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Forcing and Anti-Forcing Numbers of $(3, 6)$-Fullerenes*

Lingjuan Shi, Heping Zhang†

School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, P. R. China

shilj15@lzu.edu.cn, zhanghp@lzu.edu.cn

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Dedicated to Professor Fuji Zhang on the occasion of his 80th birthday

Abstract

Let G be a connected graph with at least one perfect matching. The forcing number of G is the smallest number of edges simultaneously contained in a unique perfect matching of G, denoted by $f(G)$. The anti-forcing number of G is the smallest number of edges whose removal from G results in a subgraph with a unique perfect matching, denoted by $af(G)$. In this paper, we obtain that for a $(3, 6)$-fullerene graph G, $f(G) \geq 1$ and $af(G) \geq 2$, and any equality holds if and only if it either has connectivity 2 or is isomorphic to K_4. Further we mainly determine all the $(3, 6)$-fullerenes with the anti-forcing number 3.

1 Introduction

A perfect matching of a graph coincides with a Kekulé structure in organic chemistry. Klein and Randić [9] introduced innate degree of freedom of a Kekulé structure of benzenoid graphs (or hexagonal systems), smallest number of double bonds determining the entire Kekulé structure, which plays an important role in the resonance theory in chemistry (see also [16]). Afterwards this number associated a Kekulé structure was called the forcing number of a perfect matching of a graph by Harary et al. [8]. Zhang and Li [21] determined the benzenoid graphs with a forcing edge (i.e. an edge contained in a unique

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†Corresponding author.

perfect matching). Further in 1997 Li [13] determined the benzenoid graphs with a forcing single edge (i.e. anti-forcing edge). In general, Vukičević and Trinajstić [17,18] introduced the anti-forcing number of a graph as the smallest number of edges whose removal from G results in a subgraph with a unique perfect matching. Algorithms for computing the anti-forcing number of hexagonal chains and double hexagonal chains were designed by Deng [1,2].

Like the forcing number, recently Lei, Yeh and Zhang [14] proposed the anti-forcing number of any single perfect matching of a graph, showed that the forcing number of every perfect matching is no more than the anti-forcing number, and in particular showed that the maximum anti-forcing number of benzenoid graphs always equals the fries number. Klein and Rosenfeld [10] also proposed the same concept under the name "(e)-forcing". Deng and Zhang [3] proved that the anti-forcing numbers of of all perfect matchings of a cata-condensed hexagonal system form an integer interval. When considering the forcing number, the same result no longer holds. By applying Z-transformation graph (or resonance graph), Zhang and Deng [22] proved that the forcing numbers of all Kekulé structures of benzenoid graphs with a forcing edge (or the minimum forcing number one) form either the integer interval from 1 to the Clar number of H or with only the gap 2.

For $k \geq 3$ an integer, a $(k,6)$-fullerene graph is a connected cubic plane (or sphere) graph whose faces are only k-gons and hexagons. The only values of k for which a $(k,6)$-fullerene exists are 3, 4 and 5 (see [4]). A (5,6)-fullerene is the usual fullerene as the molecular graph of sphere carbon fullerene. A $(4,6)$-fullerene is the molecular graph of a boron-nitrogen fullerene.

For a fullerene graph, Zhang et al. [23] showed that the minimum forcing number has at least three and this lower bound can be achieved by infinitely many fullerene graphs. Yang et al. [20] proved that a fullerene graph has the anti-forcing number at least 4. They give a procedure to construct all fullerenes whose anti-forcing numbers achieve the lower bound 4. Furthermore, they showed that, for every even $n \geq 20$ $(n \neq 22, 26)$, there exists a fullerene with n vertices that has the anti-forcing number 4, and the fullerene with 26 vertices has the anti-forcing number 5. For BN-fullerene graphs (or (4,6)-fullerenes), Jiang and Zhang [11] obtained the forcing spectrum of a tubule with cyclic edge-connectivity 3. They also showed that all perfect matchings of any BN-fullerene graph have the forcing numbers at least two and constructed all seven BN-fullerene graphs with the minimum forcing number two.

