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More on L-Borderenergetic Graphs*

Bo Deng†, **Xueliang Li**

Center for Combinatorics and LPMC,
Nankai University, Tianjin 300071, China
e-mail: `dengbo450@163.com` , `lxl@nankai.edu.cn`

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Abstract

The energy $\mathcal{E}(G)$ of a graph G is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. If a graph G of order n has the same energy as the complete graph K_n does, i.e., if $\mathcal{E}(G) = 2(n-1)$, then G is said to be borderenergetic. Similarly, for the Laplacian energy $\mathcal{LE}(G)$ of a graph G, F. Tura proposed the concept of L-borderenergetic graphs recently. That is, a graph G of order n is L-borderenergetic if it has the same Laplacian energy as the complete graph K_n does. In this paper, we first show that a kind of threshold graphs are L-borderenergetic. Then we use tensor product to construct regular L-borderenergetic graphs. At last, all the connected non-complete and pairwise non-isomorphic L-borderenergetic graphs of small order n are depicted for n with $4 \leq n \leq 9$. All these results are different from those in Tura's paper.

1 Introduction

All graphs considered in this paper are simple and undirected. Let G be a graph with its edge set $E(G)$ and vertex set $V(G)$, whose order is denoted by $|V(G)|$. The complete graph and the cycle of order n are denoted by K_n and C_n, respectively. The union of two vertex-disjoint graphs G_1 and G_2 is denoted by $G_1 \cup G_2$.

Let $A(G)$ be an adjacency matrix of G. The spectrum of G is the non-increasing sequence $Sp(G) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, which is composed of the eigenvalues of the adjacency matrix $A(G)$. If $D(G)$ is the diagonal matrix of the vertex degrees of G, $L(G) = D(G) - A(G)$

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†corresponding author.

is defined to be the Laplacian matrix of G. The spectrum of $L(G)$ is the sequence of its eigenvalues displayed in non-increasing order, denoted by $LSp(G) = \{\mu_1, \mu_2, \cdots, \mu_n\}$. It is well known that $L(G)$ is a positive semidefinite and singular matrix. So, for $i = 1, 2, \cdots, n-1$, $\mu_i \geq 0$ and $\mu_n = 0$. Besides, when each Laplacian eigenvalue is an integer, G is said to be a Laplacian integral graph. For details on spectral graph theory, see [2].

The energy of a graph G, denoted by $\mathcal{E}(G)$, is defined as [6, 7]

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| .$$

For additional information on graph energy and its applications in chemistry, we refer to [7–9, 15].

Recently, Gong et al. [5] proposed the concept of *borderenergetic* graphs, namely graphs of order n satisfying $\mathcal{E}(G) = 2(n-1)$. Some related results on borderenergetic graphs can be seen in [4, 13, 19–21]. In fact, analogous topics on energy of graphs have been researched [1, 10, 11, 14, 16–18].

For the Laplacian energy of a graph G [12], similarly, F. Tura [22] proposed the concept of L-borderenergetic graphs. That is, a graph G of order n is L-borderenergetic if $\mathcal{LE}(G) = \mathcal{LE}(K_n)$, where $\mathcal{LE}(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}|$ and μ_i and $\overline{d}$ are the Laplacian eigenvalue and the average degree of G, respectively. Note that $\mathcal{LE}(K_n) = 2(n-1)$. Several classes of L-borderenergetic graphs [22] are obtained including result that for each integer $r \geq 1$, there are $2r+1$ graphs, of order $n = 4r+4$, which are pairwise L-noncospectral and L-borderenergetic graphs.

It is of interest to find more L-borderenergetic graphs, especially, connected and to establish their structural differences. Of course, we can use some graph operations to construct them, such as tensor product of graphs. However, the problem of finding all L-borderenergetic graphs on n vertices becomes rather difficult when $n > 7$. Indeed, using a computer, it took several seconds for the case $n \leq 7$. But in other cases, it took dramatically long time, about 1 day for $n = 8$, and about 3 days for $n = 9$. Our final results are shown in Table 1.

n	4	5	6	7	8	9
number	2	1	11	5	33	23

Table 1. The numbers of connected non-complete and pairwise non-isomorphic L-borderenergetic graphs on n vertices for $4 \leq n \leq 9$.

