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On the Minimal Energy of Tetracyclic Graphs

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Abstract

The energy of a graph is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. In this paper, we characterize the tetracyclic graph of order n with minimal energy. By this, the validity of a conjecture for the case $e = n+3$ proposed by Caporossi et al. (1999) has been confirmed.

1 Introduction

Let G be a simple graph with n vertices and $A(G)$ the adjacency matrix of G. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(G)$ are said to be the eigenvalues of the graph G. The energy of G is defined as

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The characteristic polynomial of $A(G)$ is also called the characteristic polynomial of G, denoted by $\phi(G,x) = \det(xI - A(G)) = \sum_{i=0}^{k} a_i(G)x^{n-i}$. Using these coefficients of $\phi(G,x)$, the energy of G can be expressed as the Coulson integral formula [14]:

$$E(G) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{x^2} \ln \left[\left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i a_{2i}(G) x^{2i} \right)^2 + \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i a_{2i+1}(G) x^{2i+1} \right)^2 \right] dx. \quad (1)$$

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For convenience, write $b_{2i}(G) = (-1)^i a_{2i}(G)$ and $b_{2i+1}(G) = (-1)^i a_{2i+1}(G)$ for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Since the energy of a graph can be used to approximate the total π-electron energy of the molecular, it has been intensively studied. For details on graph energy, we refer the readers to the book [24], two reviews [11, 13] and some recent papers [5–10, 12, 15–23, 25–37].

A connected graph on n vertices with e edges is called an (n, e)-graph. We call an (n, e)-graph a unicyclic graph, a bicyclic graph, a tricyclic graph, and a tetracyclic graph if $e = n, n+1, n+2$ and $n+3$, respectively. Follow [35], let $S_{n,e}$ be the graph obtained by the star S_n with $e - n + 1$ additional edges all connected to the same vertex, and $B_{n,e}$ be the bipartite (n, e)-graph with two vertices on one side, one of which is connected to all vertices on the other side.

In [1], Caporossi et al. gave the following conjecture:

Conjecture 1.1. [1] *Connected graphs G with $n \geq 6$ vertices, $n-1 \leq e \leq 2(n-2)$ edges and minimum energy are $S_{n,e}$ for $e \leq n + [(n-7)/2]$, and $B_{n,e}$ otherwise.*

This conjecture is true when $e = n-1$, $2(n-2)$ [1], and when $e = n$ for $n \geq 6$ [15]. Li et al. [26] showed that $B_{n,e}$ is the unique bipartite graph of order n with minimal energy for $e \leq 2n-4$. Hou [16] proved that for $n \geq 6$, $B_{n,n+1}$ has the minimal energy among all bicyclic graphs of order n with at most one odd cycle. Let $\mathcal{G}_{n,e}$ be the set of connected graphs with n vertices and e edges. Let $\mathcal{G}^1_{n,e}$ be the subset of $\mathcal{G}_{n,e}$ which contains no disjoint two odd cycles of length p and q with $p + q \equiv 2 \pmod 4$, and $\mathcal{G}^2_{n,e} = \mathcal{G}_{n,e} \setminus \mathcal{G}^1_{n,e}$. Zhang and Zhou [36] characterized the graphs with minimal, second-minimal and third-minimal energy in $\mathcal{G}^1_{n,n+1}$ for $n \geq 8$. Combining the results (Lemmas 5-9) in [36] with the fact that $E(B_{n,n+1}) < E(S_{n,n+1})$ for $5 \leq n \leq 7$, we can deduce the following theorem.

Theorem 1.2. [36] *The graph with minimal energy in $\mathcal{G}^1_{n,n+1}$ is $S_{n,n+1}$ for $n = 4$ or $n \geq 8$, and $B_{n,n+1}$ for $5 \leq n \leq 7$, respectively.*

Li et al. [22] proved that $B_{n,n+2}$ has minimal energy in $\mathcal{G}^1_{n,n+2}$ for $7 \leq n \leq 9$, and for $n \geq 10$, they wanted to characterize the graphs with minimal and second-minimal energy in $\mathcal{G}^1_{n,n+2}$, but left four special graphs without determining their ordering. Huo et al. solved this problem in [18], and the results on minimal energy can be restated as follows.

Theorem 1.3. *The graph with minimal energy in* $\mathcal{G}^1_{n,n+2}$ *is* $B_{n,n+2}$ *for* $7 \leq n \leq 9$ [22], *and* $S_{n,n+2}$ *for* $n \geq 10$ [18], *respectively.*

In [35], the authors claimed that they gave a complete solution to conjecture 1.1 for $e = n + 1$ and $e = n + 2$ by showing the following two results.