In this paper we consider the forcing number and anti-forcing number of (3,6)-fullerene graphs. The remainder is organized as follows. In Section 2, we present some basic results and preliminaries about $(3,6)$-fullerenes. In Section 3, we obtain that for a $(3,6)$-fullerene graph, its minimum forcing number is at least one and anti-forcing number is at least two, and each lower bound can be achieved if and only if it has the connectivity 2 or it is isomorphic to K_4 (i.e. it is a nanotube with each end capped a pair of triangles with a common edge). In such case, by the way we show that its forcing spectrum and anti-forcing spectrum are $\text{Spec}_f(G) = \{1, \frac{n}{4}\}$, $\text{Spec}_{af}(G) = \{2, \frac{n}{4}+1\}$ respectively, where n is the number of vertices of this graph. Further, in Section 4, we determine all the $(3,6)$-fullerenes with the anti-forcing number 3.

2 Preliminary

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A *perfect matching* of G is a set of edges M of G such that each vertex is incident with exactly one edge of M. A subset S of a perfect matching M of G *forces* M (or S is a forcing set of M) if S is contained in no other perfect matching of G. The *forcing number* of M, denoted by $f(G,M)$, is the cardinality of a smallest set S that forces M. Let $f(G)$ and $F(G)$, respectively, denote the minimum and maximum values of $f(G,M)$ over all perfect matchings M of G. An edge is called a *forcing edge* if it is contained in a unique perfect matching of G.

For a perfect matching M of G, a subset $S \subseteq E(G) \setminus M$ is called an *anti-forcing set of* M if $G - S$ has a unique perfect matching M. The cardinality of a smallest anti-forcing set of M is called the *anti-forcing number of* M, denoted by $af(G,M)$. The anti-forcing number of G is the minimum value over anti-forcing numbers of all perfect matchings of G, denoted by $af(G)$. Whereas, $Af(G)$:=$max\{af(G,M) : M$ is a perfect matching of $G\}$ is defined as the *maximum anti-forcing number* of G.

The *forcing spectrum* and *anti-forcing spectrum* of a graph G are respectively defined as Spec_f(G)=$\{f(G,M) : M$ is a perfect matching of $G\}$, and $\text{Spec}_{af}(G) = \{af(G,M) : M$ is a perfect matching of $G\}$.

For a $(3,6)$-fullerene G, by Euler's formula it has exactly four faces of size 3 and $\frac{n}{2} - 2$ faces of size 6, and has the connectivity 2 or 3.

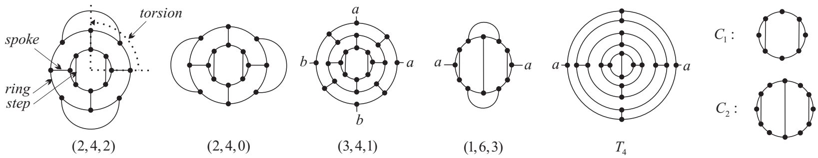

Figure 1: Five (3, 6)-fullerenes and two caps C_1 and C_2.

Let T_l ($l \geq 1$) be the tube consisting of l cyclic chains each of two hexagons, capped on each end by a cap of two adjacent triangles. For example, T_4 is depicted in Fig. 1. Note that the degenerate case T_0 (i.e. K_4, the complete graph with 4 vertices) is allowed sometimes. Yang and Zhang obtained the following result.

Theorem 2.1. [19] *A* $(3,6)$*-fullerene* G *has the connectivity* 2 *if and only if* $G \cong T_l$ *for some integer* $l \geq 1$.

If a (3,6)-fullerene G other than K_4 has the connectivity 3, then its structure is also a tube determined only by three parameters r, s, t, where $r \geq 1$ is the radius (number of rings), s is the size (number of spokes in each layer and $s \geq 4$ is even) and t is the twist (torsion, $0 \leq t \leq s$, $t \equiv r \pmod 2$) [5–7]. So we denote it by (r, s, t). For example, (2, 4, 0), (2, 4, 2), (3, 4, 1), (1, 6, 3) are depicted in Fig. 1, C_1 is a cap of (2, 4, 0), (2, 4, 2) and (3, 4, 1), and C_2 is a cap of (1, 6, 3). In other words, a 3-connected $(3,6)$-fullerene (r, s, t) consists of $r - 1$ concentric layers of hexagons (i.e. each layer is a cyclic chain of s hexagons) and two caps with torsion t on ends. If we find a cap with k steps in G, then $G \cong (r, 2k, t)$. This result will be applied repeatedly in Sections 4 and 5.