In this paper, we first show that a kind of threshold graphs are L-borderenergetic. Then we use tensor product to construct regular L-borderenergetic graphs. At last, all the connected non-complete and pairwise non-isomorphic L-borderenergetic graphs of small order n are depicted for n with $4 \leq n \leq 9$.

2 Threshold graphs

Including several classes of L-borderenergetic graphs have been constructed by Tura in [22], here we will find a class of threshold graphs which are also L-borderenergetic.

At first, let's recall the definitions of threshold graphs and Ferrers-Sylvester diagrams. A *threshold graph* is obtained through an iterative process which starts with an isolated vertex, and at each step, either a new isolated vertex is added, or a vertex adjacent to all previous vertices (dominating vertex) is added. A *Ferrers-Sylvester diagram* (see Figure 2) is a grid representing a degree sequence $(d) = (d_1, d_2, \cdots, d_n)$ in which the ith row of the grid contains d_i boxes. The *conjugate of a degree sequence* (d) is the sequence $(d^*) = (d_1^*, d_2^*, \cdots, d_k^*)$ where $d_i^* = |\{d_j \geq i\}|$. Visually speaking, the value for d_i^* is the number of boxes in the ith column of the Ferrers-Sylvester diagram.

Let S_n^1 be the graph with m edges obtained from an n-order star S_n by adding an edge. Obviously, S_n^1 is a unicyclic and threshold graph (see Figure 1).

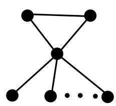


Figure 1. The graph S_n^1.

Lemma 1. *[3] Let G be a connected graph of order n with m edges. In addition, let d_i^* be the ith conjugate degree of G. Then*

$$\mathcal{LE}(G) \leq \sum_{i=1}^{n} |d_i^* - 2m/n|$$

with equality holding if and only if G is a threshold graph.

Theorem 2. *The graph S_n^1 is L-borderenergetic.*

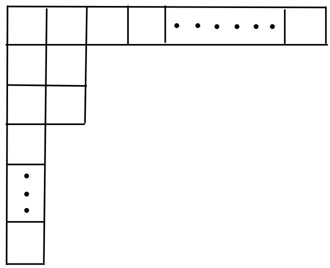


Figure 2. The Ferrers-Sylvester diagram of S_n^1.

Proof. Since S_n^1 is a threshold graph, by the condition of the equality holding in Lemma 1, we have

$$\mathcal{LE}(S_n^1) = \sum_{i=1}^{n} |d_i^* - 2m/n| \tag{1}$$

As S_n^1 is a unicyclic graph, we get $m = n$. From the Ferrers-Sylvester diagram of S_n^1 (see Figure 2), it can be seen that

$$d_1^* = n,\ d_2^* = 3,\ d_3^* = d_4^* = \cdots = d_{n-1}^* = 1,\ d_n^* = 0$$

So by (1), we obtain

$$\mathcal{LE}(S_n^1) = (n-2) + 1 + (n-3) + 2 = 2(n-1).$$

$\square$

Note that from [22] one can only get that for some even integers, there are L-borderenergetic graphs. Since the order n of the graph S_n^1 can be any integer (even or odd), we immediately get the following result, which is stronger than Tura's result.

Theorem 3. *For any integer $n \geq 4$, there is an L-borderenergetic graph.*

3 Regular graphs

In this section, we use tensor product to construct some regular L-borderenergetic graphs.

The tensor product of two graphs G_1 and G_2, denoted by $G_1 \otimes G_2$, has vertex set $V(G_1) \times V(G_2)$, in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if both the edges $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. Then, it is easy to see that the order of $G_1 \otimes G_2$ is $|V(G_1)||V(G_2)|$. A result in [5] on the energy of tensor product of two graphs is given below.

