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Resolvent Estrada and Signless Laplacian Estrada Indices of Graphs

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Abstract

The aim of this paper is to first establish some new bounds for the resolvent Estrada index of graphs. Finally, we introduce the resolvent signless Laplacian Estrada index of a graph and the extremal trees together with some bounds in general with respect to this new invariant are presented.

1 Introduction

Let G be a finite simple graph with vertex set $V(G) = \{v_1, \cdots, v_n\}$. The adjacency matrix of G is denoted by $\mathbf{A}(G)$ and $\mathbf{D}(G) = diag(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of G, where $d_i = deg(v_i)$, $1 \leq i \leq n$. The matrix $\mathbf{Q}(G) = \mathbf{D}(G) + \mathbf{A}(G)$ is called the signless Laplacian matrix of G. The set of all eigenvalues of $\mathbf{A}(G)$ and $\mathbf{Q}(G)$ are denoted by $Spec(G)$ and $\mathbf{Q}{-}Spec(G)$, respectively.

Suppose $Spec(G) = \{\lambda_1, \ldots, \lambda_n\}$, $\mathbf{Q}{-}Spec(G) = \{q_1, \ldots, q_n\}$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $q_1 \geq q_2 \geq \ldots \geq q_n \geq 0$. The resolvent Estrada index of G was put forward by Estrada and Higham [15] as $EE_r(G) = \sum_{i=1}^{n}(1 - \frac{\lambda_i}{n-1})^{-1}$.

It is well known that $\lambda_1 = n - 1$ if and only if G is isomorphic to the complete graph K_n [7, 8]. Thus, for all non-complete graphs of order n, $|\frac{\lambda_i}{n-1}| < 1$. Consequently, for all

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non-complete graphs, due to the Taylor series, we have:

$$EE_r(G) = \sum_{k\geq 0} \frac{M_k(G)}{(n-1)^k} \tag{1}$$

where $M_k(G)$ is the $k-$th spectral moment of G, i.e. $M_k(G) = \sum_{i=1}^{n} \lambda_i^k$. It is well known that $M_0(G) = n$, $M_1(G) = 0$, $M_2(G) = 2m$, $M_3(G) = 6\,t$ and $M_k(G)$ is the number of closed walks of length k in G [7,8].

In [3], Chen and Qian proved that if G is a non-complete graph and $e \in E(G)$, then $EE_r(G) > EE_r(G-e)$. As an immediate consequence, the graph $K_n - e$ and the empty graph $\overline{K_n}$ have maximal and minimal resolvent Estrada index, respectively. Chen and Qian also determined the first thirteen trees with the greatest resolvent Estrada index, and characterized the multipartite graphs having the maximal resolvent Estrada index. The extremal trees with respect to EE_r-values are investigated in [3, 19] and several bounds for this quantity in terms of the number of vertices and edges are presented in [5, 20]. On the other hand, Gutman et al. [20] found trees, unicyclic, bicyclic, and tricyclic graphs with minimum and maximum values of EE_r.

In this paper, we intend to further pursue the development of the work done on this issue. In Section 2, we recall some important inequalities that are crucial throughout this article. Afterwards, in Section 3, we will give several bounds for resolvent Estrada index of graphs, and specially for the cases of trees and bipartite graphs. Finally, in the last section we introduce the resolvent signless Laplacian Estrada index ($SLEE_r$) of graphs and present the extremal trees together with some bounds in general with respect to it.

2 Preliminaries

In this section, some important analytical inequalities are presented. The first result is the Grüss type discrete inequality [2].

Remark 2.1 *Suppose $\bar{a} = (a_1, a_2, \ldots, a_n)$ and $\bar{b} = (b_1, b_2, \ldots, b_n)$ such that there are real numbers m_1, M_1, m_2, M_2 with $m_1 \leq a_i \leq M_1, m_2 \leq b_i \leq M_2, 1 \leq i \leq n$. Then*

$$|C_n(\bar{a}, \bar{b})| \leq \frac{1}{n^2}\left[\frac{n^2}{4}\right](M_1 - m_1)(M_2 - m_2)\,,$$

where

$$C_n(\bar{a}, \bar{b}) := \frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{1}{n^2}\sum_{i=1}^{n} a_i . \sum_{i=1}^{n} b_i\ .$$

The Diaz−Metcalf inequality is a reverse for the well-known Cauchy−Schwarz inequality. Diaz and Metcalf proved their inequality in the more general case of an inner product space over the real or complex number field, but its simple form is as follows:

Remark 2.2 (Diaz−Metcalf inequality) *[13]*
If $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are n-tuples of real numbers with $a_i \neq 0$, and if $m \leq \frac{b_i}{a_i} \leq M, 1 \leq i \leq n$, then

$$\sum_{i=1}^{n} b_i^2 + mM \sum_{i=1}^{n} a_i^2 \leq (M+m) \sum_{i=1}^{n} a_i b_i\,,$$

with equality if and only if for all i, $1 \leq i \leq n$, either $b_i = m a_i$ or $b_i = M a_i$.

The Pólya−Szegö, Shisha−Mond and Ozeki−Izumino−Mori−Seo inequalities are three other reverses for the Cauchy−Schwarz inequality which is important throughout this paper.

Remark 2.3 *If $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are n-tuples of real numbers with $0 < m_1 \leq a_i \leq M_1, 0 < m_2 \leq b_i \leq M_2, 1 \leq i \leq n$, then we have the following classical inequalities:*

- **Pólya−Szegö inequality** *[12, 25]*

$$\frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{\left(\sum_{i=1}^{n} a_i b_i\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2 .$$

  with equality if and only if $\nu = \frac{n M_1 m_2}{M_1 m_2 + M_2 m_1}$ is an integer and if ν of the a_i are equal to m_1 and the others equal to M_1, with the corresponding b_i being M_2, m_2 respectively.

