MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **76** (2016) 669-692

ISSN 0340 - 6253

On Ordering of Complements of Graphs with Respect to Matching Numbers. II*

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(Received January 21, 2016)

Abstract

An extensively studied quasi-order, defined in terms of matching numbers of graphs, is investigated further in this paper and applied it to graph complements. Some transformations on the complements of graphs are presented. As an application, we determine the maximum and minimum graphs with respect to the quasi-order in the set of the complements of unicyclic graphs with given order and in the set of unicyclic graphs with given order and girth, respectively.

1 Introduction

All graphs considered in this paper are undirected and simple (i.e., no multiple edges and loops). Let $G = (V(G), E(G))$ be such a graph, with vertex set $V(G)$ and edge set $E(G)$. A matching of G is a set of pairwise nonadjacent edges in $E(G)$. A k-matching is a matching consisting of k edges. By $m(G, k)$ we denote the number of k-matchings of G. It is both consistent and convenient to define $m(G, 0) = 1$ as well as $m(G, k) = 0$ for $k < 0$ and $k > n/2$, where $n = |V(G)|$ is the order of G.

*Supported by the National Natural Science Foundation of China (No. 11561032 and No. 11201198), the Natural Science Foundation of Jiangxi Province (No. 20142BAB211013), the Sponsored Program for Cultivating Youths of Outstanding Ability in Jiangxi Normal University.

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Many results pertaining to the matching numbers could be expressed by means of the *matching polynomial* [7], which is usually defined as

$$\alpha(G,\lambda) = \sum_{k\geq 0} (-1)^k \, m(G,k) \, \lambda^{n-2k} \, .$$

Details of the theory of matching polynomial can be found in the monographs [5,14].

There is a natural ordering with respect to the matching numbers, introduced in the 1970s by Gutman [10,11] and eventually elaborated in cooperation with Zhang [19,20,33–35]. If for two graphs G_1 and G_2 the relations $m(G_1,k) \geq m(G_2,k)$ are satisfied for all k, then we write $G_1 \succeq G_2$ (or $G_2 \preceq G_1$). If $G_1 \succeq G_2$ and $m(G_1,k) > m(G_2,k)$ for some k, then we write $G_1 \succ G_2$ (or $G_2 \prec G_1$). If both $G_1 \succeq G_2$ and $G_2 \succeq G_1$ hold, then we write $G_1 \sim G_2$.

As a binary relation on graphs, $\preceq$ is reflexive and transitive, but not anti-symmetric because there are non-isomorphic graphs G_1 and G_2 such that $G_1 \sim G_2$. Hence $\preceq$ is a quasi-order. Since there exist graphs for which neither $G_1 \succeq G_2$ nor $G_2 \succeq G_1$ holds, which means that G_1 and G_2 are *incomparable* w.r.t. the relations $\succeq$, the ordering implied by this relation is not complete.

The quasi-order was extensively used since introduced, especially in connection with the energy of trees [17,25], for which the relation

$$E(T) = \frac{2}{\pi} \int_0^\infty \frac{1}{x^2} \ln \left[\sum_{k\geq 0} m(T,k) \, x^{2k} \right] dx \, . \tag{1}$$

was shown to hold [10]. The integral on the right hand side of Eq. (1) is increasing in all the coefficients $m(G,k)$. From Eq. (1), it immediately follows that if $T_1 \succeq T_2$ holds for two trees T_1 and T_2, then $E(T_1) \geq E(T_2)$.

Another straightforward application of the quasi-order is for comparing Hosoya indices. The Hosoya index of a graph G is defined as the total number of matchings in G, i.e., as $Z(G) = \sum_k m(G,k)$; for details and further references see [29,36]. At this point it is worth noting that via the Hosoya index, the matching numbers $m(G,k)$ have been related also to certain types of entropy [6,21,24,27].

In 2012, Gutman and Wagner [18] extended the applicability of formula (1) to all graphs, by conceiving the concept of *matching energy*, defined as

$$ME(G) = \frac{2}{\pi} \int_0^\infty \frac{1}{x^2} \ln \left[\sum_{k\geq 0} m(G,k) \, x^{2k} \right] dx \, . \tag{2}$$

Evidently, if G is a tree, then $ME(G) = E(G)$. The matching energy is nowadays the subject of extensive studies, see the survey [15], the recent papers [1–4, 23], and the references cited therein.

The quasi-order $\succeq$ has been studied for various classes of graphs: acyclic [10, 11], unicyclic [12], bicyclic [12,13], tricyclic [16], and many others [19,28,30,31]. For these classes the maximal and minimal elements with respect to $\succeq$ could be determined. In particular, the maximum and minimum elements in the class of connected graphs with n vertices are the complete graph K_n and the star S_n, respectively [18]. So and Wang [28] determined the minimum elements among all connected graphs of order n and size m for $n-1 \leq m \leq 2n-3$ and $\frac{n(n-1)}{2} - (n-2) \leq m \leq \frac{n(n-1)}{2}$.

This paper is the continuation of the work of [22], which presents some transformations on the complements of graphs, one of which concerns grafting two pendent paths attached at different vertices. The corresponding problem of grafting of two pendent paths attached at a vertex has been solved in [22]. Combining all these results enables us to find the maximum and minimum graphs with respect to the quasi-order in the set of the complements of all unicyclic graphs with given order and in the set of all unicyclic graphs with given order and girth, respectively.

2 Main results

First a few necessary definitions and auxiliary lemmas are provided. Let u and v be two distinct vertices of a graph G. A path $P = uw_1w_2 \ldots w_tv$ between u and v is called an *internal path* from u to v in G if all internal vertices are of degree two, i.e., $d_G(w_i) = 2$ for $i = 1, \ldots, t$. The length of the path P is $t + 1$. If the internal path between u and v is of length one, then u and v are actually adjacent. If v is of degree one, the internal path $uw_1w_2 \cdots w_tv$ is also said to be a *pendent path* (attaching at u) of length $t+1$. For a graph G with u a non-isolated vertex, let $G(u; a_1, \ldots, a_t)$ denote the graph obtained from G by attaching t pendent paths of length $a_1, \ldots a_t$ respectively at the vertex u. Especially when $a_1 = \cdots = a_t = 1$, $G(u; a_1, \ldots, a_t)$ is simply written as $G(u;^* t)$. Similarly, the notation $G(u, v; a, b)$ stands for the graph obtained from G by attaching two pendent paths of length a and b at u and v, respectively.

Recall a standard notation in graph theory. For a graph G and v a vertex of G, let $N_G(v)$ denote the set of vertices in G adjacent to v.

Lemma 1. [5,14] *If $e = uv$ is an arbitrary edge of G with end vertices u and v, then for all non-negative integers k,*

$$m(G,k) = m(G-e,k) + m(G-u-v,k-1) \tag{3}$$

$$m(G,k) = m(G-u,k) + \sum_{v \in N_G(u)} m(G-u-v,k-1)\,. \tag{4}$$

The following result can be immediately obtained by applying (3) of Lemma 1 on edges incident with u, whose proof is simple and so omitted.

Lemma 2. *If u is an arbitrary vertex of G, then for any vertices $v_1, \ldots, v_s$ adjacent to u in G and all non-negative integers k, we have*

$$m(G,k) = m(G-uv_1-\cdots-uv_s,k) + \sum_{i=1}^{s} m(G-u-v_i,k-1)\,.$$

As usual, by $\overline{G}$ we denote the complement of the graph G. Let K_p be the complete graph of order p. By straight observation we can get some simple properties on the matching numbers of the complement.