Lemma 4. *[5] If G_1 and G_2 are any two graphs, then $\mathcal{E}(G_1 \otimes G_2) = \mathcal{E}(G_1)\mathcal{E}(G_2)$.*

For regular graphs, we have

Theorem 5. *If G is a d-regular graph, then $\mathcal{LE}(G) = \mathcal{E}(G)$.*

Proof. Obviously, the average degree of G is d and the Laplacian eigenvalue of G possessing the form of $d - \lambda_i$, where $i = 1, 2, \cdots, n$. Then, we have

$$\mathcal{LE}(G) = \sum_{i=1}^{n} |\mu_i - d| = \sum_{i=1}^{n} |d - \lambda_i - d| = \sum_{i=1}^{n} |\lambda_i| = \mathcal{E}(G)$$

□

Theorem 6. *Let G be an L-borderenergetic graph. Suppose that G is obtained from the tensor product of two L-integral graphs G_1 and G_2, where G_1 and G_2 are r_1-regular and r_2-regular, respectively. Then both $|V(G_1)|$ and $|V(G_2)|$ are odd.*

Proof. Since G_1 and G_2 are all regular, by the definition of tensor product, G is also regular. Then from Theorem 6 we get $\mathcal{LE}(G) = \mathcal{E}(G)$, $\mathcal{LE}(G_1) = \mathcal{E}(G_1)$ and $\mathcal{LE}(G_2) = \mathcal{E}(G_2)$. By Lemma 4, we see that

$$\mathcal{LE}(G) = \mathcal{E}(G) = \mathcal{E}(G_1 \otimes G_2) = \mathcal{E}(G_1)\mathcal{E}(G_2) = \mathcal{LE}(G_1)\mathcal{LE}(G_2) \tag{2}$$

Since the energy of a graph is never an odd integer, there exist two integers t_1 and t_2 satisfying $\mathcal{E}(G_i) = 2(|V(G_i)| - t_i)$, and then we have $\mathcal{LE}(G_i) = 2(|V(G_i)| - t_i)$ for $i = 1, 2$. Thus, by (2) we can see that

$$2(|V(G_1)||V(G_2)| - 1) = 4(|V(G_1)| - t_1)(|V(G_2)| - t_2) \tag{3}$$

By (3), we obtain

$$|V(G_1)||V(G_2)| = 2t_1|V(G_2)| + 2t_2|V(G_1)| - 2t_1t_2 - 1 \tag{4}$$

From above equation, we note that its right hand is odd and its left hand is the product of $|V(G_1)|$ and $|V(G_2)|$. So, we know that both $|V(G_1)|$ and $|V(G_2)|$ are odd. □

Using Theorem 6, we can construct some regular L-borderenergetic graphs with small orders. Assume that $G_1 = K_{|V(G_1)|}$ and $|V(G_1)| = |V(G_2)| > 1$. Then, $t_1 = 1$ and $t_2 = (|V(G_1)| - 1)/2$ by (4). From (2) and (3), we obtain $\mathcal{LE}(G_2) = |V(G_1)| + 1$.

For $|V(G_1)| = 3$ and $|V(G_1)| = 7$, we can verify that graphs $K_3 \otimes K_3$ and $K_7 \otimes \{K_3 \cup C_4\}$ are both L-borderenergetic.

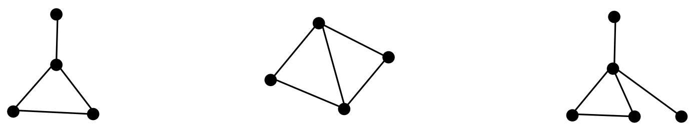
4 L-borderenergetic graphs of small orders

In this section, we will depict all the connected non-complete and pairwise non-isomorphic L-borderenergetic graphs of small order n with $4 \leq n \leq 9$, and give their L-spectra and average degrees.