- **Shisha−Mond inequality** *[26]*

$$\frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i b_i} - \frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} b_i^2} \leq \left(\sqrt{\frac{M_1}{m_2}} - \sqrt{\frac{m_1}{M_2}}\right)^2 .$$

- **A Grüss type inequality**

$$\Big(\sum_{i=1}^{n} a_i^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{n} b_i^2\Big)^{\frac{1}{2}} - \sum_{i=1}^{n} a_i b_i \leq \frac{\sqrt{M_1 M_2}\big(\sqrt{M_1 M_2} - \sqrt{m_1 m_2}\big)^2}{2\sqrt{m_1 m_2}} \min\left\{\frac{M_1}{m_1}, \frac{M_2}{m_2}\right\}.$$

- **Ozeki−Izumino−Mori−Seo inequality** *[22, 24]*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \frac{n^2}{3}\big(M_1 M_2 - m_1 m_2\big)^2.$$

3 Bounds on resolvent Estrada index

Let G be a simple graph of order n and size m. The **first Zagreb index**, $Zg_1(G)$, and **second Zagreb index**, $Zg_2(G)$, of the graph G is defined as $Zg_1(G) = \sum_{v\in V(G)} d^2(v)$ and $Zg_2(G) = \sum_{e=uv\in E(G)} d(u)d(v)$, where $d(u)$ denotes the degree of vertex u in G [18]. In this section, some upper bounds for the resolvent Estrada index of graphs are presented. For the sake of completeness we mention here a result of Chen and Qian [4] as follows:

Lemma 3.1 (Bounds of Closed Walks) *Let $M_k(G)$ be the k-th spectral moment of G with degree sequence $(d_1, d_2, \ldots, d_n)$. Then*

1. $M_k(G) \leq n\,\Delta^{k-1}$, *for each* $k \geq 2$.
2. $M_k(G) \leq 2\,m\,\Delta^{k-2}$, *for each* $k \geq 2$.
3. $M_k(G) \leq Zg_1(G)\,\Delta^{k-3}$, *for each* $k \geq 3$.
4. $M_k(G) \leq 2\,Zg_2(G)\,\Delta^{k-4}$, *for each* $k \geq 4$.
5. $M_k(G) \leq \sum_{i=1}^{n} d_i^{k-1}$, *for each* $k \geq 2$.

Each of the equalities holds in (1)–(5) if and only if k is even and each component of G is the complete bipartite graph $K_{\Delta,\Delta}$.

Lemma 3.2 *Let G be a graph with degree sequence $(d_1, d_2, \ldots, d_n)$ and $\Delta < n-1$. Then,*

1. $EE_r(G) < n\left(1 + \frac{\Delta}{(n-1)(n-1-\Delta)}\right)$.
2. $EE_r(G) < n + \frac{2m}{(n-1)(n-1-\Delta)}$.
3. $EE_r(G) < n + \frac{2\,m}{(n-1)^2} + \frac{Zg_1(G)}{(n-1)^2(n-1-\Delta)}$.
4. $EE_r(G) < n + \frac{2\,m}{(n-1)^2} + \frac{6\,t}{(n-1)^3} + \frac{2\,Zg_2(G)}{(n-1)^3(n-1-\Delta)}$.
5. $EE_r(G) < n + \frac{1}{n-1}\sum_{i=1}^{n} \frac{d_i}{n-1-d_i} = n - \frac{n}{n-1} + \sum_{i=1}^{n} \frac{1}{n-1-d_i}$.

Proof. The proof of all parts are independent from others and so we first prove part (5), then (4), (3), (2) and (1).

1. By (1), Lemma 3.1 (5) and the fact that $M_0(G) = n, M_1(G) = 0$, we have:
$$EE_r(G) = M_0(G) + \frac{M_1(G)}{(n-1)} + \sum_{k\geq 2} \frac{M_k(G)}{(n-1)^k} < n + \sum_{k\geq 2} \frac{\sum_{i=1}^n d_i^{k-1}}{(n-1)^k}$$
$$= n + \sum_{i=1}^n \sum_{k\geq 2} \frac{d_i^{k-1}}{(n-1)^k} = n + \sum_{i=1}^n \frac{\frac{d_i}{(n-1)^2}}{1 - \frac{d_i}{n-1}}$$
$$= n + \frac{1}{n-1} \sum_{i=1}^n \frac{d_i}{n-1-d_i} = n - \frac{n}{n-1} + \sum_{i=1}^n \frac{1}{n-1-d_i} .$$
This completes the proof of part 5.

2. To prove part 4, we apply (1), Lemma 3.1 (4), and the fact that $M_0(G) = n, M_1(G) = 0, M_2(G) = 2\,m, M_3(G) = 6\,t$. So,
$$EE_r(G) = M_0(G) + \frac{M_1(G)}{(n-1)} + \frac{M_2(G)}{(n-1)^2} + \frac{M_3(G)}{(n-1)^3} + \sum_{k\geq 4} \frac{M_k(G)}{(n-1)^k}$$
$$< n + \frac{2\,m}{(n-1)^2} + \frac{6\,t}{(n-1)^3} + \frac{2\,Zg_2(G)}{\Delta^4} \sum_{k\geq 4} (\frac{\Delta}{n-1})^k$$
$$= n + \frac{2\,m}{(n-1)^2} + \frac{6\,t}{(n-1)^3} + \frac{2\,Zg_2(G)}{\Delta^4} (\frac{\frac{\Delta^4}{(n-1)^4}}{1 - \frac{\Delta}{n-1}})$$
$$= n + \frac{2\,m}{(n-1)^2} + \frac{6\,t}{(n-1)^3} + \frac{2\,Zg_2(G)}{(n-1)^3\,(n-1-\Delta)},$$
as desired.

3. Again by (1) and Lemma 3.1 (3),
$$EE_r(G) = M_0(G) + \frac{M_1(G)}{(n-1)} + \frac{M_2(G)}{(n-1)^2} + \sum_{k\geq 3} \frac{M_k(G)}{(n-1)^k}$$
$$< n + \frac{2\,m}{(n-1)^2} + \frac{Zg_1(G)}{\Delta^3} \sum_{k\geq 3} (\frac{\Delta}{n-1})^k$$
$$= n + \frac{2\,m}{(n-1)^2} + \frac{Zg_1(G)}{\Delta^3} (\frac{\frac{\Delta^3}{(n-1)^3}}{1 - \frac{\Delta}{n-1}})$$
$$= n + \frac{2\,m}{(n-1)^2} + \frac{Zg_1(G)}{(n-1)^2(n-1-\Delta)}.$$
Hence the part 3.

4. The parts (1) and (2) are similar and can be deduced as follows:
$$EE_r(G) = M_0(G) + \frac{M_1(G)}{(n-1)} + \sum_{k\geq 2} \frac{M_k(G)}{(n-1)^k} < n + \frac{2\,m}{\Delta^2} \sum_{k\geq 2} (\frac{\Delta}{n-1})^k$$

$$
\begin{aligned}
&= n + \frac{2\,m}{\Delta^2}\Big(\frac{\frac{\Delta^2}{(n-1)^2}}{1-\frac{\Delta}{n-1}}\Big) = n + \frac{2\,m}{(n-1)(n-1-\Delta)},\\
EE_r(G) &= M_0(G) + \frac{M_1(G)}{(n-1)} + \sum_{k\geq 2}\frac{M_k(G)}{(n-1)^k} < n + \frac{n}{\Delta}\sum_{k\geq 2}\Big(\frac{\Delta}{n-1}\Big)^k\\
&= n + \frac{n}{\Delta}\Big(\frac{\frac{\Delta^2}{(n-1)^2}}{1-\frac{\Delta}{n-1}}\Big) = n + \frac{n\,\Delta}{(n-1)(n-1-\Delta)}\\
&= n\Big(1 + \frac{\Delta}{(n-1)(n-1-\Delta)}\Big).
\end{aligned}
$$

This completes our argument. ■

Since the star graph S_n is the unique $n-$vertex bipartite graph with $\Delta = n-1$, all other bipartite graphs are having $\Delta < n-1$. In the following result, some upper bounds for EE_r of such graphs are presented.