For a graph G with $X \subseteq V(G)$ and $S \subseteq E(G)$, let $G[X]$ and $G[S]$ denote the subgraph of G induced by X and S respectively. Let $e(X)$ be the number of edges in $G[X]$. Let ∂X be the set of edges with only one end in X. Then ∂X is an edge-cut of G. An edge-cut S of G is *trivial* if its edges are all incident with the same vertex, and *cyclic* if $G - S$ has at least two components, each containing a cycle. A cyclic k-edge-cut is trivial if its deletion gives a cycle of length k as one component. The *cyclic edge-connectivity* of G, denoted by $c\lambda(G)$, is the minimum size of cyclic edge-cuts of G. We say G is cyclically k-edge-connected if $k \leq c\lambda(G)$.

Lemma 2.2. [19] *Let* G *be a 3-connected* $(3,6)$*-fullerene graph other than* K_4*. Then the four triangles of* G *are pairwise nonadjacent; and each triangular face of* G *is adjacent to* *three hexagons, both of which intersect at exactly one edge; every cyclic 3-edge-cut of G is trivial.*

Lemma 2.3. *Suppose that S is a 3-edge-cut of a 3-connected $(3,6)$-fullerene graph G other than K_4. Then S is a trivial edge-cut or a trivial cyclic 3-edge-cut of G.*

Proof. There exist a proper subset $X \subset V(G)$ such that $S = \partial X$. Without loss of generality, assume that $|X| \leq |\overline{X}|$, where $\overline{X} = V(G) \setminus X$. Note that $e(X) = \frac{3|X|-3}{2} = |X| + \frac{|X|-3}{2}$. If S is not a trivial edge-cut, then $|X| \geq 3$ since $|X|$ is odd. So $e(X) \geq |X|$. This implies that $G[X]$ and thus $G[\overline{X}]$ each contains a cycle. That is, S is a cyclic 3-edge-cut of G. Further, By Lemma 2.2 S is a trivial cyclic edge-cut. □

Lemma 2.4. *Let G be a 3-connected $(3,6)$-fullerene graph other than K_4. Then neither 4-cycles nor 5-cycles exist in G, and each 3-cycle or 6-cycle is the boundary of a face.*

Proof. Since G is 3-connected, each 3-cycle bounds a face. If G has a 4-cycle C, then by Lemma 2.2 C is an induced cycle of G. By definition C is not the boundary of a face. This implies that G has a 2 or 1-edge-cut, contradicting the 3-connectivity of G. Thus G has no 4-cycles. Similarly, we have that G has no 5-cycles.

Let C be a 6-cycle in G. Since G has no 4-cycles or 5-cycles, C is an induced cycle in G. We shall prove that C bounds a face of G. If not, then since G is 3-connected and C has 6 vertices, ∂X is disjoint union of two 3-edge-cuts of G, where $X = V(C)$, which lie outside and inside C respectively. By Lemma 2.3, any such 3-edge-cut is trivial edge-cut or trivial cyclic edge-cut of G. Either case gives the existence of 4-cycle or 5-cycle or a pair of adjacent triangles. This contradiction shows that any 6-cycle bounds a face of G. □

The following classical results are useful in our discussions.

Theorem 2.5. [15] *Let G be an r-regular and $(r-1)$-edge-connected graph with an even number of vertices. Then the deletion of any $r-1$ edges of G results in a graph with a perfect matching.*

Theorem 2.6. [12] *Let G be a connected graph with a unique perfect matching. Then G has a cut-edge belonging to the perfect matching.*

3 Lower bounds for $f(G)$ and $af(G)$ of (3,6)-fullerenes

Lemma 3.1. *If G is a $(3,6)$-fullerene graph, then $f(G) \geq 1$ and $af(G) \geq 2$.*

Proof. Since G is 2-connected cubic graph, Theorem 2.5 implies that each edge of G is contained in a perfect matching. So G has distinct perfect matchings and $f(G) \geq 1$. Further deletion of any edge of G results in a subgraph with at least two perfect matchings. So $af(G) \geq 2$. □

Let G be a graph with a perfect matching M. A cycle of G is M-alternating if its edges alternate in M and $E(G) \setminus M$. It is obvious that M is a unique perfect matching of G if and only if G has no M-alternating cycles. Let $c(M)$ denote the maximum number of disjoint M-alternating cycles of G. A *compatible M-alternating set* of G is a set of M-alternating cycles such that any two members are either disjoint or intersect only at edges in M. Let $c'(M)$ denote the maximum cardinality of compatible M-alternating sets of G.