4.1. L-borderenergetic graphs of orders $n = 4$ and 5

There are exactly two such L-borderenergetic graphs for $n = 4$ and only one for $n = 5$. These graphs are shown in Figure 3. The corresponding L-spectra are given as follows.

$$LSp(G_4^1) = \{4, 3, 1, 0\};$$
$$LSp(G_4^2) = \{4, 4, 2, 0\};$$
$$LSp(G_5^1) = \{5, 3, 1, 1, 0\};$$



G_4^1 G_4^2 G_5^1

Figure 3. The L-borderenergetic graphs of $n = 4$ and 5.

4.2. L-borderenergetic graphs of order $n = 6$

There are exactly 11 such L-borderenergetic graphs of order $n = 6$. These graphs are presented in Figure 4. The L-spectra of them are shown below.

$$LSp(G_6^1) = \{6, 4, 4, 2, 2, 0\};$$
$$LSp(G_6^2) = \{6, 5, 4, 3, 2, 0\};$$
$$LSp(G_6^3) = \{6, 6, 6, 4, 4, 0\};$$
$$LSp(G_6^4) = \{6, 5, 5, 3, 3, 0\};$$
$$LSp(G_6^5) = \{6, 6, 5, 4, 3, 0\};$$
$$LSp(G_6^6) = \{6, 6, 4, 3, 3, 0\};$$
$$LSp(G_6^7) = \{6, 3, 1, 1, 1, 0\};$$

$$LSp(G_6^8) = \{6, 4, 3, 2, 1, 0\};$$

$$LSp(G_6^9) = \{6, 4, 4, 3, 1, 0\};$$

$$LSp(G_6^{10}) = \{6, 3, 3, 1, 1, 0\};$$

$$LSp(G_6^{11}) = \{6, 5, 3, 3, 1, 0\};$$

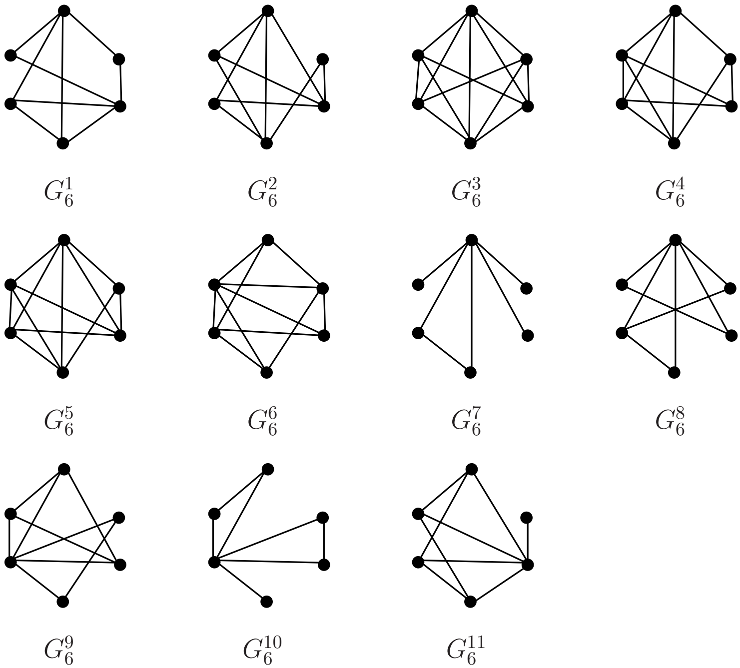


Figure 4. The L-borderenergetic graphs of order $n = 6$.