Lemma 3. *For any simple graph G of order n, with H as its subgraph on t vertices, the following results hold.*
(1) If H is a spanning subgraph of G, then $\overline{H} \succeq \overline{G}$ with equality if and only if $H = G$.
(2) If H is an induced subgraph of G, then $\overline{H} \preceq \overline{G}$ with equality if and only if $G = H \vee K_{n-t}$.

Proof. If H is a spanning subgraph of G, then $V(H) = V(G)$ and $E(H) \subseteq E(G)$. It is easy to see that $E(\overline{H}) \supseteq E(\overline{G})$ and then $m(\overline{H},k) \geq m(\overline{G},k)$ for all k. In this case, $m(\overline{H},1) = m(\overline{G},1)$ if and only if $H = G$.

If H is an induced subgraph of G, then $\overline{H}$ is an induced subgraph of $\overline{G}$ too. Thus $m(\overline{H},k) \leq m(\overline{G},k)$ for all k. If $m(\overline{H},1) = m(\overline{G},1)$, then $E(\overline{H}) = E(\overline{G})$ and so $\overline{G} = \overline{H} \cup (n-t)P_1$. Therefore $G = \overline{\overline{G}} = \overline{\overline{H} \cup (n-t)P_1} = H \vee K_{n-t}$. $\square$

In the theory of matching polynomials it has been shown that the matching polynomial of $\overline{G}$ can be computed from the matching polynomial of the graph G as follows.

Lemma 4. [8] *Let $\overline{G}$ be the complement of the graph G. Then*

$$\alpha(\overline{G},\lambda) \;=\; \sum_{k\geq 0} m(G,k)\,\alpha(K_{n-2k},\lambda) \tag{5}$$

$$m(\overline{G},k) \;=\; \frac{1}{(n-2k)!\,\sqrt{2\pi}} \int_{-\infty}^{+\infty} \alpha(G,\lambda)\,\alpha(K_{n-2k},\lambda)\,e^{-\lambda^2/2}\,d\lambda\,. \tag{6}$$

Formulas (5) and (6) were discovered by Zaslavsky [32] and Godsil [8], respectively. Directly from Eq. (5) we get

Lemma 5. [26] *Let G be a simple graph with n vertices and $\overline{G}$ its complement. Then*

$$m(\overline{G},k) = \sum_{\ell\geq 0} (-1)^{\ell}\, m(G,\ell)\, m(K_{n-2\ell},k-\ell)\,. \tag{7}$$

Some results on the matching numbers of the complement have been obtained in our earlier paper [22], some of which will be used in the paper and listed here.

Theorem 1. *Let u and v be adjacent vertices of a graph G. If G_1 (resp., G_2) is the graph obtained from G by inserting t vertices into the edge uv (resp., by joining the vertex u to an end vertex of a path P_t) then $\overline{G_1} \succeq \overline{G_2}$. If in addition $d_G(u) \geq 2$, then $\overline{G_1} \succ \overline{G_2}$.*

Recall some notations from [22]. For an arbitrary edge $e = uv$ of a graph G with $d_G(u) > 1$ and $d_G(v) > 1$, let $G(u \circ v)$ denote the graph obtained from G by deleting the edge e and then identifying u and v, and adding a pendent edge at the identified vertex.

Theorem 2. *Let G be a simple graph and uv an edge of G such that $N_G(u) \cap N_G(v) = \emptyset$, and $d_G(u), d_G(v) > 1$. Then $\overline{G} \succ \overline{G(u \circ v)}$.*

For two graphs G and H, the notation $G(u,v)H$ stands for the graph obtained by identifying the vertex u of G and the vertex v of H.

Theorem 3. *Suppose that G is an arbitrary graph and T is a tree with $t+1$ vertices, with u being a vertex of G and v a vertex of T. Then*

$$\overline{G(u;^* t)} \preceq \overline{G(u,v)T} \preceq \overline{G(u;t)}$$

where the left–hand side equality holds if and only if $T \cong S_{t+1}$ with v as its center whereas the right–hand side equality holds if and only if $T \cong P_{t+1}$ with v as its end vertex.

Theorem 4. *Let G, H' and H'' be three connected graphs, $u, v \in V(G)$, $u' \in V(H')$ and $u'' \in V(H'')$, where $|V(H')|, |V(H'')| \geq 2$. Let $G_{u,v}$ be the graph obtained from G, H' and H'' by identifying u with u', and v with u'', and G_u (resp., G_v) be obtained by identifying u (resp., v) with both u' and u''. If $G - u \cong G - v$, then $\overline{G_{u,v}} \succ \overline{G_u}$.*

The following result comes from the proof of Theorem 5 in [22] and will be used several times here.

Lemma 6. *Let G be a simple graph and u be a non-isolated vertex of G. Let a, b ($a \leq b$) be two positive integers and $G(u;a,b)$ defined as previously. Then $m(G(u;a,b),k) - m(G(u;a-1,b+1),k) = (-1)^a \sum_{u' \in N_G(u)} m(G-u-u' \cup P_{b-a}, k-a-1)$ and equal to zero for $k = 0, 1, \ldots, a$.*

Edge grafting on two pendent paths attached at a vertex was investigated in our paper [22]. Now we shall discuss the general case and the case of grafting two pendent paths at two adjacent vertices will be given first as follows.

Lemma 7. *Let G be a simple graph and uv an edge of G with $d_G(u) > 1$ or $d_G(v) > 1$. Then for any positive integers a and b, we have*

$$\overline{G(u,v;a,b)} \prec \overline{G(v;a+b)} \quad \text{and} \quad \overline{G(u,v;a,b)} \prec \overline{G(u;a+b)}.$$

Proof. For convenience, let $G_1 = G(u,v;a,b)$ and $G_2 = G(v;a+b)$, and assume that G_1 (and so G_2) has n vertices. If $d_G(v) = 1$, then $d_G(u) > 1$ must hold by the condition that u and v cannot both be pendent vertices. In this case, note that $G_1 = G(u;a,b+1)$ and $G_2 = G(u;a+b+1)$, and so $\overline{G_1} \prec \overline{G_2}$ by Theorem 5 in [22]. Now assume that $d_G(v) \geq 2$. By Lemma 1, we have

$$m(G_1,k) = m((G-uv)(u,v;a,b),k) + m(P_a \cup P_b \cup G-u-v, k-1)$$

$$\begin{aligned}
&=m(P_a \cup (G-uv)(v;b),k) + m(P_{a-1} \cup (G-u)(v;b),k-1) \\
&+m(P_a \cup P_b \cup G-u-v,k-1) \\
m(G_2,k) &=m(P_a \cup G(v;b),k) + m(P_{a-1} \cup G(v,b-1),k-1) \\
&=m(P_a \cup (G-uv)(v;b),k) + m(P_a \cup P_b \cup G-u-v,k-1) \\
&+m(P_{a-1} \cup G(v;b-1),k-1)\,.
\end{aligned}$$

Consequently, we have

$$m(G_1,k)-m(G_2,k) = m(P_{a-1}\cup(G-u)(v;b),k-1)-m(P_{a-1}\cup G(v;b-1),k-1)\,. \quad (8)$$