Lemma 3.2. [14] *$f(G, M) \geq c(M)$ and $af(G, M) \geq c'(M)$.*

Let M_0 be the set of edges consisting of all spokes of a (3,6)-fullerene graph G (T_l or (r, s, t)) and all steps in two caps. Then it is a perfect matching of G. Note that M_0 is fixed in this section.

Theorem 3.3. *Let G be a $(3,6)$-fullerene graph $T_l, l \geq 0$, with n vertices. Then* $\mathrm{Spec}_f(G) = \{1, \frac{n}{4}\}$ *and* $\mathrm{Spec}_{af}(G) = \{2, \frac{n}{4} + 1\}$.

Proof. For T_0, the theorem holds obviously. Next we suppose that $G \cong T_l, l \geq 1$. Since each M_0-alternating cycle of T_l contains all edges of M_0, each edge in M_0 is a forcing edge of M_0 and any pair of adjacent edges not in M_0 form an anti-forcing set of M_0. Together with Lemma 3.1 we have $f(T_l, M_0) = 1$ and $af(T_l, M_0) = 2$.

For any perfect matching M of G other than M_0, it follows that $M \cap M_0 = \emptyset$. Clearly, $G - M_0$ consists of $l+1$ concentric 4-cycles, which are disjoint M-alternating. By Lemma 3.2 $f(T_l, M) \geq l + 1 = \frac{n}{4}$. Choosing exactly one edge of M from each concentric 4-cycle, we obtain an edge set $S \subseteq M$. It is obvious that S forces M and $|S| = \frac{n}{4}$. So $f(G, M) = \frac{n}{4}$.

Further, $M \cup M_0$ forms an M-alternating cycle, denoted by C. Then C and the $l + 1$ concentric 4-cycles of T_l form a compatible M-alternating set. By Lemma 3.2 $af(G, M) \geq c'(M) \geq l + 2 = \frac{n}{4} + 1$. Let S' be a set of edges consisting of one edge from

M_0 and one edge not in M from each concentric 4-cycle. Since $G-S'$ has no M-alternating cycles, S' is an anti-forcing set of M. Thus $af(G,M)=\frac{n}{4}+1$. □

From the proof we can see that $T_l, l\geq 1$, has exactly $2^{l+1}+1$ perfect matchings, where exactly one perfect matching has forcing number 1 and anti-forcing number 2 and 2^{l+1} perfect matchings have forcing number $l+1$ and anti-forcing number $l+2$.

A *patch* of a (3, 6)-fullerene graph G is a 2-connected subgraph of G whose all interior faces are hexagons and triangles and all vertices not on the outer face have degree 3.

Lemma 3.4. *If G is a $(3,6)$-fullerene graph other than T_l, $l\geq 0$, then we have*
(i) $G-\{v_1,v_2\}$ is 2-edge-connected for $e=v_1v_2$ which is not an edge of a triangle;
(ii) $G-\{v_1,v_2,v_3,v_4\}$ is 2-edge-connected for $\{v_1,v_2,v_3,v_4\}$ which induce T' (see Fig. 2) in G.

Proof. (i) Since G is a 3-connected cubic plane graph, each edge of G belongs to exactly two faces. Suppose e is shared by two hexagonal faces f' and f'' of G. Let f_1 and f_2 be the facial cycles in G sharing exactly one endvertex of e. So f_1 and f_2 are disjoint. Otherwise, a nontrivial cyclic 3-edge-cut would happen, contradicting Lemma 2.2. Hence the boundary of each face in $G-\{v_1,v_2\}$ is a cycle, and $G-\{v_1,v_2\}$ is 2-edge-connected.
(ii) Suppose that $G[\{v_1,v_2,v_3,v_4\}]\cong T'$. By Lemma 2.2 G has patch B as shown in

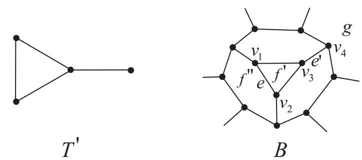

Figure 2: Illustration for the proof of Lemma 3.4.