Theorem 1.4. (Theorem 1 of [35]) *Let* G *be a connected graph with* n *vertices and* $n+1$ *edges. Then*

$$E(G) \geq E(S_{n,n+1})$$

with equality if and only if $G \cong S_{n,n+1}$.

Theorem 1.5. (Theorem 2 of [35]) *Let* G *be a connected graph with* n *vertices and* $n+2$ *edges. Then*

$$E(G) \geq E(S_{n,n+2})$$

with equality if and only if $G \cong S_{n,n+2}$.

Note that $E(B_{n,n+1}) < E(S_{n,n+1})$ for $5 \leq n \leq 7$, and $E(B_{n,n+2}) < E(S_{n,n+2})$ for $6 \leq n \leq 9$. Hence Theorems 1.4 and 1.5 do not hold for smaller n, respectively. Moreover, even for large n, there is a little gap in the original proofs of Theorems 1.4 and 1.5 in [35], respectively. For completeness, we will prove the following two results in Section 2.

Theorem 1.6. $S_{n,n+1}$ *if* $n = 4$ *or* $n \geq 8$, $B_{n,n+1}$ *if* $5 \leq n \leq 7$ *has minimal energy in* $\mathcal{G}_{n,n+1}$.

Theorem 1.7. *The complete graph* K_4 *if* $n = 4$, $S_{n,n+2}$ *if* $n = 5$ *or* $n \geq 10$, $B_{n,n+2}$ *if* $6 \leq n \leq 9$ *has minimal energy in* $\mathcal{G}_{n,n+2}$. *Furthermore,* $S_{6,8}$ *has second-minimal energy in* $\mathcal{G}_{6,8}$.

Li and Li [21] discussed the graph with minimal energy in $\mathcal{G}^1_{n,n+3}$, and claimed that the graph with minimal energy in $\mathcal{G}^1_{n,n+3}$ is $B_{n,n+3}$ for $9 \leq n \leq 17$, and $S_{n,n+3}$ for $n \geq 18$, respectively. Note that $E(S_{n,n+3}) < E(B_{n,n+3})$ for $n \geq 12$. In Section 3, we will first illustrate the correct version of this result, and then we will show the following theorem.

Theorem 1.8. *The wheel graph* W_5 *if* $n = 5$, *the complete bipartite graph* $K_{3,3}$ *if* $n = 6$, $B_{n,n+3}$ *if* $7 \leq n \leq 11$, $S_{n,n+3}$ *if* $n \geq 12$ *has minimal energy in* $\mathcal{G}_{n,n+3}$. *Furthermore,* $S_{n,n+3}$ *has second-minimal energy in* $\mathcal{G}_{n,n+3}$ *for* $6 \leq n \leq 7$.

Lemma 1.9. [35] $E(S_{n,e}) < E(B_{n,e})$ *if* $n - 1 \leq e \leq \frac{3}{2}n - 3$; $E(B_{n,e}) < E(S_{n,e})$ *if* $\frac{3}{2}n - \frac{5}{2} \leq e \leq 2n - 4$.

From Lemma 1.9, we know that the bound $e \leq n + [(n-7)/2]$ in Conjecture 1.1 should be understood that $e \leq n + \lceil (n-7)/2 \rceil$. With Theorems 1.6, 1.7 and 1.8, we give a complete solution to Conjecture 1.1 for $e = n+1, n+2$ and $n+3$.

2 The graphs with minimal energy in $\mathcal{G}_{n,n+1}$ and $\mathcal{G}_{n,n+2}$

The following three lemmas are needed in the sequel.

Lemma 2.1. [9] *If F is an edge cut of a simple graph G, then $E(G-F) \leq E(G)$, where $G-F$ is the subgraph obtained from G by deleting the edges in F.*

Lemma 2.2. [35] *(1) Suppose that $n_1, n_2 \geq 3$ and $n = n_1 + n_2$. Then*

$$E(S_{n_1,n_1} \cup S_{n_2,n_2}) \geq E(S_{n-3,n-3} \cup C_3)$$

with equality if and only if $\{n_1, n_2\} = \{3, n-3\}$.

(2) $E(S_{n-3,n-3} \cup C_3) > E(S_{n,n+1})$ for $n \geq 6$.

(3) $E(S_{n,n+1}) > E(S_{n,n})$ for $n \geq 4$.