Note that $a \geq 1$ and $b \geq 1$. If $b = 1$, then in this case $G(v;0) = G$. Since $m(G_1,\ell) = m(G_2,\ell)$ for $\ell = 0,1$, and by Lemma 5 and Eq. (8), we have

$$\begin{aligned}
&m(\overline{G_1},k) - m(\overline{G_2},k) \\
&= \sum_{\ell \geq 0} (-1)^{\ell}\, m(K_{n-2\ell},k-\ell)(m(G_1,\ell) - m(G_2,\ell)) \\
&= \sum_{\ell \geq 1} (-1)^{\ell}\, m(K_{n-2\ell},k-\ell)\, m(P_{a-1} \cup (G-u)(v;1),\ell-1) \\
&- \sum_{\ell \geq 1} (-1)^{\ell}\, m(K_{n-2\ell},k-\ell)\, m(P_{a-1} \cup G,\ell-1) \\
&= \sum_{\ell' \geq 0} (-1)^{\ell'+1}\, m(K_{n-2-2\ell'},k-1-\ell')\, m(P_{a-1} \cup (G-u)(v;1),\ell') \\
&- \sum_{\ell' \geq 0} (-1)^{\ell'+1}\, m(K_{n-2-2\ell'},k-1-\ell')\, m(P_{a-1} \cup G,\ell') \\
&= -m(\overline{P_{a-1} \cup (G-u)(v;1)},k-1) + m(\overline{P_{a-1} \cup G},k-1)\,.
\end{aligned}$$

Let us observe that $(G-u)(v;1)$ can be obtained from G by deleting all edges but uv incident with u in G, so $(G-u)(v;1)$ is a proper spanning subgraph of G. Thus by Lemma 3, $\overline{P_{a-1} \cup (G-u)(v;1)} \succ \overline{P_{a-1} \cup G}$. Thus $m(\overline{G_1},k) \leq m(\overline{G_2},k)$, and strict inequality holds for at least one k, say $k = 2$. Thus we are done in this case.

Then we may assume that $b \geq 2$ hereafter. By Lemma 1, we have

$$\begin{aligned}
&m(P_{a-1} \cup (G-u)(v;b),k-1) \\
&= m(P_{a-1} \cup (G-u)(v;b-1),k-1) + m(P_{a-1} \cup (G-u)(v;b-2),k-2) \\
&= m(P_{a-1} \cup (G-u)(v;b-1),k-1) + m(P_{a-1} \cup G-u-v \cup P_{b-1},k-2) \\
&+ \sum_{v' \in N_G(v)\setminus\{u\}} m(P_{a-1} \cup G-u-v-v' \cup P_{b-2},k-3) \qquad (9)
\end{aligned}$$

where the last equality follows by applying Lemma 2 on $P_{a-1} \cup (G-u)(v; b-2)$ to all edges but uv incident with v in G, and

$$\begin{aligned}
&m(P_{a-1} \cup G(v; b-1), k-1) \\
&= m(P_{a-1} \cup (G-u)(v; b-1), k-1) + \sum_{u' \in N_G(u)} m(P_{a-1} \cup (G-u-u')(v; b-1), k-2) \\
&= m(P_{a-1} \cup (G-u)(v; b-1), k-1) + m(P_{a-1} \cup G-u-v \cup P_{b-1}, k-2) \\
&+ \sum_{u' \in N_G(u) \setminus \{v\}} m(P_{a-1} \cup (G-u-u')(v; b-1), k-2)\,. \qquad (10)
\end{aligned}$$

Substituting (9) and (10) into Eq. (8), we come to

$$\begin{aligned}
&m(G_1, k) - m(G_2, k) \\
&= \sum_{v' \in N_G(v) \setminus \{u\}} m(P_{a-1} \cup G-u-v-v' \cup P_{b-2}, k-3) \\
&- \sum_{u' \in N_G(u) \setminus \{v\}} m(P_{a-1} \cup (G-u-u')(v; b-1), k-2)\,. \qquad (11)
\end{aligned}$$

Note that $m(G_1, 0) = 1 = m(G_2, 0)$ obviously and $m(G_1, 1) = |E(G_1)| = |E(G_2)| = m(G_2, 1)$. From the first equality above we know that $m(G_1, 2) = m(G_2, 2)$ since $m(P_{a-1} \cup (G-u)(v; b), 1) = m(P_{a-1} \cup G(v; b-1), 1)$. As the convention $m(\cdot, k) = 0$ for $k < 0$ is adopted, the above formula is valid for any k. By Lemma 5,

$$\begin{aligned}
&m(\overline{G_1}, k) - m(\overline{G_2}, k) \\
&= \sum_{\ell \geq 0} (-1)^\ell\, m(K_{n-2\ell}, k-\ell)\, (m(G_1, \ell) - m(G_2, \ell)) \\
&= \sum_{\ell \geq 3} (-1)^\ell\, m(K_{n-2\ell}, k-\ell) \sum_{v' \in N_G(v) \setminus \{u\}} m(P_{a-1} \cup G-u-v-v' \cup P_{b-2}, \ell-3) \\
&- \sum_{\ell \geq 2} (-1)^\ell\, m(K_{n-2\ell}, k-\ell) \sum_{u' \in N_G(u) \setminus \{v\}} m(P_{a-1} \cup (G-u-u')(v; b-1), \ell-2) \\
&= - \sum_{v' \in N_G(v) \setminus \{u\}} m(\overline{P_{a-1} \cup G-u-v-v' \cup P_{b-2}}, k-3) \\
&- \sum_{u' \in N_G(u) \setminus \{v\}} m(\overline{P_{a-1} \cup (G-u-u')(v; b-1)}, k-2)\,.
\end{aligned}$$

Thus we have $m(\overline{G_1}, k) - m(\overline{G_2}, k) \leq 0$ and is strictly less than zero for at least one k, say $k = 3$. Therefore we proved that $\overline{G(u, v; a, b)} \prec \overline{G(v; a+b)}$.

By the same way, $\overline{G(u, v; a, b)} \prec \overline{G(u; a+b)}$ can be proved. $\square$

By a more subtle method, we can get main result on grafting two pendent paths at two vertices connected by an internal path.

Theorem 5. *Let G be a simple graph and u, v be its two vertices with $d_G(u) > 1$ or $d_G(v) > 1$. If there exists an internal path between u and v, then for all positive integers a, b we have*

$$\overline{G(u,v;a,b)} \prec \overline{G(u;a+b)}, \quad \text{if} \quad a \geq 2$$

and

$$\overline{G(u,v;a,b)} \prec \overline{G(v;a+b)}, \quad \text{if} \quad b \geq 2\,.$$

Proof. For convenience, let $G_1 = G(u,v;a,b)$ and $G_2 = G(v;a+b)$, and assume that G_1 (and so G_2) has n vertices. By the same reason as in the proof of Lemma 7, we can always assume that $d_G(u) > 1$ and $d_G(v) > 1$. Suppose the shortest internal path between u and v is of length t. Choose a shortest internal path P between u and v, say $P = uw_1w_2\cdots w_tv$. Assume the two pendent paths in $G(u,v;a,b)$ attached at u and v are $uu_1\cdots u_a$ and $vv_1\cdots v_b$ respectively, and the pendent path in $G(v;a+b)$ attached at v is $vv_1\cdots v_{a+b}$.

If $t = 1$, the conclusion follows from Lemma 7.