Fig. 2. If g is a triangle, then clearly g and f'' are disjoint. If g is a hexagon, then g and f'' are also disjoint, otherwise, G has a nontrivial cyclic 3-edge-cut, a contradiction. So the boundary of each face in $G-\{v_1,v_2,v_3,v_4\}$ is a cycle and $G-\{v_1,v_2,v_3,v_4\}$ is 2-edge-connected. □

Lemma 3.5. *If G is a $(3,6)$-fullerene graph other than T_l, $l\geq 0$, then $f(G)\geq 2$.*

Proof. By Lemma 3.1 it suffices to show that any edge $e=v_1v_2$ of G cannot be a forcing edge. That is, $G-\{v_1,v_2\}$ has at least two perfect matchings. Since G is a 3-connected cubic plane graph, any two faces of G share at most one edge and each edge belongs to exactly two faces. Let f' and f'' be distinct faces of G which share edge e.

If both f' and f'' are hexagons, then $G - \{v_1, v_2\}$ is 2-edge-connected by Lemma 3.4 and thus has at least two distinct perfect matchings by Theorem 2.6. Since G has no adjacent triangles, next we may suppose that exactly one of f' and f'' is a triangle, say f'. By Lemma 2.2 G has patch B as shown in Fig. 2. $G-\{v_1, v_2, v_3, v_4\}$ is 2-edge-connected by Lemma 3.4 and has at least two distinct perfect matchings. $\square$

The following example shows that the lower bound of Lemma 3.5 is sharp.

Lemma 3.6. *Let M be any perfect matching of the $(3,6)$-fullerene graph $(2,4,2)$. Then $f(G,M) = 2$.*

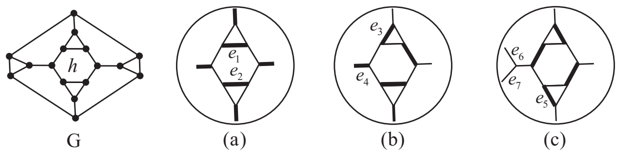

Figure 3: Illustration for the proof of Lemma 3.6 (thick edges belong to M).

Proof. Let G be the (3,6)-fullerene (2,4,2) with the specific hexagon h as shown in Fig. 3. A hexagon of G is called *resonant* if it is an alternating cycle with respect to some perfect matching M of G. It is obvious that every hexagon of G is not resonant. So h has at least two vertices which is covered by the edges in M but not in h. In fact h has exactly two such vertices. By the symmetry, we only need to consider the three cases (a)-(c) in Fig. 3.

For Fig. 3(a), $\{e_1, e_2\}$ is a forcing set of M; For Fig. 3(b), $\{e_3, e_4\}$ is a forcing set of M; For Fig. 3(c), $\{e_5, e_6\}$ or $\{e_5, e_7\}$ is a forcing set of M. So $f(G,M) \leq 2$. By Lemma 3.5, we conclude that $f(G,M) = 2$. $\square$

For a $(3,6)$-fullerene graph G other than T_l, we have $af(G) \geq 2$. The following result will show that the anti-forcing number has a larger lower bound.

Lemma 3.7. *If G is a $(3,6)$-fullerene graph other than T_l, $l \geq 0$, then $af(G) \geq 3$.*

Proof. Suppose to the contrary that G has an anti-forcing set $S = \{e_1, e_2\}$. Let M be a unique perfect matching of $G - S$. Then $G - S$ is connected and has a cut-edge e in M by Theorem 2.6. So $S' := \{e_1, e_2, e\}$ is an edge-cut of G. We only need to consider the following two cases by Lemma 2.3:

Case 1: S' is a trivial edge-cut of G. Since $f(G) \geq 2$ by Lemma 3.5, single edge e is contained in at least two perfect matchings M_1 and M_2 of G. Since e_1 and e_2 are adjacent to e, neither e_1 nor e_2 belong to M_1 (resp. M_2). So M_1 and M_2 are two perfect matchings of $G - S$, contradicting that S is an anti-forcing set of G.