Lemma 3.3 *Let G be a bipartite graph such that $\Delta < n-1$. Then,*

1. $EE_r(G) \leq n\big(1 + \frac{\Delta}{(n-1)^2 - \Delta^2}\big)$,
2. $EE_r(G) \leq n + \frac{2m}{(n-1)^2 - \Delta^2}$,
3. $EE_r(G) \leq n + \frac{2\,m}{(n-1)^2} + \frac{\Delta Zg_1(G)}{(n-1)^4 - \Delta^2(n-1)^2}$,
4. $EE_r(G) \leq n + \frac{2\,m}{(n-1)^2} + \frac{2Zg_2(G)}{(n-1)^4 - \Delta^2(n-1)^2}$,
5. $EE_r(G) \leq n + \sum_{i=1}^{n}\frac{d_i}{(n-1)^2 - d_i^2}$.

In each part, the equality is satisfied if and only if all components of G are the complete bipartite graph $K_{\Delta,\Delta}$.

Proof. It is clear that if G is a bipartite graph, then it does not have closed walks of odd length and so, $M_{2k+1}(G) = 0$, $k \geq 0$. To prove part (1), we apply Lemma 3.1 (1). Then,

$$
\begin{aligned}
EE_r(G) &= M_0(G) + \sum_{k\geq 1}\frac{M_{2k}(G)}{(n-1)^{2k}} \leq n + n\sum_{k\geq 1}\frac{\Delta^{2k-1}}{(n-1)^{2k}}\\
&= n + n\Big(\frac{\frac{\Delta}{(n-1)^2}}{1-\frac{\Delta^2}{(n-1)^2}}\Big) = n\big(1 + \frac{\Delta}{(n-1)^2-\Delta^2}\big).
\end{aligned}
$$

Other parts can be deduced by a similar argument. ■

If G is connected then in each part of Lemma 3.3, the equality is satisfied if and only if n is even and $G \cong K_{\frac{n}{2},\frac{n}{2}}$. On the other hand, all inequalities of Lemma 3.3 are strict, when G is acyclic. In [4], it is shown that if G is a tree (or a forest) with $\Delta \geq 2$, then for any positive integer k with $k \geq 3$,

$$M_k(G) < n\left(\sqrt{4(\Delta-1)}\right)^{k-1}. \tag{2}$$

So, (2) holds for $k = 1, 2$.

Lemma 3.4 *Let G be a tree (or a forest) with $\Delta \geq 2$. Then,*

$$EE_r(G) < n\left(1 + \frac{n\sqrt{4(\Delta-1)}}{(n-1)^2 - 4(\Delta-1)}\right).$$

Proof. The result can be proved directly by a similar argument as Lemma 3.3(1) and Equations (1) and (2). ■

Up to now, many lower and upper bounds for the largest and least eigenvalues of graphs were given. In [23, 28], bounds on the spectral radius $\lambda_1(G)$ of a connected graph G in terms of n and m are reported as:

$$\frac{2m}{n} \leq \sqrt{\frac{Zg_1(G)}{n}} \leq \lambda_1(G) \leq min\{\Delta, \sqrt{n-1}\} \tag{3}$$

In [27], it is proved that for the least eigenvalue $\lambda_n(G)$, we have:

$$-\frac{n}{2} \leq \lambda_n(G) < -\frac{1+\sqrt{1+\frac{4(n-3)}{n-1}}}{2} \tag{4}$$

The empty graph $\overline{K_n}$ is the only graph with exactly one eigenvalue. Moreover, the graph G has spectrum $\{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}\}$, $\lambda_1 > \lambda_2$, if and only if G is the direct sum of m_1 complete graphs of order $\lambda_1 + 1$. It is well-known that a connected graph with exactly two distinct eigenvalues is isomorphic to the complete graph K_n. In the following next three lemmas, some upper bounds for the resolvent Estrada index of graphs are obtained.

Lemma 3.5 *Let G be a non-complete connected graph or empty graph with n vertices, then*

$$EE_r(G) \leq \left\lceil \frac{n^2}{4} \right\rceil \frac{(\lambda_1 - \lambda_n)^2}{n(n-1-\lambda_1)(n-1-\lambda_n)} + n$$

with equality if and only if $G \cong \overline{K_n}$.

Proof. Define $a_i = \frac{1}{n-1-\lambda_i}$ and $b_i = n-1-\lambda_i$, $1 \le i \le n$. Note that $M_1(G) = 0$ and $n = EE_r(\overline{K_n}) \le EE_r(G)$. This shows that if $M_1^{-1} = m_2 = n-1-\lambda_1$ and $m_1^{-1} = M_2 = n-1-\lambda_n$ then by Remark 2.1, we have

$$|C_n(\bar{a},\bar{b})| \le \frac{1}{n^2}\left[\frac{n^2}{4}\right]\left(\frac{1}{n-1-\lambda_1} - \frac{1}{n-1-\lambda_n}\right)(\lambda_1-\lambda_n).$$

On the other hand,

$$\begin{aligned} \left|C_n(\bar{a},\bar{b})\right| &= \left|\frac{1}{n}\sum_{i=1}^{n} 1 - \frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{n-1-\lambda_i}\sum_{i=1}^{n}(n-1-\lambda_i)\right| \\ &= \left|1-\frac{1}{n^2}\frac{EE_r(G)}{n-1}.n(n-1)\right| = \left|1-\frac{EE_r(G)}{n}\right| = \frac{EE_r(G)}{n}-1, \end{aligned}$$

as desired. ■

Lemma 3.6 *Let G be a non-complete connected or empty graph with n vertices, then*

(1). $EE_r(G) \le \dfrac{n(n-1)(n-1-\lambda_1-\lambda_n)}{(n-1-\lambda_1)(n-1-\lambda_n)}$,

(2). $EE_r(G) \le \dfrac{n(2n-2-\lambda_1-\lambda_n)^2}{4(n-1-\lambda_1)(n-1-\lambda_n)}$,

(3). $EE_r(G) \le \dfrac{n(n-1)}{n-1-\left(\sqrt{n-1-\lambda_n}-\sqrt{n-1-\lambda_1}\right)^2}$,

(4). $EE_r(G) \le \left(\sqrt{n}+\dfrac{1}{2\sqrt{n}}.\sqrt{\dfrac{n-1-\lambda_n}{n-1-\lambda_1}}\left(\sqrt{\dfrac{n-1-\lambda_n}{n-1-\lambda_1}}-1\right)^2\right)^2$,

(5). $EE_r(G) \le \dfrac{n(\lambda_1-\lambda_n)^2}{3(n-1-\lambda_1)(n-1-\lambda_n)} + n.$

In each par the equality holds if and only if $G \cong \overline{K_n}$.