Now assume that $t > 1$. By Lemma 1, we have

$$\begin{aligned}
m(G_1+uv,k) =& m(G_1,k) + m(P_a \cup P_b \cup G-u-v, k-1)\\
m(G_2+uv,k) =& m(G_2,k) + m(P_{a+b} \cup G-u-v, k-1)\\
=& m(G_2,k) + m(P_a \cup P_b \cup G-u-v, k-1)\\
&+ m(P_{a-1} \cup P_{b-1} \cup G-u-v, k-2)\,.
\end{aligned}$$

By Eq. (11) in the proof of Lemma 7,

$$\begin{aligned}
&m(G_1+uv,k) - m(G_2+uv,k)\\
=& \sum_{v' \in N_G(v)} m(P_{a-1} \cup G-u-v-v' \cup P_{b-2}, k-3)\\
-& \sum_{u' \in N_G(u)} m(P_{a-1} \cup (G-u-u')(v;b-1), k-2)\,.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
& m(G_1,k) - m(G_2,k) \\
& = m(G_1+uv,k) - m(G_2+uv,k) + m(P_{a-1} \cup P_{b-1} \cup G-u-v, k-2) \\
& = \sum_{v' \in N_G(v)} m(P_{a-1} \cup G-u-v-v' \cup P_{b-2}, k-3) \\
& - \sum_{u' \in N_G(u)} m(P_{a-1} \cup (G-u-u')(v;b-1), k-2) \\
& + m(P_{a-1} \cup P_{b-1} \cup G-u-v, k-2) \\
& = m(P_{a-1} \cup (G-u)(v;b-2), k-2) \\
& - \sum_{u' \in N_G(u)} m(P_{a-1} \cup (G-u-u')(v;b-1), k-2) \qquad (12)
\end{aligned}$$

where the last equality follows by observing

$$\begin{aligned}
& m(P_{a-1} \cup (G-u)(v;b-2), k-2) \\
& = \sum_{v' \in N_G(v)} m(P_{a-1} \cup G-u-v-v' \cup P_{b-2}, k-3) \\
& + m(P_{a-1} \cup P_{b-1} \cup G-u-v, k-2)
\end{aligned}$$

which can be obtained by applying Lemma 2 on $P_{a-1} \cup (G-u)(v;b-2)$ to all edges incident with v in G.

Now we distinguish it in two cases according to the values of t and b.

Case 1. $t \le b-1$. Eq. (12) continues as

$$\begin{aligned}
& m(G_1,k) - m(G_2,k) \\
& = m(P_{a-1} \cup (G-u)(v;b-2), k-2) - m(P_{a-1} \cup (G-u-w_1)(v;b-1), k-2) \\
& - \sum_{u' \in N_G(u) \setminus \{w_1\}} m(P_{a-1} \cup (G-u-u')(v;b-1), k-2)\,.
\end{aligned}$$

For convenience, let $\hat{G} = (G - P \setminus \{v\}) \cup P_{a-1}$. Then we observe that $P_{a-1} \cup (G-u)(v;b-2) \cong \hat{G}(v;t,b-2)$ and $P_{a-1} \cup (G-u-w_1)(v;b-1) \cong \hat{G}(v;t-1,b-1)$. If $t \le b-2$, then by Lemma 6, we have

$$\begin{aligned}
& m(\hat{G}(v;t,b-2), k-2) - m(\hat{G}(v;t-1,b-1), k-2) \\
& = (-1)^t \sum_{v' \in N_{\hat{G}}(v)} m(\hat{G}-v-v' \cup P_{b-t-2}, k-t-3)
\end{aligned}$$

which is equal to zero for any $k = 2, \ldots, t+2$ and is identically zero for any k if $t = b-1$ because $\hat{G}(v; t, b-2) \cong \hat{G}(v; t-1, b-1)$ in this case. Further by Lemma 5, we have

$$\begin{aligned}
& m(\overline{G_1}, k) - m(\overline{G_2}, k) = \sum_{\ell \geq 0} (-1)^{\ell}\, m(K_{n-2\ell}, k-\ell)\,(m(G_1, \ell) - m(G_2, \ell)) \\
& = \sum_{\ell \geq t+3} (-1)^{\ell}\, m(K_{n-2\ell}, k-\ell)(-1)^{t} \sum_{v' \in N_{\hat{G}}(v)} m(\hat{G} - v - v' \cup P_{b-t-2}, \ell - t - 3) \\
& - \sum_{\ell \geq 2} (-1)^{\ell}\, m(K_{n-2\ell}, k-\ell) \sum_{u' \in N_G(u) \setminus \{w_1\}} m(P_{a-1} \cup (G-u-u')(v; b-1), \ell - 2) \\
& = - \sum_{v' \in N_{\hat{G}}(v)} m(\overline{\hat{G} - v - v' \cup P_{b-t-2}}, k - t - 3) \\
& - \sum_{u' \in N_G(u) \setminus \{w_1\}} m(\overline{P_{a-1} \cup (G-u-u')(v; b-1)}, k-2)\,.
\end{aligned}$$

Thus $m(\overline{G_1}, k) - m(\overline{G_2}, k) \leq 0$ and is strictly less than zero at least for $k = 2$ and $k = t+3$. Therefore $\overline{G_1} \prec \overline{G_2}$.

Case 2. $t \geq b$. Choose an arbitrary vertex $\dot{u} \in N_G(u) \setminus \{w_1\}$. From Eq. (12), we have

$$\begin{aligned}
& m(G_1, k) - m(G_2, k) \\
& = m(P_{a-1} \cup (G-u-\dot{u})(v; b-2), k-2) \\
& + \sum_{\dot{u}' \in N_G(\dot{u}) \setminus \{u\}} m(P_{a-1} \cup (G-u-\dot{u}-\dot{u}')(v; b-2), k-3) \\
& - \sum_{u' \in N_G(u) \setminus \{\dot{u}\}} m(P_{a-1} \cup (G-u-u')(v; b-1), k-2) \\
& - m(P_{a-1} \cup (G-u-\dot{u})(v; b-1), k-2)\,.
\end{aligned} \tag{13}$$

Let us observe that in the above equality the last term $m(P_{a-1} \cup (G-u-\dot{u})(v; b-1), k-2)$ ca be expressed as

$$\begin{aligned}
& m(P_{a-1} \cup (G-u-\dot{u})(v; b-1), k-2) \\
& = m(P_{a-1} \cup (G-u-\dot{u})(v; b-2), k-2) + m(P_{a-1} \cup (G-u-\dot{u})(v; b-3), k-3)\,.
\end{aligned}$$

Since $b \geq 2$ and if $b = 2$, $(G-u-\dot{u})(v; -1) = G - u - \dot{u} - v$ can be understood without confusion. Meanwhile, in the right–side of Eq. (13) the third term can be expressed as

$$\sum_{u' \in N_G(u) \setminus \{\dot{u}\}} m(P_{a-1} \cup (G-u-u')(v; b-1), k-2)$$

$$
\begin{aligned}
&= \sum_{u' \in N_G(u)\setminus\{\dot{u}, w_1\}} m(P_{a-1} \cup (G-u-u')(v; b-1), k-2) \\
&+ m(P_{a-1} \cup (G-u-w_1)(v; b-1), k-2) \\
&= \sum_{u' \in N_G(u)\setminus\{\dot{u}, w_1\}} m(P_{a-1} \cup (G-u-u')(v; b-1), k-2) \\
&+ m(P_{a-1} \cup (G-u-w_1-\dot{u})(v; b-1), k-2) \\
&+ \sum_{\dot{u}' \in N_G(\dot{u})\setminus\{u\}} m(P_{a-1} \cup (G-u-w_1-\dot{u}-\dot{u}')(v; b-1), k-3)\,. \qquad (14)
\end{aligned}
$$