Case 2: S' is a trivial cyclic 3-edge-cut of G. Let e_3 be the edge in a triangle which is adjacent to both e_1 and e_2. Then $e_1, e_2 \notin M$ and $e \in M$ imply that $e_3 \in M$. By Lemma 3.5, we can find at least two distinct perfect matchings M_3 and M_4 of G such that e_3 belongs to both M_3 and M_4. So $e_1, e_2 \notin M_i, i = 3, 4$. That is, M_3 and M_4 are two distinct perfect matchings of $G - S$, a contradiction. □

In summary, by Theorem 3.3 and Lemmas 3.7 and 3.5 we obtain the following theorem.

Theorem 3.8. *Let G be a $(3,6)$-fullerene graph. Then the following statements are equivalent: (i) $f(G) = 1$, (ii) $af(G) = 2$, and (iii) $G \cong T_l$ for some integer $l \geq 0$.*

4 All $(3,6)$-fullerenes with the anti-forcing number 3

In this section we mainly determine all the $(3,6)$-fullerenes with the anti-forcing number 3. At first we describe the structure of a minimum anti-forcing set as follows.

Lemma 4.1. *Let G be a 3-connected $(3,6)$-fullerene graph other than K_4 and $S = \{e_1, e_2, e_3\}$ be an anti-forcing set. Then $G[S]$ is neither a triangle nor star $K_{1,3}$.*

Proof. Since $G[S]$ is not isomorphic to star $K_{1,3}$, it suffices to show that it is not a triangle. We can check that $(2,4,0)$, $(2,4,2)$ and $(1,6,3)$ has no such anti-forcing set (see Fig. 1). Next, we consider other $(3,6)$-fullerenes. Let M be a unique perfect matching of $G - S$.

If $G[S]$ is a triangle, then G has patch P_1 as shown in Fig. 4, and $\{u_1v_1, u_2v_2, u_3v_3\} \subset M$. If f_1, f_2 and f_3 are pairwise disjoint faces, then $G - \{u_1, v_1, u_2, v_2, u_3, v_3\}$ is 2-edge-connected since each face is bounded by a cycle, and has more than one perfect matchings by Theorem 2.6, a contradiction. So suppose that f_1 and f_2 share one edge. Then G has patch P_2. So $G \cong (r, 4, t)$, $r \geq 3$ (for $r = 2$, G is either $(2,4,0)$ or $(2,4,2)$),

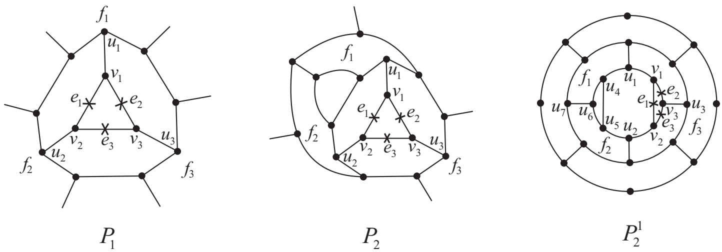

Figure 4: Illustration for the proof of Theorem 4.1.

and G has patch P_2^1. Further, $\{u_1v_1, u_2v_2, u_3v_3, u_4u_5, u_6u_7\} \subset M$. We can check that $G - \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, u_5, u_6, u_7\}$ is 2-edge-connected and thus has more than one perfect matchings. A contradiction would occur. $\square$

Lemma 4.2. *Let G be a 3-connected $(3,6)$-fullerene graph. Then G has subgraph T as shown in Fig. 5 if and only if for some positive integer k, G is isomorphic to either $(2,s,0)$, $s = 2k \geq 4$, or $(1,s,3)$, $s = 2k \geq 8$, or $(1,s,\frac{s}{2})$, $s = 4k+2 \geq 6$.*