Proof. In Remarks 2.2 and 2.3, we define $a_i = \sqrt{n-1-\lambda_i}$, $b_i = \frac{1}{\sqrt{n-1-\lambda_i}}$, $1 \le i \le n$, $m = \frac{1}{n-1-\lambda_n}$, $M = \frac{1}{n-1-\lambda_1}$, $m_1 = M_2^{-1} = \sqrt{n-1-\lambda_1}$ and $M_1 = m_2^{-1} = \sqrt{n-1-\lambda_n}$. We prove each part separately as follows:

1. Applying Diaz-Metcalf inequality, we have

$$\sum_{i=1}^{n}\frac{1}{n-1-\lambda_i} + \frac{\sum_{i=1}^{n}(n-1-\lambda_i)}{(n-1-\lambda_1)(n-1-\lambda_n)} \le n\Big(\frac{1}{n-1-\lambda_1}+\frac{1}{n-1-\lambda_n}\Big).$$

So,

$$\frac{EE_r(G)}{n-1} + \frac{n(n-1)}{(n-1-\lambda_1)(n-1-\lambda_n)} \le n\Big(\frac{2n-2-\lambda_1-\lambda_n}{(n-1-\lambda_1)(n-1-\lambda_n)}\Big).$$

Therefore,

$$EE_r(G) \leq \frac{n(n-1)(n-1-\lambda_1-\lambda_n)}{(n-1-\lambda_1)(n-1-\lambda_n)}$$

which completes part (1).

2. By Pólya−Szegö inequality, we have

$$\frac{1}{n^2}\sum_{i=1}^{n}(n-1-\lambda_i)\sum_{i=1}^{n}\frac{1}{n-1-\lambda_i} \leq \frac{1}{4}\Big(\sqrt{\frac{n-1-\lambda_n}{n-1-\lambda_1}}+\sqrt{\frac{n-1-\lambda_1}{n-1-\lambda_n}}\Big)^2.$$

Therefore,

$$EE_r(G) \leq \frac{n(2n-2-\lambda_1-\lambda_n)^2}{4(n-1-\lambda_1)(n-1-\lambda_n)}.$$

This proves part (2).

3. Apply Shisha−Mond inequality, we can see that

$$\frac{1}{n}\sum_{i=1}^{n}(n-1-\lambda_i) - n\Big(\sum_{i=1}^{n}\frac{1}{n-1-\lambda_i}\Big)^{-1} \leq \Big(\sqrt{n-1-\lambda_n}-\sqrt{n-1-\lambda_1}\Big)^2.$$

Therefore,

$$n-1-n\Big(\frac{EE_r(G)}{n-1}\Big)^{-1} \leq \Big(\sqrt{n-1-\lambda_n}-\sqrt{n-1-\lambda_1}\Big)^2$$

and hence,

$$EE_r(G) \leq \frac{n(n-1)}{n-1-\Big(\sqrt{n-1-\lambda_n}-\sqrt{n-1-\lambda_1}\Big)^2},$$

proving part (3).

4. By Grüss inequality,

$$\begin{aligned}&\left(\sum_{i=1}^{n}(n-1-\lambda_i)\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\frac{1}{n-1-\lambda_i}\right)^{\frac{1}{2}}-n\\ \leq\quad & \frac{\sqrt[4]{\frac{n-1-\lambda_n}{n-1-\lambda_1}}\left(\sqrt[4]{\frac{n-1-\lambda_n}{n-1-\lambda_1}}-\sqrt[4]{\frac{n-1-\lambda_1}{n-1-\lambda_n}}\right)^2}{2\sqrt[4]{\frac{n-1-\lambda_1}{n-1-\lambda_n}}}\sqrt{\frac{n-1-\lambda_n}{n-1-\lambda_1}}.\end{aligned}$$

Therefore,

$$EE_r(G) \leq \Big(\sqrt{n}+\frac{1}{2\sqrt{n}}.\sqrt{\frac{n-1-\lambda_n}{n-1-\lambda_1}}\Big(\sqrt{\frac{n-1-\lambda_n}{n-1-\lambda_1}}-1\Big)^2\Big)^2,$$

as desired.

5. By Ozeki$-$Izumino$-$Mori$-$Seo inequality, we have
$$\sum_{i=1}^{n}(n-1-\lambda_i)\sum_{i=1}^{n}\frac{1}{n-1-\lambda_i}-n^2\leq\frac{n^2}{3}(\frac{n-1-\lambda_n}{n-1-\lambda_1}+\frac{n-1-\lambda_1}{n-1-\lambda_n}-2).$$
Therefore,
$$EE_r(G)\leq\frac{n(\lambda_1-\lambda_n)^2}{3(n-1-\lambda_1)(n-1-\lambda_n)}+n.$$
which completes part (5). Finally, the above equalities hold when G has at most two distinct eigenvalues and so G is the empty graph $\overline{K_n}$.

■

Lemma 3.7 *Let G be a non-complete connected or empty graph with n vertices, then*

(1). $EE_r(G)<\left[\frac{n^2}{4}\right]\cdot\frac{(n+2\gamma)^2}{2n(3n-2)(n-1-\gamma)}+n,$

(2). $EE_r(G)<\frac{n(n-1)(3n-2-2\gamma)}{(3n-2)(n-1-\gamma)},$

(3). $EE_r(G)<\frac{n(5n-2\gamma-4)^2}{8(3n-2)(n-1-\gamma)},$

(4). $EE_r(G)<\frac{n(n-1)}{n-1-\left(\sqrt{\frac{3}{2}n-1}-\sqrt{n-1-\gamma}\right)^2},$

(5). $EE_r(G)<\left(\sqrt{n}+\frac{\sqrt{2}}{8\sqrt{n}}\cdot\sqrt{\frac{3n-2}{n-1-\gamma}}\left(\sqrt{\frac{3n-2}{n-1-\gamma}}-\sqrt{2}\right)^2\right)^2,$

(6). $EE_r(G)<\frac{n(2\gamma+n)^2}{6(3n-2)(n-1-\gamma)}+n,$

where, $\gamma=min\{\Delta,\sqrt{n-1}\}$.