(4) $E(S_{n-3,n-3} \cup C_3) > E(S_{n,n+2})$ for $n \geq 6$.

Lemma 2.3. *(1)* [15] *$S_{n,n}$ has minimal energy in $\mathcal{G}_{n,n}$ for $n = 3$ or $n \geq 6$.*

(2) $B_{n,n}$ and $S_{n,n}$ have, respectively, minimal and second-minimal energy in $\mathcal{G}_{n,n}$ for $4 \leq n \leq 5$. In particular, $S_{n,n}$ is the unique non-bipartite graph in $\mathcal{G}_{n,n}$ with minimal energy for $4 \leq n \leq 5$.

Proof. By Table 1 of [3], there are two $(4,4)$-graphs and five $(5,5)$-graphs. By simple computation, we can obtain the result (2). ∎

Proof of Theorem 1.6: By Theorem 1.2, it suffices to prove that $E(G) > E(S_{n,n+1})$ when $n = 4$ or $n \geq 8$, and $E(G) > E(B_{n,n+1})$ when $5 \leq n \leq 7$ for $G \in \mathcal{G}^2_{n,n+1}$.

Suppose that $G \in \mathcal{G}^2_{n,n+1}$. As there is nothing to prove for the case $n \leq 5$, we suppose that $n \geq 6$. Then G has a cut edge f such that $G - f$ contains exactly two components, say G_1 and G_2, which are non-bipartite unicyclic graphs. Let $|V(G_1)| = n_1$, $|V(G_2)| = n_2$, and $n_1 + n_2 = n$. By Lemmas 2.1, 2.2 and 2.3, we have

$$\begin{aligned} E(G) &\geq E(G_1 \cup G_2) && (2)\\ &\geq E(S_{n_1,n_1} \cup S_{n_2,n_2}) && (3)\\ &\geq E(S_{n-3,n-3} \cup C_3) && (4)\\ &> E(S_{n,n+1}). && (5) \end{aligned}$$

In particular, $E(G) > E(S_{n,n+1}) > E(B_{n,n+1})$ for $6 \leq n \leq 7$. The proof is thus complete. ∎

Remark 2.4. *The proof of Theorem 1.6 (for large n) is similar to that of Theorem 1.4 except that in [35], the authors did not point out that G_1 and G_2 are non-bipartite unicyclic graphs. Without this assumption, we know that the inequality* (3) *does not hold when n_1 or n_2 equals to* 4 *or* 5 *by Lemma 2.3 (2). Moreover, the inequality $E(G_1 \cup G_2) \geq E(S_{n-3,n-3} \cup C_3)$ does not hold. For example: $E(C_4 \cup S_{n-4,n-4}) < E(S_{n-3,n-3} \cup C_3)$ for $n \geq 7$, since $E(C_4) = E(C_3) = 4$ and $E(S_{n-4,n-4}) < E(S_{n-3,n-3})$ by Lemma 2.1.*

Lemma 2.5. *$S_{n,n+1}$ is the unique non-bipartite graph in $\mathcal{G}_{n,n+1}$ with minimal energy for $5 \leq n \leq 7$. Furthermore, $S_{n,n+1}$ has second-minimal energy in $\mathcal{G}_{n,n+1}$ for $n = 5$ or 7, and $S_{6,7}$ has third-minimal energy in $\mathcal{G}_{6,7}$.*

Proof. By Table 1 of [3], there are five $(5,6)$-graphs. By simple calculation, we can prove the theorem for $n = 5$. By Table 1 of [4], there are 19 $(6,7)$-graphs. By direct computation, we can prove the theorem for $n = 6$. By the results (Lemmas 5-9) in [36], we can obtain that $S_{7,8}$ has second-minimal energy in $\mathcal{G}^1_{7,8}$. On the other hand, from the proof of Theorem 1.6, $E(G) > E(S_{7,8})$ for $G \in \mathcal{G}^2_{7,8}$. Therefore $S_{7,8}$ has second-minimal energy in $\mathcal{G}_{7,8}$, and so the theorem is true for $n = 7$. ■

Proof of Theorem 1.7: Since K_4 is the unique graph in $\mathcal{G}_{4,6}$, the theorem holds for $n = 4$. By Table 1 of [3], there are four $(5,7)$-graphs. By simple calculation, we can prove the theorem for $n = 5$. By Table 1 of [4], there are 22 $(6,8)$-graphs. By direct computation, we can prove the theorem for $n = 6$. Now suppose that $n \geq 7$. By Theorem 1.3, it suffices to prove that $E(G) > E(S_{n,n+2})$ when $n \geq 10$, and $E(G) > E(B_{n,n+2})$ when $7 \leq n \leq 9$ for $G \in \mathcal{G}^2_{n,n+2}$.