Proof. It is sufficient to show the necessity. Since G is a 3-connected $(3,6)$-fullerene graph, $G \cong (r,s,t)$, where $r \geq 1$, $s = 2k \geq 4$, $t \equiv r (\text{mod } 2)$ and $0 \leq t \leq \frac{s}{2}$. So G has the cap as shown in Fig. 5, the length of the cycle C_1 is at least 8, and the distance between v_i and u_i on C_1 is at least 2 for $i = 1$ or 3. If v_2 is adjacent to u_2, then

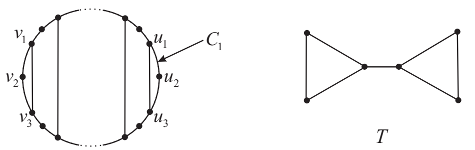

Figure 5: Illustration for the proof of Lemma 4.2.

$|V(G)| = 2rs = 2s = 2 + 2 \times (4k+1)$ and $G \cong (1,s,\frac{s}{2})$, $s = 4k+2 \geq 6$. If v_2 and u_2 are not adjacent, then by the 3-connectivity of G, $G \cong (1,s,3)$, $s = 2k \geq 8$ or $(2,s,0)$, $s = 2k \geq 4$. $\square$

Now, we obtain the following main result in this paper.

Theorem 4.3. *Let G be a $(3,6)$-fullerene graph. Then $af(G) = 3$ if and only if G is isomorphic to $(2,s,0)$, $s = 2k \geq 4$, or $(1,s,3)$, $s = 2k \geq 8$, or $(1,s,\frac{s}{2})$, $s = 4k+2 \geq 6$, or $(2,4,2)$, or $(3,4,1)$.*

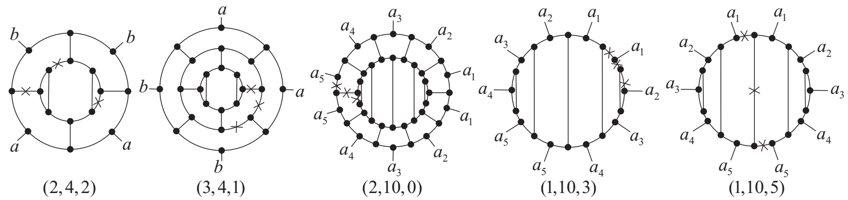

Figure 6: Five (3,6)-fullerene graphs with the anti-forcing number 3.

Proof. We first consider the sufficiency. For such graph G, $af(G) \geq 3$ by Theorem 3.8. If $G \cong (2,4,2)$ or $(3,4,1)$, then G has an anti-forcing set of 3 edges, marked by crosses as shown in Fig. 6. For the other three cases, each of such three graphs has an anti-forcing set of size 3 (for example, see Fig. 6 in case $s = 10$). Hence $af(G) = 3$.

Conversely, by Theorem 3.8 suppose that G is 3-connected and is not isomorphic to K_4 and $S = \{e_1, e_2, e_3\}$ is an anti-forcing set of G. Then it follows that $G-S$ is connected. Let M be a unique perfect matching of $G-S$. By Theorem 2.6 $G-S$ has a cut-edge $e = v_1v_2$ in M. Let G_1 and G_2 be the two components of $G - S \cup \{e\}$ with $|V(G_1)| \leq |V(G_2)|$, $v_i \in V(G_i), i = 1, 2$.

Since G_1 has odd number of vertices, $S' = S \cup \{e\}$ contains properly a 3-edge-cut of G, say $\{e_1, e_2, e\}$. By Lemma 2.3 $\{e_1, e_2, e\}$ is a trivial edge-cut or a trivial cyclic 3-edge-cut of G. The remaining proof is based on the following two lemmas.

Lemma 4.4. *If* $\{e_1, e_2, e\}$ *is a trivial cyclic* 3*-edge-cut of* G*, then* G *has subgraph* T *or* $G \cong (2,4,2)$.

Lemma 4.5. *If* $\{e_1, e_2, e\}$ *is a trivial edge-cut of* G*, then* G *has subgraph* T*, or* $G \cong (2,4,2)$ *or* $(3,4,1)$.

By Lemmas 4.4 and 4.5, the necessary holds. □

Next, we prove Lemma 4.4, but the proof of Lemma 4.5 is tedious and put in the final section.