4.3. L-borderenergetic graphs of order $n = 7$

There are exactly 5 such L-borderenergetic graphs of order $n = 7$. These graphs are depicted in Figure 5. The following is their L-spectra.

$$LSp(G_7^1) = \{7, 3, 1, 1, 1, 1, 0\};$$

$$LSp(G_7^2) = \{7, 5, 5, 5, 4, 2, 0\};$$

$$LSp(G_7^3) = \{7, 6, 5, 4, 4, 2, 0\};$$

$$LSp(G_7^4) = \{7, 6, 5, 4, 4, 2, 0\};$$

$$LSp(G_7^5) = \{7, 6, 5, 4, 3, 3, 0\};$$

4.4. *L*-borderenergetic graphs of order $n = 8$

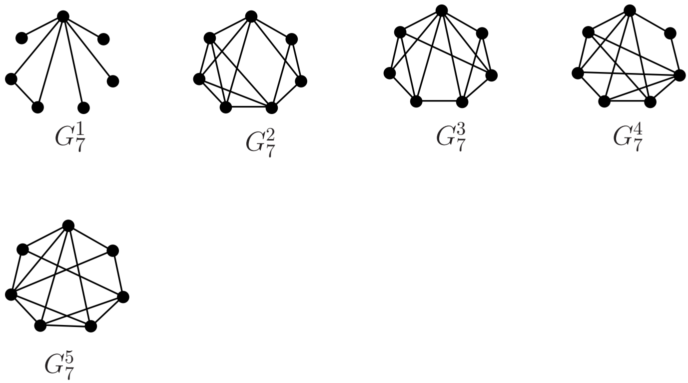


Figure 5. The L-borderenergetic graphs of order $n = 7$.

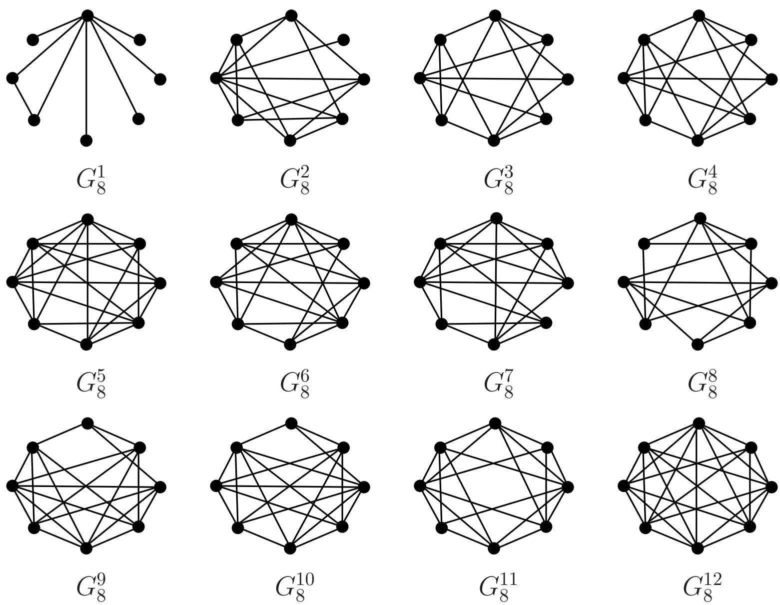
There are exactly 33 such L-borderenergetic graphs of order $n = 8$. These graphs are shown in Figure 6. The corresponding L-spectra are given as follows.