Thus the equation (13) continues as

$$
\begin{aligned}
&m(G_1, k) - m(G_2, k) \\
&= \sum_{\dot{u}' \in N_G(\dot{u})\setminus\{u\}} m(P_{a-1} \cup (G-u-\dot{u}-\dot{u}')(v; b-2), k-3) \\
&- \sum_{u' \in N_G(u)\setminus\{\dot{u}, w_1\}} m(P_{a-1} \cup (G-u-u')(v; b-1), k-2) \\
&- m(P_{a-1} \cup (G-u-w_1-\dot{u})(v; b-1), k-2) \\
&- \sum_{\dot{u}' \in N_G(\dot{u})\setminus\{u\}} m(P_{a-1} \cup (G-u-w_1-\dot{u}-\dot{u}')(v; b-1), k-3) \\
&- m(P_{a-1} \cup (G-u-\dot{u})(v; b-3), k-3)\,.
\end{aligned}
$$

For convenience, let $G_{\dot{u}'} = P_{a-1} \cup (G - P \setminus \{v\} - \dot{u} - \dot{u}')$ and then we observe that $P_{a-1} \cup (G-u-\dot{u}-\dot{u}')(v; b-2) = G_{\dot{u}'}(v; b-2, t)$ and $P_{a-1} \cup (G-u-w_1-\dot{u}-\dot{u}')(v; b-1) = G_{\dot{u}'}(v; b-1, t-1)$.

On the one hand, by Lemma 6, we have

$$
\begin{aligned}
&\sum_{\dot{u}' \in N_G(\dot{u})\setminus\{u\}} (m(G_{\dot{u}'}(v; b-2, t), k-3) - m(G_{\dot{u}'}(v; b-1, t-1), k-3)) \\
&= \sum_{\dot{u}' \in N_G(\dot{u})\setminus\{u\}} (-1)^b \sum_{v' \in N_{G_{\dot{u}'}}(v)} m(G_{\dot{u}'} - v - v' \cup P_{t-b}, k-b-3)\,.
\end{aligned}
$$

On the other hand, if $b \geq 3$ and then $t \geq 3$ (since $t \geq b$), we have

$$
\begin{aligned}
&m(P_{a-1} \cup (G-u-\dot{u})(v; b-3), k-3) \\
&= m(P_{a-1} \cup (G-u-\dot{u}-w_1)(v; b-3), k-3) \\
&+ m(P_{a-1} \cup (G-u-\dot{u}-w_1-w_2)(v; b-3), k-4) \\
&= m(P_{a-1} \cup (G-u-\dot{u}-w_1-w_2)(v; b-3), k-3) \\
&+ m(P_{a-1} \cup (G-u-\dot{u}-w_1-w_2-w_3)(v; b-3), k-4)
\end{aligned}
$$

$$+\,m(P_{a-1}\cup(G-u-\dot{u}-w_1-w_2)(v;b-3),k-4)$$

and then

$$\begin{aligned}
&m(P_{a-1}\cup(G-u-w_1-\dot{u})(v;b-1),k-2)\\
&+m(P_{a-1}\cup(G-u-\dot{u}-w_1-w_2)(v;b-3),k-3)\\
&=m(P_{a-1}\cup(G-u-w_1-\dot{u})(v;b-1)+w_2v_{b-2},k-2)
\end{aligned}$$

which follows by applying Lemma 1 to the edge w_2v_{b-2} in $P_{a-1}\cup(G-u-w_1-\dot{u})(v;b-1)+w_2v_{b-2}$.

If $b=2$, we have

$$\begin{aligned}
&m(P_{a-1}\cup(G-u-w_1-\dot{u})(v;1),k-2)\\
&+m(P_{a-1}\cup(G-u-\dot{u}-v),k-3)\\
&=m(P_{a-1}\cup(G-u-w_1-\dot{u})(v;1),k-2)\\
&+m(P_{a-1}\cup(G-u-\dot{u}-v-w_1),k-3)\\
&+m(P_{a-1}\cup(G-u-\dot{u}-v-w_1-w_2),k-4)\\
&=m(P_{a-1}\cup(G-u-w_1-\dot{u})(v;^*2),k-2)\\
&+m(P_{a-1}\cup(G-u-\dot{u}-v-w_1-w_2),k-4)
\end{aligned}$$

where the last equality follows by applying Lemma 1 to one of two pendent edges at v in $P_{a-1}\cup(G-u-w_1-\dot{u})(v;^*2)$.

Combining all these arguments above, we have if $b\geq 3$

$$\begin{aligned}
&m(G_1,k)-m(G_2,k)\\
&=\sum_{\dot{u}'\in N_G(\dot{u})\setminus\{u\}}(-1)^b\sum_{v'\in N_{G_{\dot{u}'}}(v)}m(G_{\dot{u}'}-v-v')\cup P_{t-b},k-b-3)\\
&-\sum_{u'\in N_G(u)\setminus\{\dot{u},w_1\}}m(P_{a-1}\cup(G-u-u')(v;b-1),k-2)+\Omega
\end{aligned}\tag{15}$$

where if $b=2$,

$$\begin{aligned}
\Omega=&m(P_{a-1}\cup(G-u-w_1-\dot{u})(v;^*2),k-2)\\
&+m(P_{a-1}\cup(G-u-\dot{u}-v-w_1-w_2),k-4)
\end{aligned}$$

and if $b \geq 3$,

$$\Omega = m(P_{a-1} \cup (G-u-w_1-\dot{u})(v;b-1) + w_2v_{b-2}, k-2)$$
$$+ m(P_{a-1} \cup (G-u-\dot{u}-w_1-w_2-w_3)(v;b-3), k-4)$$
$$+ m(P_{a-1} \cup (G-u-\dot{u}-w_1-w_2)(v;b-3), k-4)\,.$$