Suppose that $G \in \mathcal{G}^2_{n,n+2}$ and C_p, C_q are two disjoint odd cycles with $p + q \equiv 2 \pmod 4$. Then there are at most two edge disjoint paths in G connecting C_p and C_q.

Case 1. There exists exactly an edge disjoint path P connecting C_p and C_q. Then there exists an edge e of P such that $G - e = G_1 \cup G_2$, where G_1 is an non-bipartite bicyclic graph with $n_1 \geq 4$ vertices and G_2 is an non-bipartite unicyclic graph with $n_2 \geq 3$ vertices. By Lemmas 2.1, 2.2, 2.3, 2.5 and Theorem 1.6, we have

$$\begin{aligned} E(G) &\geq E(G_1 \cup G_2) \geq E(S_{n_1,n_1+1} \cup S_{n_2,n_2}) > E(S_{n_1,n_1} \cup S_{n_2,n_2}) \\ &\geq E(S_{n-3,n-3} \cup C_3) > E(S_{n,n+2}). \end{aligned}$$

In particular, $E(G) > E(S_{n,n+2}) > E(B_{n,n+2})$ for $7 \leq n \leq 9$.

Case 2. There exist exactly two edge disjoint paths P^1 and P^2 connecting C_p and C_q. Then there exist two edges e_1 and e_2 such that e_i is an edge of P^i for $i = 1, 2$, and $G - \{e_1, e_2\} = G_3 \cup G_4$, where G_3 and G_4 are non-bipartite unicyclic graphs. Let $|V(G_3)| = n_1$ and $|V(G_4)| = n_2$. Then by Lemmas 2.1, 2.2 and 2.3, we have

$$E(G) \geq E(G_3 \cup G_4) \geq E(S_{n_1,n_1} \cup S_{n_2,n_2}) \geq E(S_{n-3,n-3} \cup C_3) > E(S_{n,n+2}).$$

In particular, $E(G) > E(S_{n,n+2}) > E(B_{n,n+2})$ for $7 \leq n \leq 9$. The proof is thus complete. ∎

Remark 2.6. *The proof of Theorem 1.7 (for large n) is similar to that of Theorem 1.5 except that in [35], the authors did not point out that G_1 and G_2 are non-bipartite graphs.*

3 The graph with minimal energy in $\mathcal{G}_{n,n+3}$

Li and Li [21] discussed the graph with minimal energy in $\mathcal{G}^1_{n,n+3}$, and we first restate their results.

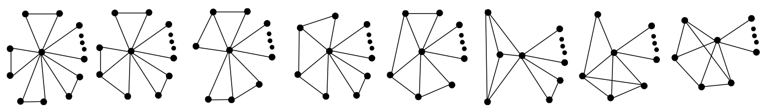

Figure 1. Graphs $G_1, G_2, G_3, G_4, G_5, G_6, G_7$ and G_8.

Follow [21], let $G_1, G_2, \ldots, G_8$ be eight special graphs in $\mathcal{G}_{n,n+3}$ as shown in Figure 1. Let $\mathscr{I}_n = \{S_{n,n+3}, B_{n,n+3}, G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8\}$.

Lemma 3.1. [21] *If $G \in \mathcal{G}^1_{n,n+3}$ and $G \notin \mathscr{I}_n$, then $E(G) > E(B_{n,n+3})$ for $n \geq 9$.*

In fact, Lemma 3.1 is also true for $n = 8$.

Lemma 3.2. *If $G \in \mathcal{G}^1_{8,11}$ and $G \notin \mathscr{I}_8 \setminus \{G_1\}$, then $E(G) > E(B_{8,11})$.*

Proof. By the results (see the proofs of Lemma 2.2 and Proposition 2.3) of [21], all we need is to show that $b_4(G) - b_4(B_{8,11}) > 0$ when G contains exactly i ($i = 10, 12, 13, 14, 15$) cycles (see Case 7 of Lemma 2.2). From [21], we have

$$b_4(G) - b_4(B_{8,11}) \geq \frac{1}{2}n^2 + \frac{3}{2}n - 12 - 2s - (5n - 35),$$

where s is the number of quadrangles in G. It is easy to check that in this case, G has at most 13 quadrangles. Therefore

$$b_4(G) - b_4(B_{8,11}) \geq \frac{1}{2}n^2 + \frac{3}{2}n - 12 - 26 - (5n - 35) = \frac{1}{2}n(n-7) - 3 = 1 > 0.$$