Proof. Define
$$f(\lambda_1,\lambda_n)=\frac{(\lambda_1-\lambda_n)^2}{(n-1-\lambda_1)(n-1-\lambda_n)}$$
By Equations (3) and (4), one can see that $f_{\lambda_1}^{'}\geq 0$ and $f_{\lambda_n}^{'}\leq 0$. Therefore, $f(\lambda_1,\lambda_n)<f(\gamma,-\frac{n}{2})$. Now the proof of part (1) can be deduced by applying Lemma 3.5. Define:
$$g(\lambda_1,\lambda_n)=\frac{n-1-\lambda_1-\lambda_n}{(n-1-\lambda_1)(n-1-\lambda_n)}.$$
By a simple calculations and applying Equations 3 and 4, we obtain $g_{\lambda_1}^{'}>0$ and $g_{\lambda_n}^{'}<0$. Therefore, $g(\lambda_1,\lambda_n)<g(\gamma,-\frac{n}{2})$. Finally, we apply Lemma 3.5 to prove of part (2). Other parts can be proved in a similar way as parts (1) and (2). ■

4 Resolvent signless Laplacian Estrada index

In this section, some bounds for a new resolvent Estrada index so-called **resolvent signless Laplacian Estrada index** are presented. To do this, we assume that G is a simple graph with signless Laplacian eigenvalues $q_1 \geq q_2 \geq \ldots \geq q_n$. It is well-known that $q_1 = 2\,n - 2$ if and only if G is a complete graph K_n. Hence, we have to consider non-complete graphs in definition of resolvent signless Laplacian Estrada index of G. Define:

$$SLEE_r(G) = \sum_{i=1}^{n} \left(1 - \frac{q_i}{2n-2}\right)^{-1} . \tag{5}$$

Notice that if $G \ncong K_n$, then for each $i = 0, 1, \ldots, n$, $q_i < 2n - 2$, and therefore $0 \leq \frac{q_i}{2n-2} < 1$. Thus, we may use the Maclaurin series for $(1 - \frac{q_i}{2n-2})^{-1}$ to evaluate $SLEE_r(G)$. In an exact phrase,

$$SLEE_r(G) = \sum_{k \geq 0} \frac{T_k(G)}{(2n-2)^k} , \tag{6}$$

where $T_k(G)$ denotes to the $k-$th signless Laplacian spectral moment of the graph G, i.e. $T_k(G) = \sum_{i=1}^{n} q_i^k$. It is well-known that $T_k(G)$ equals to the number of closed semi-edge walks of length k [9] and so for some small values of k, it is possible to evaluate $T_k(G)$ in terms of some graph parameters. For example, $T_0(G) = n$, $T_1(G) = 2\,m$, $T_2(G) = Zg_1(G) + 2\,m$ and $T_3(G) = 6\,t + 3\,Zg_1(G) + \sum_{v \in V(G)} d^3(v)$.

In the following Lemma, resolvent signless Laplacian Estrada index $SLEE_r$ is computed by the characteristic polynomial of $\mathbf{Q}$.

Lemma 4.1 *Let G be a non-complete $n-$vertex graph. Then,*

$$SLEE_r(G) = (2n-2)\frac{\Phi'_G(2n-2)}{\Phi_G(2n-2)} ,$$

where $\Phi_G(x)$ is the characteristic polynomial of the matrix $\mathbf{Q}$.

Proof. By definition,

$$\frac{\Phi'_G(x)}{\Phi_G(x)} = \sum_{i=1}^{n} (x - q_i)^{-1} .$$

The proof can be completed by substituting $x = 2n - 2$ in above formula. ■

4.1 $SLEE_r$ of Trees

In this subsection, the quantity $SLEE_r$ for some trees are computed. We start by an example applicable in chemistry.

Example 4.2 *Apply Lemma 4.1 to calculate the resolvent signless Laplacian Estrada index of the molecular graphs of Methane and Ethane molecules. It can easily seen that* $\Phi_{CH_4}(x) = x^5 - 8x^4 + 18x^3 - 16x^2 + 5x$ *and* $\Phi_{C_2H_6}(x) = x^8 - 14x^7 + 73x^6 - 182x^5 + 244x^4 - 182x^3 + 73x^2 - 14x + 1$. *Hence, by Lemma 4.1,*

$$SLEE_r(CH_4) = 8.\frac{5x^4 - 32x^3 + 54x^2 - 32x + 5}{x^5 - 8x^4 + 18x^3 - 16x^2 + 5x}\Big|_{x=8} \simeq 7.09524$$

$$SLEE_r(C_2H_6) = 14.\frac{8x^7 - 98x^6 + 438x^5 - 910x^4 + 976x^3 - 546x^2 + 146x - 14}{x^8 - 14x^7 + 73x^6 - 182x^5 + 244x^4 - 182x^3 + 73x^2 - 14x + 1}\Big|_{x=14} \simeq 9.37856.$$

From (6), we can see that if G_1 and G_2 are two non-complete graphs on the same number of vertices and $T_k(G_1) \leq T_k(G_2)$, for all $k \geq 0$, then $SLEE_r(G_1) \leq SLEE_r(G_2)$. On the other hand, it is proved in [14] that $T_k(G-e) < T_k(G)$, for each $k \geq 1$. Therefore, the next result immediately follows.

Lemma 4.3 *Let G be a graph and $e \in E(G)$. Then $SLEE_r(G-e) < SLEE_r(G)$.*

Lemma 4.4 *Let T be an $n-$vertex tree on n. Then,*

$$\sum_{i=1}^{n} \frac{n-1}{n-2-cos(\frac{\pi i}{n})} \leq SLEE_r(T) \leq \frac{2n^3 - 4n^2 - n + 4}{2n^2 - 7n + 6}$$

with left equality if and only if $T \cong P_n$ and right equality if and only if $T \cong S_n$.

Proof. By [14], if T is a tree on n vertices, then $SLEE_r(P_n) \leq SLEE_r(T) \leq SLEE_r(S_n)$. It is well-known that the Laplacian eigenvalues of P_n are equal to $2\left(1 + cos(\frac{\pi i}{n})\right)$, $i = 1, ..., n$. Since the Laplacian and signlees Laplacian spectra of bipartite graphs coincide, the signlees Laplacian eigenvalues of P_n are equal to $2\left(1 + cos(\frac{\pi i}{n})\right)$, $i = 1, ..., n$. Also, $\mathbf{Q}-$spectrum of S_n is $\{[n]^1, [1]^{n-2}, [0]^1\}$. Thus the proof is a direct consequence of (5). ■

Apply Lemmas 4.3 and 4.4 to prove the following result:

Lemma 4.5 *Let G be an $n-$vertex non-complete graph on n. Then,*

$$n = SLEE_r(\overline{K_n}) \leq SLEE_r(G) \leq SLEE_r(K_n - e).$$

Furthermore, if G is a connected then

$$SLEE_r(P_n) \leq SLEE_r(G) \leq SLEE_r(K_n - e).$$

4.2 Lower Bounds on $SLEE_r$

Up to now, many lower and upper bounds for the largest and least signless Laplacian eigenvalues q_1 and q_n were given. In [9, 10], bounds on the signless Laplacian spectral radius $q_1(G)$ of a connected graph G in terms of n and m are given. One of the important bounds are as follows:

$$\frac{4m}{n} \leq q_1(G) \leq \frac{2m}{n-1} + n - 2 \tag{7}$$

in which left equality holds if and only if G is a regular graph and right equality holds if and only if G is S_n or K_n. The following bounds are also presented for the least $\mathbf{Q}-$eigenvalue $q_n(G)$ [11, 17]:

$$\frac{2m}{n-2} - n + 1 \leq q_n(G) < \delta. \tag{8}$$

It is well known that the empty graph $\overline{K_n}$ is the unique graph with exactly one $\mathbf{Q}-$eigenvalue. Cvetković [6] proved that if G is a connected graph with r distinct signless Laplacian eigenvalues and diameter d, then $d \leq r - 1$. On the other hand, the complete graph K_n is the unique connected graph with exactly two $\mathbf{Q}-$eigenvalues. In the following Lemma, connected graphs with three distinct $\mathbf{Q}-$eigenvalues are characterized.

Lemma 4.6 *[1] Let G be a connected graph of order $n \geq 4$. Then G has exactly three distinct $\mathbf{Q}-$eigenvalues if and only if G is one of the graphs $K_n - e$, S_n, $K_{\frac{n}{2},\frac{n}{2}}$, $\overline{K_3 + S_4}$ or $\overline{K_1 + 2K_3}$.*

In what follows, we apply previous Lemma to compute the signless laplacian spectrum of some graphs.

$$\begin{aligned}
\mathbf{Q}-Spec(K_n - e) &= \{[\frac{3n-6 \pm \sqrt{n^2+4n-12}}{2}]^1, [n-2]^{n-2}\} \\
\mathbf{Q}-Spec(S_n) &= \{[n]^1, [1]^{n-2}, [0]^1\} \\
\mathbf{Q}-Spec(K_{\frac{n}{2},\frac{n}{2}}) &= \{[n]^1, [\frac{n}{2}]^{n-2}, [0]^1\} \\
\mathbf{Q}-Spec(\overline{K_3 + S_4}) &= \{[9]^1, [4]^{n-2}, [1]^1\} \ ; n = 7 \\
\mathbf{Q}-Spec(\overline{K_1 + 2K_3}) &= \{[9]^1, [4]^{n-2}, [1]^1\} \ ; n = 7
\end{aligned}$$

We are now ready to present some lower bounds for the resolvent signless Laplacian Estrada index of graphs.

Lemma 4.7 *Let G be a graph with n vertices and m edges, and let $I \subseteq \{1, 2, \dots, n\}$, then*

$$SLEE_r(G) \geq \sum_{j \in I} \frac{2n-2}{2n-2-q_j} + \frac{(2n-2)(n-n')^2}{(2n-2)(n-n') - 2m + \sum_{j \in I} q_j}$$

where $n' = n(I)$, and equality holds if and only if $q_i = q_j$, for all $i, j \notin I$.

Proof. Apply the Cauchy-Schwarz inequality,

$$\begin{aligned} SLEE_r(G) &= \sum_{i \in I} \frac{2n-2}{2n-2-q_i} + \sum_{i \notin I} \frac{2n-2}{2n-2-q_i} \\ &\geq \sum_{i \in I} \frac{2n-2}{2n-2-q_i} + \frac{(2n-2)(n-n')^2}{\sum_{i \notin I}(2n-2-q_i)} \\ &= \sum_{j \in I} \frac{2n-2}{2n-2-q_j} + \frac{(2n-2)(n-n')^2}{(2n-2)(n-n') - 2m + \sum_{j \in I} q_j}, \end{aligned}$$

proving the result. ■

Theorem 4.8 *Let G be a non-complete connected or empty graph with n vertices and m edges, and let $1 \leq s < r \leq n$. Then,*

(1). $SLEE_r(G) \geq \dfrac{n^2\,(n-1)}{n\,(n-1) - m}$,

(2). $SLEE_r(G) \geq \dfrac{n\,(n-1)}{n\,(n-1) - 2m} + \dfrac{n\,(n-1)^3}{n\,(n-1)^2 + 2m - mn}$,

(3). $SLEE_r(G) \geq \dfrac{2\,n-2}{2\,n-2-q_2} + \dfrac{2\,n-2}{2\,n-2-q_s} + \dfrac{(2\,n-2)\,(n-2)^2}{(2\,n-2)(n-2) - 2\,m + q_r + q_s}$,

(4). $$\begin{aligned} SLEE_r(G) &> \frac{2\,n-2}{2\,n-2-2\,\overline{\overline{d}}} + \frac{2\,n-2}{2\,n-1-\overline{\overline{d}}} + \frac{(2\,n-2)\,(n-2)^2}{(2\,n-2)\,(n-2) - 2\,m + n - 2 + 2\,\Delta} \\ &\geq \frac{2\,n-2}{2\,n-2-2\,\delta} + \frac{2\,n-2}{2\,n-1-\delta} + \frac{(2\,n-2)\,(n-2)^2}{(2\,n-2)\,(n-2) - 2\,m + n - 2 + 2\,\Delta}. \end{aligned}$$

The equalities in parts (1) and (2) hold if and only if $G \cong \overline{K_n}$, and in (3) the equality is satisfied if and only if $G \cong \overline{K_n}$, $K_n - e$, S_n, $K_{\frac{n}{2},\frac{n}{2}}$, $\overline{K_3 + S_4}$ or $\overline{K_1 + 2K_3}$.

Proof. Apply Lemma 4.7. Our main proof will consider four separate parts as follows:

1. By setting I as a empty set, the inequality in part (1) is obtained, and the equality holds only when $q_1 = q_2 = \cdots = q_n$. On the other hand, $\overline{K_n}$ is the unique graph with exactly one $\mathbf{Q}-$eigenvalue, as desired.

2. By considering $I = \{1\}$, we have $SLEE_r(G) \geq \frac{2n-2}{2n-2-q_1} + \frac{2\,(n-1)^3}{2\,(n-1)^2 + q_1 - 2m}$. The equality in part (2) holds when either $q_1 = q_2 = \cdots = q_n$ or $q_1 \neq q_2 = \cdots = q_n$. As already mentioned above, K_n is the unique graph with exactly two distinct $\mathbf{Q}-$eigenvalues,

a contradiction. Therefore, the equality in this part holds if and only if $G \cong \overline{K_n}$. On the other hand, by Equation (7), $q_1 \geq \frac{4m}{n}$ with equality if and only if G is a regular graph. Define:

$$f(x) = \frac{2n-2}{2n-2-x} + \frac{2\,(n-1)^3}{2\,(n-1)^2 + x - 2m}; \qquad \frac{4\,m}{n} \leq x < 2n-2$$

Then f is an increasing function on the interval $\left[\frac{4\,m}{n}, 2\,n-2\right)$. Therefore, $f(q_1) \geq f(\frac{4m}{n})$, which completes part (2).