By Lemma 5 and Eq. (15), we have if $b \geq 3$

$$\begin{aligned}
&m(\overline{G_1},k) - m(\overline{G_2},k)\\
&= \sum_{\ell \geq 0} (-1)^{\ell}\, m(K_{n-2\ell}, k-\ell)\,(m(G_1,\ell) - m(G_2,\ell))\\
&= \sum_{\ell \geq b+3} (-1)^{\ell+b}\, m(K_{n-2\ell}, k-\ell) \sum_{\dot{u}' \in N_G(\dot{u})\setminus\{u\}} \sum_{v' \in N_{G_{\dot{u}'}}(v)} m(G_{\dot{u}'}-v-v' \cup P_{t-b}, \ell-b-3)\\
&- \sum_{\ell \geq 2} (-1)^{\ell}\, m(K_{n-2\ell}, k-\ell) \sum_{u' \in N_G(u)\setminus\{\dot{u},w_1\}} m(P_{a-1} \cup (G-u-u')(v;b-1), \ell-2)\\
&- \sum_{\ell \geq 2} (-1)^{\ell}\, m(K_{n-2\ell}, k-\ell)\, m(P_1 \cup P_{a-1} \cup (G-u-w_1-\dot{u})(v;b-1)\\
&+ w_2v_{b-2}, k-2), \ell-2) - \sum_{\ell \geq 4} (-1)^{\ell}\, m(K_{n-2\ell}, k-\ell)\cdot\\
&\cdot\, m(P_1 \cup P_{a-1} \cup (G-u-\dot{u}-w_1-w_2-w_3)(v;b-3), k-4), \ell-4)\\
&- \sum_{\ell \geq 4} (-1)^{\ell}\, m(K_{n-2\ell}, k-\ell)\, m(P_{a-1} \cup (G-u-\dot{u}-w_1-w_2)(v;b-3), k-4), \ell-4)\\
&= - \sum_{\dot{u}' \in N_G(\dot{u})\setminus\{u\}} \sum_{v' \in N_{G_{\dot{u}'}}(v)} m(\overline{G_{\dot{u}'}-v-v' \cup P_{t-b}}, k-b-3)\\
&- \sum_{u' \in N_G(u)\setminus\{\dot{u},w_1\}} m(\overline{P_{a-1} \cup (G-u-u')(v;b-1)}, k-2)\\
&- m(\overline{P_1 \cup P_{a-1} \cup (G-u-w_1-\dot{u})(v;b-1) + w_2v_{b-2}}, k-2)\\
&- m(\overline{P_1 \cup P_{a-1} \cup (G-u-\dot{u}-w_1-w_2-w_3)(v;b-3)}, k-4)\\
&- m(\overline{P_{a-1} \cup (G-u-\dot{u}-w_1-w_2)(v;b-3)}, k-4)
\end{aligned}$$

where P_1 is added in two places of the third and fourth terms because in this situation the order of $P_1 \cup P_{a-1} \cup (G-u-w_1-\dot{u})(v;b-1) + w_2v_{b-2}, k-2)$ is equal to $n-2\ell+2(\ell-2) = n-4$ and that of $P_1 \cup P_{a-1} \cup (G-u-\dot{u}-w_1-w_2-w_3)(v;b-3)$ is equal to $n-2\ell+2(\ell-4) = n-8$, and if $b = 2$

$$m(\overline{G_1},k) - m(\overline{G_2},k)$$

$$= \sum_{\ell \geq 0} (-1)^\ell \, m(K_{n-2\ell}, k-\ell) \left(m(G_1, \ell) - m(G_2, \ell) \right)$$
$$= \sum_{\ell \geq 5} (-1)^\ell \, m(K_{n-2\ell}, k-\ell) \sum_{\dot{u}' \in N_G(\dot{u}) \setminus \{u\}} \sum_{v' \in N_{G_{\dot{u}'}}(v)} m(G_{\dot{u}'} - v - v' \cup P_{t-2}, \ell - 5)$$
$$- \sum_{\ell \geq 2} (-1)^\ell \, m(K_{n-2\ell}, k-\ell) \sum_{u' \in N_G(u) \setminus \{\dot{u}, w_1\}} m(P_{a-1} \cup (G - u - u')(v; 1), \ell - 2)$$
$$- \sum_{\ell \geq 2} (-1)^\ell \, m(K_{n-2\ell}, k-\ell) \, m(P_{a-1} \cup (G - u - w_1 - \dot{u})(v;^* 2), \ell - 2)$$
$$- \sum_{\ell \geq 4} (-1)^\ell \, m(K_{n-2\ell}, k-\ell) \, m(P_{a-1} \cup (G - u - \dot{u} - v - w_1 - w_2), \ell - 4)$$
$$= - \sum_{\dot{u}' \in N_G(\dot{u}) \setminus \{u\}} \sum_{v' \in N_{G_{\dot{u}'}}(v)} m(\overline{G_{\dot{u}'} - v - v' \cup P_{t-2}}, k - 5)$$
$$- \sum_{u' \in N_G(u) \setminus \{\dot{u}, w_1\}} m(\overline{P_{a-1} \cup (G - u - u')(v; 1)}, k - 2)$$
$$- m(\overline{P_{a-1} \cup (G - u - w_1 - \dot{u})(v;^* 2)}, k - 2)$$
$$- m(\overline{P_{a-1} \cup (G - u - \dot{u} - v - w_1 - w_2)}, k - 4) \, .$$

From above, it follows immediately that $m(\overline{G_1}, k) \leq m(\overline{G_2}, k)$ for all k and strict inequality holds for at least one k, say $k = 2$. Therefore $\overline{G_1} \prec \overline{G_2}$ is proved.

Similarly, $\overline{G(u, v; a, b)} \prec \overline{G(u; a + b)}$ can be shown. □

Theorem 6. *Let G be a simple graph and u, v be its two vertices with $d_G(v) = 2$. If there exists an internal path between u and v, then for all positive integers a we have*

$$\overline{G(u, v; a, 1)} \prec \overline{G(v; a + 1)} \, .$$

Proof. Choose a shortest internal path P from u to v, say $P = uw_1 \cdots w_t v$, and we can assume $t \geq 1$ by Lemma 7. Then as in Case 2 of Theorem 5, choose a vertex $\dot{u} \in N_G(u) \setminus \{w_1\}$, by Eq. (13), we have

$$\begin{aligned} & m(G_1, k) - m(G_2, k) \\ & = m(P_{a-1} \cup G - u - \dot{u} - v, k - 2) \\ & + \sum_{\dot{u}' \in N_G(\dot{u}) \setminus \{u\}} m(P_{a-1} \cup G - u - \dot{u} - \dot{u}' - v, k - 3) \\ & - \sum_{u' \in N_G(u) \setminus \{\dot{u}\}} m(P_{a-1} \cup (G - u - u'), k - 2) \\ & - m(P_{a-1} \cup (G - u - \dot{u}), k - 2) \, . \end{aligned} \tag{16}$$

By Lemma 1,

$$\begin{aligned}m(P_{a-1}\cup(G-u-\dot{u}),k-2)=&m(P_{a-1}\cup(G-u-\dot{u}-v),k-2)\\&+\sum_{v'\in N_G(v)} m(P_{a-1}\cup(G-u-\dot{u}-v-v'),k-3)\,.\end{aligned}$$

Together with Eq. (14), the Equation (16) continues as

$$\begin{aligned}&m(G_1,k)-m(G_2,k)\\&=\sum_{\dot{u}'\in N_G(\dot{u})\setminus\{u\}} m(P_{a-1}\cup G-u-\dot{u}-\dot{u}'-v,k-3)\\&-\sum_{u'\in N_G(u)\setminus\{\dot{u},w_1\}} m(P_{a-1}\cup(G-u-u'),k-2)\\&-m(P_{a-1}\cup(G-u-w_1-\dot{u}),k-2)\\&-\sum_{\dot{u}'\in N_G(\dot{u})\setminus\{u\}} m(P_{a-1}\cup(G-u-w_1-\dot{u}-\dot{u}'),k-3)\\&-\sum_{v'\in N_G(v)} m(P_{a-1}\cup(G-u-\dot{u}-v-v'),k-3)\,.\end{aligned}\tag{17}$$

With the condition that $d_G(v)=2$, we can assume that $N_G(v)=\{w_t,\dot{v}\}$. Add a new vertex $\hat{v}$ and join it to v and $\dot{v}$ in $P_{a-1}\cup(G-u-w_1-\dot{u})$, and denote by H the resulting graph obtained from $P_{a-1}\cup(G-u-w_1-\dot{u})$ with two new edges $\hat{v}v$ and $\hat{v}\dot{v}$. Applying Lemma 1 to the vertex $\hat{v}$ in H, we have

$$\begin{aligned}m(H,k-2)&=m(H-\hat{v},k-2)+m(H-\hat{v}-\dot{v},k-3)+m(H-\hat{v}-v,k-3)\\&=m(P_{a-1}\cup(G-u-w_1-\dot{u}),k-2)\\&+m(P_{a-1}\cup(G-u-\dot{u}-v-\dot{v}),k-3)\\&+m(P_{a-1}\cup(G-u-\dot{u}-v-w_1),k-3)\end{aligned}$$

where the last equality follows by observing that $H-\hat{v}-\dot{v}\cong P_{a-1}\cup(G-u-\dot{u}-v-\dot{v})$ and $H-\hat{v}-v\cong P_{a-1}\cup(G-u-\dot{u}-v-w_1)$.