The proof is thus complete. ∎

Lemma 3.3. [21] *For each $G_j \in \mathscr{I}_n$ $(j = 1, \ldots, 8)$, $E(S_{n,n+3}) < E(G_j)$ for $n \geq 9$ and $E(B_{n,n+3}) < E(G_j)$ for $9 \leq n \leq 17$.*

By the proof of Lemma 2.4 of [21], we can get the following result for $n = 8$.

Lemma 3.4. *For each $G_j \in \mathscr{I}_n \setminus \{G_1\}$ $(j = 2, \ldots, 8)$, $E(B_{n,n+3}) < E(G_j)$ for $n = 8$.*

Li and Li [21] claimed that the graph with minimal energy in $\mathcal{G}^1_{n,n+3}$ is $B_{n,n+3}$ for $9 \leq n \leq 17$, and $S_{n,n+3}$ for $n \geq 18$, respectively. However, from Lemma 1.9, we can obtain the following result.

Corollary 3.5. *$E(S_{n,n+3}) < E(B_{n,n+3})$ for $n \geq 12$, and $E(B_{n,n+3}) < E(S_{n,n+3})$ for $7 \leq n \leq 11$.*

The authors of [21] failed to get the above result in that (in the proof of Proposition 2.5 of [21]) they used the wrong formula $b_4(S_{n,n+3}) = 4n - 18$ instead of the correct one $b_4(S_{n,n+3}) = 4n - 24$. In fact, by Lemmas 3.1, 3.2, 3.3,3.4 and Corollary 3.5, we can characterize the graph with minimal energy in $\mathcal{G}^1_{n,n+3}$ as follows.

Theorem 3.6. *The graph with minimal energy in $\mathcal{G}^1_{n,n+3}$ is $B_{n,n+3}$ for $8 \leq n \leq 11$, and $S_{n,n+3}$ for $n \geq 12$, respectively.*

To prove Theorem 1.8, we need the following two lemmas.

Lemma 3.7. *(1) $E(K_4) > E(S_{4,4})$, and $E(B_{n,n+2}) > E(S_{n,n})$ for $7 \leq n \leq 9$.*
(2) $E(S_{n,n+2}) > E(S_{n,n})$ for $n \geq 5$.

Proof. (1) It is easy to obtain that $E(K_4) = 6$, $E(S_{4,4}) \doteq 4.96239$, $E(B_{7,9}) \doteq 7.21110$, $E(S_{7,7}) \doteq 6.64681$, $E(B_{8,10}) \doteq 7.91375$, $E(S_{8,8}) \doteq 7.07326$, $E(B_{9,11}) \doteq 8.46834$ and $E(S_{9,9}) \doteq 7.46410$. Hence the result (1) follows.

(2) Since $6 = E(S_{5,7}) > E(S_{5,5}) \doteq 5.62721$, we now suppose $n \geq 6$. By direct computation, we have that $\phi(S_{n,n+2}, x) = x^n - (n+2)x^{n-2} - 6x^{n-3} + (3n-15)x^{n-4}$ and $\phi(S_{n,n}, x) = x^n - nx^{n-2} - 2x^{n-3} + (n-3)x^{n-4}$. By Eq. (1), we obtain that

$$\begin{aligned} E(S_{n,n+2}) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{x^2}\ln((1+(n+2)x^2+(3n-15)x^4)^2+(6x^3)^2)dx \\ &> \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{x^2}\ln((1+nx^2+(n-3)x^4)^2+(2x^3)^2)dx = E(S_{n,n}). \end{aligned}$$

■

Lemma 3.8. *$E(S_{n-3,n-3} \cup C_3) > E(S_{n,n+3})$ for $n \geq 6$.*

Proof. For $6 \le n \le 14$, the result follows by direct computation. Suppose that $n \ge 15$. By direct calculation, we have that $\phi(S_{n,n+3}, x) = x^n - (n+3)x^{n-2} - 8x^{n-3} + (4n-24)x^{n-4}$. Let $f(x) = x^4 - (n+3)x^2 - 8x + 4n - 24$. Then we have that $f(-\sqrt{n-1}) > 0$, $f(-2) < 0$, $f(0) > 0$, $f(2) < 0$ and $f(\sqrt{n+3}) > 0$. Hence

$$E(S_{n,n+3}) < 4 + \sqrt{n-1} + \sqrt{n+3}.$$

On the other hand, we have $E(S_{n-3,n-3} \cup C_3) > 4 + \sqrt{2} + 2\sqrt{n-4}$ [35], and so $E(S_{n-3,n-3} \cup C_3) > E(S_{n,n+3})$. ■