$LSp(G_8^1) = \{8, 3, 1, 1, 1, 1, 1, 0\}$; $LSp(G_8^2) = \{8, 7, 4, 4, 4, 4, 1, 0\}$;
$LSp(G_8^3) = \{8, 5, 5, 5, 3, 3, 3, 0\}$; $LSp(G_8^4) = \{8, 6, 5, 5, 4, 3, 3, 0\}$;
$LSp(G_8^5) = \{8, 8, 7, 6, 5, 5, 5, 0\}$; $LSp(G_8^6) = \{8, 7, 7, 5, 5, 4, 4, 0\}$;
$LSp(G_8^7) = \{8, 7, 6, 6, 5, 5, 3, 0\}$; $LSp(G_8^8) = \{7, 6, 5, 5, 4, 3, 2, 0\}$;
$LSp(G_8^9) = \{8, 7, 7, 5, 5, 5, 3, 0\}$; $LSp(G_8^{10}) = \{8, 7, 7, 5, 5, 5, 3, 0\}$;
$LSp(G_8^{11}) = \{8, 7, 6, 6, 5, 5, 3, 0\}$; $LSp(G_8^{12}) = \{8, 8, 8, 7, 6, 6, 5, 0\}$;
$LSp(G_8^{13}) = \{8, 6, 5, 5, 5, 5, 2, 0\}$; $LSp(G_8^{14}) = \{8, 7, 5, 5, 4, 4, 3, 0\}$;
$LSp(G_8^{15}) = \{8, 7, 6, 5, 4, 4, 4, 0\}$; $LSp(G_8^{16}) = \{8, 8, 6, 5, 5, 4, 4, 0\}$;
$LSp(G_8^{17}) = \{8, 6, 6, 6, 4, 4, 4, 0\}$; $LSp(G_8^{18}) = \{8, 6, 6, 6, 6, 4, 4, 0\}$;
$LSp(G_8^{19}) = \{8, 8, 8, 8, 6, 6, 6, 0\}$; $LSp(G_8^{20}) = \{8, 8, 6, 6, 6, 6, 4, 0\}$;
$LSp(G_8^{21}) = \{8, 8, 5, 5, 4, 4, 4, 0\}$; $LSp(G_8^{22}) = \{6 + \sqrt{2}, 6 - \sqrt{2}, 7, 6, 4, 4, 3, 0\}$;
$LSp(G_8^{23}) = \{8, 7, 6, 6, 5, 4, 4, 0\}$; $LSp(G_8^{24}) = \{8, 8, 6, 6, 5, 5, 4, 0\}$;
$LSp(G_8^{25}) = \{8, 4, 3, 3, 2, 1, 1, 0\}$; $LSp(G_8^{26}) = \{5 + \sqrt{3}, 5 - \sqrt{3}, 5, 4, 2, 2, 1, 0\}$;
$LSp(G_8^{27}) = \{8, 4, 4, 3, 3, 1, 1, 0\}$; $LSp(G_8^{28}) = \{8, 5, 3, 3, 3, 1, 1, 0\}$;
$LSp(G_8^{30}) = \{8, 7, 7, 6, 5, 5, 4, 0\}$; $LSp(G_8^{29}) = \{6 + \sqrt{2}, 6 - \sqrt{2}, 7, 5, 4, 4, 2, 0\}$;
$LSp(G_8^{31}) = \{8, 5, 4, 4, 3, 3, 1, 0\}$; $LSp(G_8^{32}) = \{8, 5, 5, 5, 4, 4, 1, 0\}$;
$LSp(G_8^{33}) = \{8, 3, 3, 3, 1, 1, 1, 0\}$;

4.5. L-borderenergetic graphs of order $n = 9$

There are exactly 23 such L-borderenergetic graphs of order $n = 9$. These graphs are presented in Figure 7. The L-spectra of them are shown below.