Thus the equation (17) continues as

$$\begin{aligned}&m(G_1,k)-m(G_2,k)\\&=\sum_{\dot{u}'\in N_G(\dot{u})\setminus\{u\}} m(P_{a-1}\cup G-u-\dot{u}-\dot{u}'-v,k-3)\\&-\sum_{u'\in N_G(u)\setminus\{\dot{u},w_1\}} m(P_{a-1}\cup(G-u-u'),k-2)\end{aligned}$$

$$
\begin{aligned}
&- m(H, k-2) \\
&- \sum_{\dot{u}' \in N_G(\dot{u}) \setminus \{u\}} m(P_{a-1} \cup (G - u - w_1 - \dot{u} - \dot{u}'), k-3)\,. \qquad (18)
\end{aligned}
$$

By Lemma 5 and Eq. (18), we have

$$
\begin{aligned}
& m(\overline{G_1}, k) - m(\overline{G_2}, k) \\
&= \sum_{\ell \geq 0} (-1)^\ell\, m(K_{n-2\ell}, k-\ell)\,(m(G_1, \ell) - m(G_2, \ell)) \\
&= \sum_{\ell \geq 3} (-1)^\ell\, m(K_{n-2\ell}, k-\ell) \sum_{\dot{u}' \in N_G(\dot{u}) \setminus \{u\}} m(P_{a-1} \cup G - u - \dot{u} - \dot{u}' - v, \ell - 3) \\
&- \sum_{\ell \geq 2} (-1)^\ell\, m(K_{n-2\ell}, k-\ell) \sum_{u' \in N_G(u) \setminus \{\dot{u}, w_1\}} m(P_{a-1} \cup (G - u - u'), \ell - 2) \\
&- \sum_{\ell \geq 2} (-1)^\ell\, m(K_{n-2\ell}, k-\ell)\, m(H, \ell - 2) \\
&- \sum_{\ell \geq 3} (-1)^\ell\, m(K_{n-2\ell}, k-\ell) \sum_{\dot{u}' \in N_G(\dot{u}) \setminus \{u\}} m(P_{a-1} \cup (G - u - w_1 - \dot{u} - \dot{u}'), \ell - 3) \\
&= - \sum_{\dot{u}' \in N_G(\dot{u}) \setminus \{u\}} m(\overline{P_{a-1} \cup G - u - \dot{u} - \dot{u}' - v}, k-3) \\
&- \sum_{u' \in N_G(u) \setminus \{\dot{u}, w_1\}} m(\overline{P_{a-1} \cup (G - u - u')}, k-2) - m(\overline{H}, k-2) \\
&+ \sum_{\dot{u}' \in N_G(\dot{u}) \setminus \{u\}} m(\overline{P_{a-1} \cup (G - u - w_1 - \dot{u} - \dot{u}')}, k-3)\,.
\end{aligned}
$$

Note that $G - u - \dot{u} - \dot{u}' - v$ can be obtained from $G - u - w_1 - \dot{u} - \dot{u}'$ by deleting all edges except vw_t incident with v, then $P_{a-1} \cup G - u - \dot{u} - \dot{u}' - v$ is a proper spanning subgraph of $P_{a-1} \cup (G - u - w_1 - \dot{u} - \dot{u}')$ and thus by Lemma 3, $\overline{P_{a-1} \cup G - u - \dot{u} - \dot{u}' - v} \succ \overline{P_{a-1} \cup (G - u - w_1 - \dot{u} - \dot{u}')}$. This implies that

$$
m(\overline{P_{a-1} \cup (G - u - w_1 - \dot{u} - \dot{u}')}, k) - m(\overline{P_{a-1} \cup G - u - \dot{u} - \dot{u}' - v}, k) \leq 0
$$

for all k. Therefore $m(\overline{G_1}, k) - m(\overline{G_2}, k) \leq 0$ and is less than zero for at least one k, say $k = 2$ or $k = 4$. $\square$

Theorem 7. *Let G be a (not necessarily connected) simple graph and $u, v, w \in V(G)$, where $d_G(u) = 1$, $d_G(w) \geq 2$, and u and w are adjacent to v in G. If $x \in V(G)$ and $x \neq u, v, w$, then $\overline{G + xw} \prec \overline{G + xu}$.*

Proof. By Lemma 1, we have

$$\begin{aligned} m(G+xw,k) &= m(G,k)+m(G-x-w-uv,k-1) \\ &\quad + m(G-x-w-u-v,k-2) \\ m(G+xu,k) &= m(G,k)+m(G-x-u-vw,k-1) \\ &\quad + m(G-x-u-v-w,k-2)\,. \end{aligned}$$

Consequently,

$$\begin{aligned} & m(G+xw,k)-m(G+xu,k) \\ & = m(G-x-w-uv,k-1)-m(G-x-u-vw,k-1) \\ & = -\sum_{w'\in N_G(w)\setminus\{v\}} m(G-x-u-w-w',k-2) \end{aligned}$$

where the last equality follows by applying Lemma 1 of deleting the vertex w in $G-x-u-vw$.

Further by Lemma 3, we have

$$\begin{aligned} & m(\overline{G+xw},k)-m(\overline{G+xu},k) \\ & = \sum_{\ell\geq 0}(-1)^{\ell}\, m(K_{n-2\ell},k-\ell)(m(G+xw,\ell)-m(G+xu,\ell)) \\ & = -\sum_{\ell\geq 2}(-1)^{\ell}\, m(K_{n-2\ell},k-\ell)\sum_{w'\in N_G(w)\setminus\{v\}} m(G-x-u-w-w',\ell-2) \\ & = -\sum_{w'\in N_G(w)\setminus\{v\}}\sum_{\ell'\geq 0}(-1)^{\ell'+2}\, m(K_{n-4-2\ell'},k-2-\ell')m(G-x-u-w-w',\ell') \\ & = -\sum_{w'\in N_G(w)\setminus\{v\}} m(\overline{G-x-u-w-w'},k-2)\,. \end{aligned}$$

Therefore $\overline{G+xu}\succ\overline{G+xw}$ and we are done. □

3 Applications

Unicyclic graphs are connected graphs with equal number of vertices and edges. Obviously unicyclic graphs have a unique cycle. Denote by $\mathcal{U}_n$ the set of unicyclic graphs with n vertices. Denote by $\mathcal{U}_{n,g}$ the set of unicyclic graphs with order n and girth g. $C_g^*(r_1,r_2,\ldots,r_g)$ and $C_g^{'}(r_1,r_2,\ldots,r_g)$ stand for the unicyclic graphs obtained from a cycle $C_g=v_1v_2\ldots v_gv_1$ by attaching r_i pendent edges and a pendent path of length

r_i at v_i respectively, for $i = 1, 2, \ldots, g$. For convenience, $C_g^*(n-g, 0, \ldots, 0)$ and $C_g'(n-g, 0, \ldots, 0)$ are denoted simply by $C_g^*(n-g)$ and $C_g'(n-g)$, respectively.