3. By setting $I = \{r, s\}$, $1 \leq s < r \leq n$, the inequality in part (3) is gained. The equality holds when $q_i = q_j$, for all $1 \leq i < j \leq n$ and $i, j \neq r, s$. Assume that $q_i = q_j = c$, for all $1 \leq i < j \leq n$ and $i, j \notin I$. Thus, three cases can be occurred as follows:

   (i) $q_r = q_s = c$ (G has exactly one $\mathbf{Q}-$eigenvalue).

   (ii) $q_r = q_s \neq c$ (G has exactly two $\mathbf{Q}-$eigenvalues).

   (iii) $c \neq q_r \neq q_s \neq c$ (G has exactly three $\mathbf{Q}-$eigenvalues).

   The case (ii) is satisfied if and only if $G \cong K_n$, a contradiction. Also, the case (i) is satisfied if and only if $G \cong \overline{K_n}$. On the other hand, by Lemma 4.6, our calculations before this Lemma and the fact that $q_1 \geq q_2 \geq \cdots \geq q_n$, the equality in (iii) holds when $G \cong K_n - e$, S_n, $K_{\frac{n}{2},\frac{n}{2}}$, $\overline{K_3 + S_4}$ or $\overline{K_1 + 2K_3}$, and also $s = 1, r = n$.

4. We know that if G is a connected graph of order n with minimum degree δ, average degree $\overline{d}$ and second largest signless Laplacian eigenvalue q_2, then $\overline{d} - 1 \leq q_2 \leq n-2$, with equalities if and only if $G \cong K_n$ [11,16]. Also, it is well-known that $2\,\delta \leq 2\,\overline{d} \leq q_1 \leq 2\,\Delta$, with equality if and only if G is a regular graph. Thus by setting $s = 1$ and $r = 2$ in part (3), one can easily see that part (4) holds.

■

4.3 Upper bounds on $SLEE_r$

In this section, we apply Equations (7) and (8) to obtain some upper bounds for the resolvent signless Laplacian Estrada index of graphs.

Lemma 4.9 *Let G be a non-complete connected or empty graph with n vertices and m edges. Then,*

$$SLEE_r(G) \leq \frac{n^2(n-1)}{n(n-1)-m} + \left[\frac{n^2}{4}\right] \frac{(n-1)(q_1-q_n)^2}{(n^2-n-m)(2n-2-q_1)(2n-2-q_n)}$$

with equality if and only if $G \cong \overline{K_n}$.

Proof. Substitute $a_i = 2n-2-q_i$ and $b_i = \frac{1}{2n-2-q_i}$ in Remark 2.1 to obtain $M_1^{-1} = m_2 = 2n-2-q_1$ and $m_1^{-1} = M_2 = 2n-2-q_n$. Since $T_1(G) = 2m$ and $n = SLEE_r(\overline{K_n}) \leq SLEE_r(G)$,

$$\begin{aligned} |C_n(\bar{a},\bar{b})| &\leq \frac{1}{n^2}\left[\frac{n^2}{4}\right](\frac{1}{2n-2-q_1} - \frac{1}{2n-2-q_n})(q_1-q_n) \\ &= \frac{1}{n^2}\left[\frac{n^2}{4}\right]\frac{(q_1-q_n)^2}{(2n-2-q_1)(2n-2-q_n)}. \end{aligned}$$

On the Other hand,

$$\begin{aligned} \left|C_n(\bar{a},\bar{b})\right| &= \left|\frac{1}{n}\sum_{i=1}^{n} 1 - \frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{2n-2-q_i}\sum_{i=1}^{n}(2n-2-q_i)\right| \\ &= \left|1 - \frac{1}{n^2}\frac{SLEE_r(G)}{2n-2}\Big(n(2n-2)-2m\Big)\right| \\ &= \frac{1}{n^2(n-1)}\left|n^2(n-1) - \big(n(n-1)-m\big)SLEE_r(G)\right|. \end{aligned}$$

Therefore,

$$\left|n^2(n-1) - \big(n^2-n-m\big)SLEE_r(G)\right| \leq \left[\frac{n^2}{4}\right]\frac{(n-1)(q_1-q_n)^2}{(2n-2-q_1)(2n-2-q_n)},$$

as desired. ■

Lemma 4.10 *Let G be a non-complete connected or empty graph with n vertices and m edges. Then,*

(1). $SLEE_r(G) \leq \dfrac{2(n-1)\Big(2m + n(2n-2-q_1-q_n)\Big)}{(2n-2-q_1)(2n-2-q_n)}$,

(2). $SLEE_r(G) \leq \dfrac{n^2(n-1)(4n-4-q_1-q_n)^2}{4\big(n^2-n-m\big)(2n-2-q_1)(2n-2-q_n)}$,

(3). $SLEE_r(G) \leq \dfrac{n^2(2n-2)}{n(2n-2)-2m-n\Big(\sqrt{2n-2-q_n}-\sqrt{2n-2-q_1}\Big)^2}$,

(4). $SLEE_r(G) \leq \dfrac{n-1}{n(n-1)-m}\Big(n + \frac{1}{2}.\sqrt{\dfrac{2n-2-q_n}{2n-2-q_1}}(\sqrt{\dfrac{2n-2-q_n}{2n-2-q_1}}-1)^2\Big)^2$,

(5). $SLEE_r(G) \leq \dfrac{n^2(n-1)}{n(n-1)-m}\Big(\dfrac{(q_1-q_n)^2}{3(2n-2-q_1)(2n-2-q_n)}+1\Big).$

In each part the equality is satisfied if and only if $G \cong \overline{K_n}$.