Proof of Theorem 1.8: By Table 1 of [3], there are two $(5, 8)$-graphs. By simple calculation, we can prove the theorem for $n = 5$. By Table 1 of [4], there are 20 $(6, 9)$-graphs. By direct computation, we can prove the theorem for $n = 6$. By [2], there are 132 $(7, 10)$-graphs. By direct computing, we can prove the theorem for $n = 7$. Now suppose that $n \ge 8$. By Theorem 3.6 and Corollary 3.5, it suffices to prove that $E(G) > E(S_{n,n+3})$ for $G \in \mathcal{G}^2_{n,n+3}$.

Suppose that $G \in \mathcal{G}^2_{n,n+3}$ and C_p, C_q are two disjoint odd cycles with $p + q \equiv 2 \pmod 4$. Then there are at most three edge disjoint paths in G connecting C_p and C_q.

Case 1. There exists exactly an edge disjoint path P^1 connecting C_p and C_q. Then there exists an edge e_1 of P^1 such that $G - e_1 = G_1 \cup G_2$, where either both G_1 and G_2 are non-bipartite bicyclic graphs, or G_1 is an non-bipartite tricyclic graph and G_2 is an non-bipartite unicyclic graph. Let $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$.

Subcase 1.1. Both G_1 and G_2 are non-bipartite bicyclic graphs. Then by Lemmas 2.1, 2.2, 2.5, 3.8 and Theorem 1.6, we have

$$\begin{aligned} E(G) &\ge E(G_1 \cup G_2) \ge E(S_{n_1,n_1+1} \cup S_{n_2,n_2+1}) > E(S_{n_1,n_1} \cup S_{n_2,n_2}) \\ &\ge E(S_{n-3,n-3} \cup C_3) > E(S_{n,n+3}). \end{aligned}$$

Subcase 1.2. G_1 is an non-bipartite tricyclic graph and G_2 is an non-bipartite unicyclic graph. It follows from Theorem 1.7 and Lemma 3.7 that $E(G_1) > E(S_{n_1,n_1})$. Therefore by Lemmas 2.1, 2.2, 2.3 and 3.8, we have

$$E(G) \ge E(G_1 \cup G_2) > E(S_{n_1,n_1} \cup S_{n_2,n_2}) \ge E(S_{n-3,n-3} \cup C_3) > E(S_{n,n+3}).$$

Case 2. There exist exactly two edge disjoint paths P^2 and P^3 connecting C_p and C_q. Then there exist two edges e_2 and e_3 such that e_i is an edge of P^i for $i = 2, 3$, and $G - \{e_2, e_3\} = G_3 \cup G_4$, where G_3 is an non-bipartite bicyclic graph with n_1 vertices and

G_4 is an non-bipartite unicyclic graph with n_2 vertices. By Lemmas 2.1, 2.2, 2.3, 2.5, 3.8 and Theorem 1.6, we have

$$E(G) \geq E(G_3 \cup G_4) \geq E(S_{n_1,n_1+1} \cup S_{n_2,n_2}) > E(S_{n_1,n_1} \cup S_{n_2,n_2})$$
$$\geq E(S_{n-3,n-3} \cup C_3) > E(S_{n,n+3}).$$

Case 3. There exist exactly three edge disjoint paths P^4, P^5 and P^6 connecting C_p and C_q. Then there exist three edges e_4, e_5 and e_6 such that e_i is an edge of P^i for $i = 4,5,6$, and $G - \{e_4, e_5, e_6\} = G_5 \cup G_6$, where G_5 and G_6 are non-bipartite unicyclic graphs. Let $|V(G_5)| = n_1$ and $|V(G_6)| = n_2$. Then by Lemmas 2.1, 2.2, 2.3 and 3.8, we have

$$E(G) \geq E(G_5 \cup G_6) \geq E(S_{n_1,n_1} \cup S_{n_2,n_2}) \geq E(S_{n-3,n-3} \cup C_3) > E(S_{n,n+3}).$$

The proof is thus complete. ■

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