$$LSp(G_9^1) = \{9,3,1,1,1,1,1,1,0\};\ LSp(G_9^2) = \{6,6,6,5,5,3,3,2,0\};$$
$$LSp(G_9^3) = \{7,6,6,5,4,4,3,1,0\};\ LSp(G_9^4) = \{9,6,6,5,5,5,3,3,0\};$$
$$LSp(G_9^5) = \{7,6,6,5,4,3,3,2,0\};\ LSp(G_9^6) = \{7,6,6,5,4,3,3,2,0\};$$
$$LSp(G_9^7) = \{9,7,6,6,6,6,4,4,0\};\ LSp(G_9^8) = \{9,8,7,5,5,5,5,4,0\};$$
$$LSp(G_9^9) = \{9,9,8,6,6,6,6,4,0\};\ LSp(G_9^{10}) = \{9,8,7,5,5,5,5,4,0\};$$
$$LSp(G_9^{11}) = \{9,9,7,7,6,6,5,5,0\};\ LSp(G_9^{12}) = \{9,9,7,7,6,6,6,4,0\};$$
$$LSp(G_9^{13}) = \{9,8,8,7,6,6,6,4,0\};\ LSp(G_9^{14}) = \{9,9,9,7,7,7,6,6,0\};$$
$$LSp(G_9^{16}) = \{7,6,6,5,4,3,3,2,0\};\ LSp(G_9^{15}) = \{6+\sqrt{2},6-\sqrt{2},6,6,4,3,3,2,0\};$$
$$LSp(G_9^{17}) = \{8,6,5,5,4,3,3,2,0\};\ LSp(G_9^{18}) = \{6,6,6,6,3,3,3,3,0\};$$
$$LSp(G_9^{19}) = \{8,6,6,6,5,5,3,3,0\};\ LSp(G_9^{20}) = \{7,6,5,5,5,3,3,2,0\};$$
$$LSp(G_9^{21}) = \{7,6,5,5,5,4,3,1,0\};\ LSp(G_9^{22}) = \{9,6,5,4,4,4,3,1,0\};$$
$$LSp(G_9^{23}) = \{9,7,4,4,4,4,3,1,0\};$$