Proof. Substitute $a_i = \sqrt{2n-2-q_i}$ and $b_i = \frac{1}{\sqrt{2n-2-q_i}}$ in Remarks 2.2, 2.3 and 2.3, and $m = \frac{1}{2n-2-q_n}$, $M = \frac{1}{2n-2-q_1}$, $m_1 = M_2^{-1} = \sqrt{2n-2-q_1}$ and $M_1 = m_2^{-1} = \sqrt{2n-2-q_n}$. Our main proof will consider five separate parts as follows:

1. Apply Diaz−Metcalf inequality, we have:

$$\sum_{i=1}^{n}\frac{1}{2n-2-q_i}+\frac{\sum_{i=1}^{n}(2n-2-q_i)}{(2n-2-q_1)(2n-2-q_n)} \leq n\Big(\frac{1}{2n-2-q_1}+\frac{1}{2n-2-q_n}\Big).$$

So,

$$\frac{SLEE_r(G)}{2n-2}+\frac{n(2n-2)-2m}{(2n-2-q_1)(2n-2-q_n)} \leq n.\frac{4n-4-q_1-q_n}{(2n-2-q_1)(2n-2-q_n)}.$$

Therefore,

$$SLEE_r(G) \leq \frac{2(n-1)\Big(2m+n(2n-2-q_1-q_n)\Big)}{(2n-2-q_1)(2n-2-q_n)}$$

which completes part (1).

2. By Pólya−Szegö inequality, we have:

$$\frac{1}{n^2}\sum_{i=1}^{n}(2n-2-q_i)\sum_{i=1}^{n}\frac{1}{2n-2-q_i} \leq \frac{1}{4}\Big(\sqrt{\frac{2n-2-q_n}{2n-2-q_1}}+\sqrt{\frac{2n-2-q_1}{2n-2-q_n}}\Big)^2.$$

Therefore,

$$SLEE_r(G) \leq \frac{n^2(n-1)(4n-4-q_1-q_n)^2}{4\big(n^2-n-m\big)(2n-2-q_1)(2n-2-q_n)},$$

proving part (2).

3. Apply Shisha−Mond inequality to see that

$$\frac{1}{n}\sum_{i=1}^{n}(2n-2-q_i)-n\Big(\sum_{i=1}^{n}\frac{1}{2n-2-q_i}\Big)^{-1} \leq \Big(\sqrt{2n-2-q_n}-\sqrt{2n-2-q_1}\Big)^2.$$

Therefore,

$$\frac{n(2n-2)-2m}{n}-\frac{n(2n-2)}{SLEE_r(G)} \leq \Big(\sqrt{2n-2-q_n}-\sqrt{2n-2-q_1}\Big)^2.$$

And so,

$$SLEE_r(G) \leq \frac{n^2(2n-2)}{n(2n-2)-2m-n\Big(\sqrt{2n-2-q_n}-\sqrt{2n-2-q_1}\Big)^2},$$

as desired.

4. By Grüss inequality, we can see that

$$\Big(\sum_{i=1}^{n}(2n-2-q_i)\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{n}\frac{1}{2n-2-q_i}\Big)^{\frac{1}{2}}-n\leq\frac{\sqrt[4]{\frac{2n-2-q_n}{2n-2-q_1}}\Big(\sqrt[4]{\frac{2n-2-q_n}{2n-2-q_1}}-\sqrt[4]{\frac{2n-2-q_1}{2n-2-q_n}}\Big)^2}{2.\sqrt[4]{\frac{2n-2-q_1}{2n-2-q_n}}}\sqrt{\frac{2n-2-q_n}{2n-2-q_1}}$$

Therefore,

$$SLEE_r(G)\leq\frac{n-1}{n(n-1)-m}\Big(n+\frac{1}{2}.\sqrt{\frac{2n-2-q_n}{2n-2-q_1}}\Big(\sqrt{\frac{2n-2-q_n}{2n-2-q_1}}-1\Big)^2\Big)^2$$

which proves part (4).

5. By Ozeki−Izumino−Mori−Seo inequality, we have

$$\sum_{i=1}^{n}(2n-2-q_i)\sum_{i=1}^{n}\frac{1}{2n-2-q_i}-n^2\leq\frac{n^2}{3}\Big(\frac{2n-2-q_n}{2n-2-q_1}+\frac{2n-2-q_1}{2n-2-q_n}-2\Big).$$

Therefore,

$$SLEE_r(G)\leq\frac{n^2(n-1)}{n(n-1)-m}\Big(\frac{(q_1-q_n)^2}{3(2n-2-q_1)(2n-2-q_n)}+1\Big),$$

proving part (5). Clearly, the equalities in above parts hold when G has at most two distinct $\mathbf{Q}-$eigenvalues and since $G\ncong K_n$, G is isomorphic to the empty graph $\overline{K_n}$.

■

Lemma 4.11 *Let G be a non-complete connected or empty graph with n vertices and m edges. Then,*

(1). $SLEE_r(G)<\frac{n^2(n-1)}{n(n-1)-m}+\left[\frac{n^2}{4}\right]\frac{(2n^3-9n^2+13n-2m-6)^2}{(n-2)(n^2-n-m)(n^2-n-2m)(3n^2-9n-2m+6)}$

(2). $SLEE_r(G)<\frac{(n-1)(4n^4-14n^3-4mn^2+14n^2+8m-4n)}{(n^2-n-2m)(3n^2-9n-2m+6)}$

(3). $SLEE_r(G)<\frac{n^2(4n^3-15n^2+17n-4mn+6m-6)^2}{4(n-2)(n^2-n-m)(n^2-n-2m)(3n^2-9n-2m+6)}$

(4). $SLEE_r(G)<\frac{2n^2(n-1)}{2n(n-1)-2m-n\left(\sqrt{3n-3-\frac{2m}{n-2}}-\sqrt{n-\frac{2m}{n-1}}\right)^2}$

(5). $SLEE_r(G)<\frac{n-1}{n(n-1)-m}\Big(n+\frac{1}{2}.\sqrt{\frac{3n-3-\frac{2m}{n-2}}{n-\frac{2m}{n-1}}}\Big(\sqrt{\frac{3n-3-\frac{2m}{n-2}}{n-\frac{2m}{n-1}}}-1\Big)^2\Big)^2$

(6). $SLEE_r(G)<\frac{n^2(n-1)}{n(n-1)-m}\Big(\frac{(2n^3-9n^2+13n-2m-6)^2}{3(n-1)(n-2)(n^2-n-2m)(3n^2-9n-2m+6)}+1\Big)$

Proof. The proof is similar to Lemma 3.7 and so it is omitted. ■

5 Concluding Remarks

In this paper, some new bounds for the resolvent Estrada index of graphs are presented. In the same line as the resolvent Estrada index, the resolvent signless Laplacian Estrada index together with the extremal trees with respect to this new invariant, are also presented. Gutman et al. [21], in a recently published paper introduced the concept of "resolvent energy" of an $n-$vertex graph G as $ER(G) = \sum_{i=1}^{n} \left(1 - \frac{\lambda_i}{n}\right)^{-1}$, where $Spec(G) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. In the mentioned paper, it is explained why ER is in fact a new kind of "graph energy", whereas the resolvent Estrada index EE_r is by no way related to the ordinary Estrada index. On the other hand, EE_r is not defined for all graphs, whereas ER is. Finally, the interested readers can change easily the results of this paper in the language of graph energy.

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