Theorem 8. *For any graph $G \in \mathcal{U}_n$, we have*

$$\overline{C_3^*(n-3)} \preceq \overline{G} \preceq \overline{C_n}$$

where the left equality holds if and only if $G = C_3^*(n-3)$ and the right equality holds if and only if $G = C_n$.

Proof. For any $G \in \mathcal{U}_n$, assume the unique cycle of G is $C_g = v_1 v_2 \ldots v_g v_1$, and the attached tree at v_i is of order $r_i + 1$, for $i = 1, \ldots, g$, where $g + r_1 + \cdots + r_g = n$. By Theorem 3, when an attached tree is transformed into a path and a star (centered at the root), its matching numbers of the complement increase and decrease accordingly, so we can assume that G is of the form $C_g'(r_1, r_2, \ldots, r_g)$ if $\overline{G}$ is the maximal graph and is of the form $C_g^*(r_1, r_2, \ldots, r_g)$ if $\overline{G}$ is the minimal graph.

By Theorem 1, we know that if a nontrivial pendent path in $C_g'(r_1, r_2, \ldots, r_g)$ is integrated into the cycle, its matching numbers of the complement increase strictly, and so $\overline{C_g'(r_1, r_2, \ldots, r_g)} \preceq \overline{C_n}$. Since the matching numbers of the complement increase strictly in this process, together with the arguments previously, we conclude that $\overline{C_n}$ attains uniquely the maximum matching numbers among all the complements of unicyclic graphs of order n.

Now if $\overline{G}$ is the minimal graph, then G is of the form $C_g^*(r_1, r_2, \ldots, r_g)$ as pointed out preciously. By Theorem 2, $\overline{G} \succ \overline{G(v_i \circ v_{i+1})}$ for any i $(1 \leq i \leq g,\ v_{g+1} = v_1)$ and so we can assume that $G = C_3^*(r_1', r_2', r_3')$ with $r_1' + r_2' + r_3' = n - 3$. Let H denote the graph $C_3^*(r_1', 0, 0)$. Then H with r_2' pendent edges at v_2 and r_3' pendent edges at v_3 is exactly G. Since $H - v_2 \cong H - v_3$, by Theorem 4, we have $\overline{G} \succ \overline{C_3^*(r_1', r_2' + r_3', 0)}$. Using this transformation once again, we show that $\overline{C_3^*(n-3)}$ is the minimum graph with respect to the quasi-order in the set of the complements of unicyclic graphs on n vertices. $\square$

Theorem 9. *For any graph $G \in \mathcal{U}_{n,g}$, we have*

$$\overline{C_g^*(n-g)} \preceq \overline{G} \preceq \overline{C_g'(n-g)}$$

where the left equality holds if and only if $G = C_g^*(n-g)$ and the right equality holds if and only if $G = C_g'(n-g)$.

Proof. For any graph $G \in \mathcal{U}_{n,g}$, assume that the unique cycle of G is $C_g = v_1v_2\dots v_gv_1$, and the attached tree at v_i is of order r_i+1, for $i = 1,\dots,g$, where $g+r_1+\cdots+r_g = n$. As in the proof of Theorem 8, we can always assume that G is of the form $C_g'(r_1,r_2,\dots,r_g)$ if $\overline{G}$ is the maximal element and is of the form $C_g^*(r_1,r_2,\dots,r_g)$ if $\overline{G}$ is the minimal element by Theorem 3.

By Lemma 1, we have

$$\begin{aligned}
&m(C_g^*(r_1,r_2,\dots,r_g),k)\\
&= m(C_g^*(0,r_2,\dots,r_g)\cup r_1P_1,k) + r_1m(C_g^*(0,r_2,\dots,r_g)-v_1\cup(r_1-1)P_1,k-1)\\
&\ \vdots\\
&= m(C_g\cup(n-g)P_1,k) + r_1m(C_g^*(0,r_2,\dots,r_g)-v_1\cup(r_1-1)P_1,k-1)\\
&+\cdots+r_{g-1}m(C_g^*(0,0,\dots,r_g)-v_{g-1}\cup(r_1+\cdots+r_{g-1}-1)P_1,k-1)\\
&+r_gm(P_{g-1}\cup(n-g-1)P_1,k-1)
\end{aligned}$$

and by the same way

$$m(C_g^*(n-g),k) = m(C_g\cup(n-g)P_1,k) + (n-g)m(P_{g-1}\cup(n-g-1)P_1,k-1)\,.$$

By Lemma 3, we have

$$\begin{aligned}
&m(\overline{C_g^*(r_1,r_2,\dots,r_g)},k)\\
&= \sum_{\ell\ge 0}(-1)^\ell m(K_{n-2\ell},k-\ell)m(C_g^*(r_1,r_2,\dots,r_g),\ell)\\
&= m(\overline{C_g\cup(n-g)P_1},k) - r_1m(\overline{C_g^*(0,r_2,\dots,r_g)-v_1\cup(r_1-1)P_1},k-1)\\
&-\dots+r_{g-1}m(\overline{C_g^*(0,0,\dots,r_g)-v_{g-1}\cup(r_1+\dots+r_{g-1}-1)P_1},k-1)\\
&-r_gm(\overline{P_{g-1}\cup(n-g-1)P_1},k-1)
\end{aligned}$$

and

$$m(\overline{C_g^*(n-g)},k) = m(\overline{C_g\cup(n-g)P_1},k) - (n-g)m(\overline{P_{g-1}\cup(n-g-1)P_1},k-1)\,.$$

Observe that $P_{g-1} \cup (n-g-1)P_1$ is a spanning subgraph of $C_g^*(0,\ldots,0,r_{i+1},\ldots,r_g) - v_i \cup (r_1+\ldots+r_i-1)P_1$ for all $i=1,\ldots,g$, and is a proper subgraph if $i \neq g$. Then by Lemma 3, for $i=1,\ldots,g-1$, we have

$$\overline{P_{g-1} \cup (n-g-1)P_1} \succ \overline{C_g^*(0,\ldots,0,r_{i+1},\ldots,r_g) - v_i \cup (r_1+\ldots+r_i-1)P_1}.$$

Thus $m_k(\overline{C_g^*(r_1,r_2,\ldots,r_g)}) \geq m_k(\overline{C_g^*(n-g)})$ and strict inequality holds for some k, say $k=2$. Therefore for any $G \in \mathcal{U}_{n,g}$, $\overline{C_g^*(n-g)} \preceq \overline{G}$ and equality holds if and only if $G = C_g^*(n-g)$.

Next the proof of the maximal case is relatively simple due to Theorem 5 and Theorem 6. First by Theorem 3 we can assume that $G = C_g'(r_1,r_2,\ldots,r_g)$ when $\overline{G}$ is the maximal graph. After applying Theorem 5 or Theorem 6 to any two consecutive pendent paths in G, the resulting graph has larger complement matching numbers but with the number of pendent paths decreased by one. Finally we come to the graph with maximum complement matching numbers and only one pendent path, which is exactly $C_g'(n-g)$. □

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