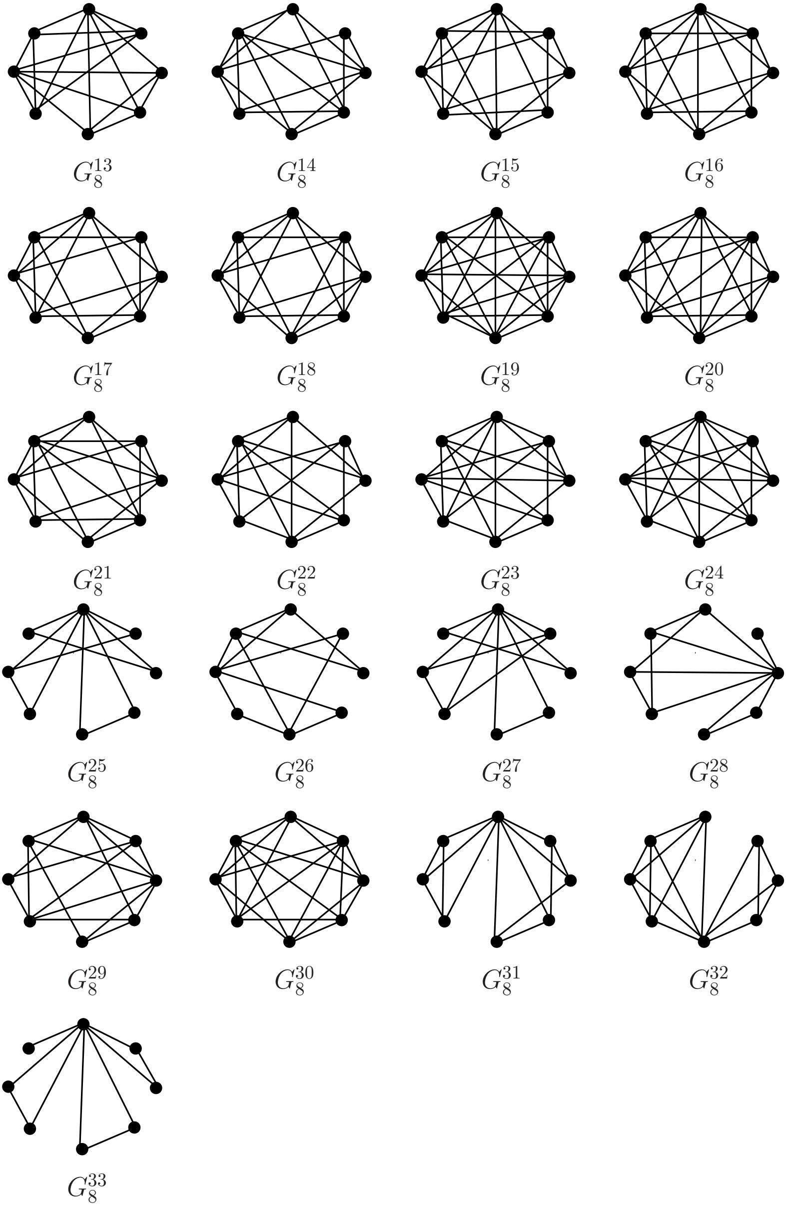


Figure 6. The L-borderenergetic graphs of order $n = 8$.

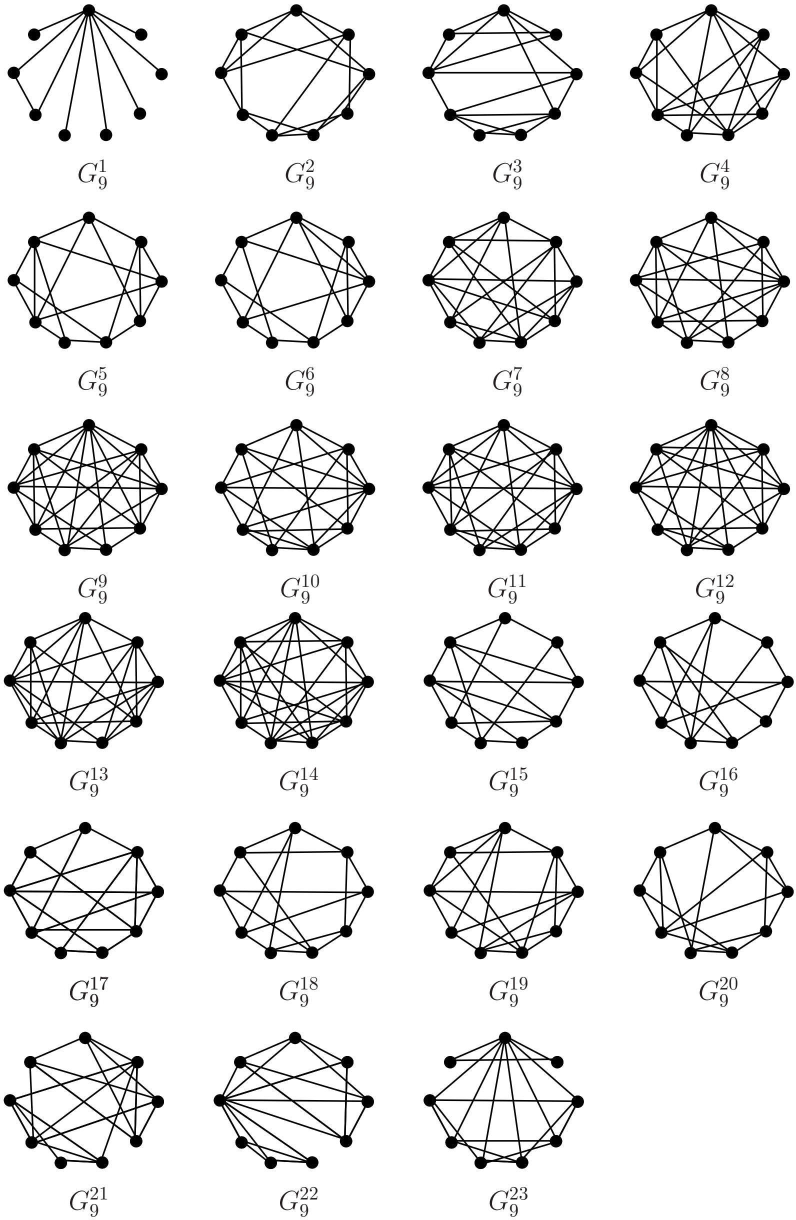


Figure 7. The L-borderenergetic graphs of order $n = 9$.

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