic graphs

In this Section we obtain more L-borderenergetic graphs including result that for each integer $r \geq 1$, there are $2r+1$ graphs, of order $n = 4r+4$, pairwise L-noncospectral and L-bordernergetic graphs. Consider the following graphs:

$$H_1 = rK_1 \cup (K_1 \nabla (r+1)K_1)$$

$$H_2 = (r+1)K_2$$

$$H_3 = rK_2 \cup 2K_1$$

$$H_4 = ((2r+1)K_1)\nabla K_1.$$

The proof of following results are similar to others above, then we will omite them.

Lemma 5 *Let $G_{1,2}$ be a graph of order $n = 4r+4$ obtained by the following join $G_{1,2} = H_1 \nabla H_2$. Then the Laplacian spectrum of $G_{1,2}$ is given by*

$$0;\ (2r+2)^{2r};\ (2r+3)^{r};\ (2r+4)^{r+1};\ 3r+4;\ 4r+4.$$

Lemma 6 *Let $G_{1,3}$ be a graph of order $n = 4r+4$ obtained by the following join $G_{1,3} = H_1 \nabla H_3$. Then the Laplacian spectrum of $G_{1,3}$ is given by*

$$0;\ (2r+2)^{2r+1};\ (2r+3)^{r};\ (2r+4)^{r};\ 3r+4;\ 4r+4.$$

Lemma 7 *Let $G_{2,3}$ be a graph of order $n = 4r+4$ obtained by the following join $G_{2,3} = H_2 \nabla H_3$. Then the Laplacian spectrum of $G_{2,3}$ is given by*

$$0;\ (2r+2)^{2r+1};\ (2r+4)^{2r+1};\ 4r+4.$$

Lemma 8 *Let $G_{2,4}$ be a graph of order $n = 4r+4$ obtained by the following join $G_{2,4} = H_2 \nabla H_4$. Then the Laplacian spectrum of $G_{2,4}$ is given by*

$$0;\ (2r+2)^{r};\ (2r+3)^{2r};\ (2r+4)^{r+1};\ (4r+4)^{2}.$$

Lemma 9 *Let $G_{3,4}$ be a graph of order $n = 4r+4$ obtained by the following join $G_{3,4} = H_3 \nabla H_4$. Then the Laplacian spectrum of $G_{3,4}$ is given by*

$$0;\ (2r+2)^{r+1};\ (2r+3)^{2r};\ (2r+4)^{r};\ (4r+4)^{2}.$$

Theorem 5 *For each integer $r \geq 1$, $G_{1,2}, G_{1,3}, G_{2,3}, G_{2,4}$ and $G_{3,4}$ are L-borderenergetic and L-noncospectral graphs.*

For integers $r \geq 1$ and $i = 0, 1, \ldots, 2r$, consider the following $2r + 1$ graphs:

$$G_{i,r} = ((2r+1)K_1)\nabla((2r+1-i)K_1) \cup (K_1 \nabla (i+1)K_1),$$

of order $n = 4r + 4$.

Lemma 10 *For integers $r \geq 1$ and $i = 0, 1, \ldots, 2r$, let $G_{i,r}$ be a graph of order $n = 4r+4$. Then the Laplacian spectrum of $G_{i,r}$ is given by*

$$0;\ (2r+1)^{2r+1-i};\ (2r+2)^{i};\ (2r+3)^{2r};\ (2r+3+i);\ (4r+4).$$

Theorem 6 *For integers $r \geq 1$ and $i = 0, 1, \ldots, 2r$, $G_{i,r}$ are L-borderenergetic and L-noncospectral graphs.*

References

[1] B. Deng, X. Li, I. Gutman, More on borderenergetic graphs, *Lin. Algebra Appl.* **497** (2016) 199–208.

[2] C. Godsil, G. Royle, *Algebraic Graph Theory*, Springer, New York, 2001.

[3] S. C. Gong, X. Li, G. H. Xu, I. Gutman, B. Furtula, Borderenergetic graphs, *MATCH Commun. Math. Comput. Chem.* **74** (2015) 321–332.

[4] R. Grone, R. Merris, Indecomposable Laplacian integral graphs, *Lin. Algebra Appl.* **428** (2008) 1565–1570.

[5] I. Gutman, B. Zhou, Laplacian energy graph of a graph, *Lin. Algebra Appl.* **414** (2006) 29–37.

[6] Y. Hou, Q Tao, Borderenergetic threshold graphs, *MATCH Commun. Math. Comput. Chem.* **75** (2016) 253–262.

[7] D. P. Jacobs, V. Trevisan, F. Tura, Eigenvalues and energy in threshold graphs, *Lin. Algebra Appl.* **465** (2015) 412–425.

[8] S. Kirkland, Constructably Laplacian integral graphs, *Lin. Algebra Appl.* **423** (2007) 3–21.

[9] X. Li, Y, Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[10] X. Li, M. Wei, S. Gong, A computer search for the borderenergetic graphs of order 10, *MATCH Commun. Math. Comput. Chem.* **74** (2015) 333–342.

[11] X. Li, H. Ma, All hypoenergetic graphs with maximum degree at most 3, *Lin. Algebra Appl.* **431** (2009) 2127–2133.

[12] X. Li, H. Ma, All connected graphs with maximum degree at most 3 whose energies are equal to the number of vertices, *MATCH Commun. Math. Comput. Chem.* **64** (2010) 7–24.

[13] X. Li, H. Ma, Hypoenergetic and strongly hypoenergetic k-cyclic graphs, *MATCH Commun. Math. Comput. Chem.* **64** (2010) 41–60.

[14] R. Merris, Laplacian graph eigenvectors, *Lin. Algebra Appl.* **278** (1998) 221–236.

[15] H. S. Ramane, H. B. Walikar, Construction of equienergetic graphs, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 203–210.

[16] Z. Shao, F. Deng, Correcting the number of borderenergetic graphs of order 10, *MATCH Commun. Math. Comput. Chem.* **75** (2016) 263–266.