Proof of Lemma 4.4. By Lemma 2.2 the four triangles of G are pairwise nonadjacent, so G has at least 12 vertices and $|V(G_2)| \geq 6$. Suppose that $\{e_1, e_2, e\}$ is a trivial cyclic 3-edge-cut of G. Then $G[V(G_1)]$ is a triangle, and G has patch A (see Fig. 7) by Lemma 2.2. So $\{v_3v_4, e\} \subset M$. Further, $e_3 \in G - \{v_1, v_2, v_3, v_4\}$, otherwise, $G - \{v_1, v_2, v_3, v_4\}$ is 2-edge-connected by Lemma 3.4 and has at least two perfect matchings, a contradiction.

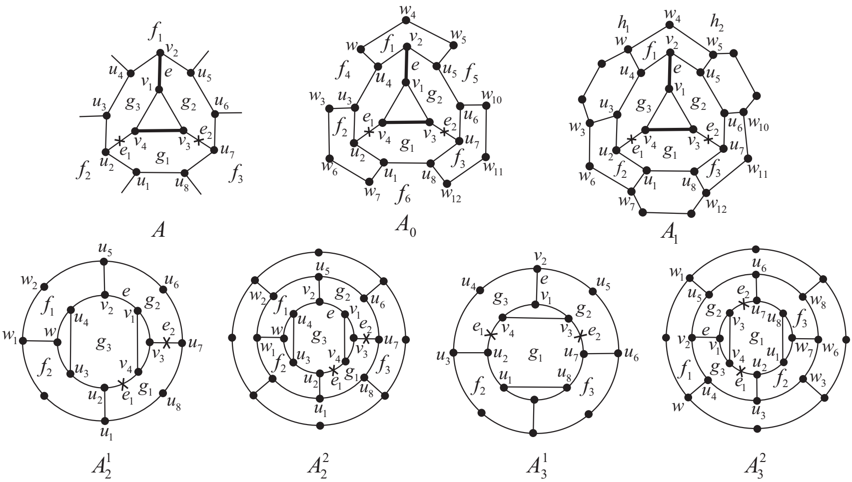

Figure 7: Illustration for the proof of Lemma 4.4.

Hence $G_2 - v_2 = G - \{v_1, v_2, v_3, v_4\} - e_3$ is connected and has a unique perfect matching $M_0 = M\mid_{G_2 - v_2}$. So it has a cut-edge e' in M_0. The two connected components of $G_2 - v_2 - e'$ are denoted by G'_1 and G'_2. Then $\{e', e_3\}$ is an edge-cut of $G - \{v_1, v_2, v_3, v_4\}$. For a subgraph H of G, $\partial_G(H)$ denotes a set of edges each of which has exactly one endvertex in H. We may assume that $|\partial_G(G'_1)| = 3$ and $|\partial_G(G'_2)| = 5$ since $|\partial_G(G'_1)|$ and $|\partial_G(G'_2)|$ are odd, $|\partial_G(G'_1)| + |\partial_G(G'_2)| = 8$, and $|\partial_G(G'_i)| \geq 3$ by the 3-connectivity of G, $i = 1, 2$. Then $\partial_G(G'_1) = \{e_3, e', a\}$, where $a \in \{e_1, e_2, v_2u_4, v_2u_5\}$. By Lemma 2.3 $\{e_3, e', a\}$ is a trivial edge-cut or a trivial cyclic 3-edge-cut of G.

Now, we suppose that G does not contain subgraph T. Then f_1, f_2 and f_3 are hexagons.

If f_1, f_2 and f_3 are mutually disjoint, then G has patch A_0, and f_4, f_5 and f_6 are distinct hexagons (see Fig. 7). It follows that f_4 and f_3 are disjoint. Otherwise, the shared edge is either $w_{12}w_{11}$ or $w_{11}w_{10}$. Further, either $w_3w_{12} \in E(G)$ or $ww_{10} \in E(G)$. So G has a 6-cycle which is not the boundary of a face, a contradiction by Lemma 2.4. So the other two vertices on f_4 are not in $V(A_0)$. Similarly, f_6 is disjoint with f_1 and f_4, otherwise, G has 4-cycle or 6-cycle which is not the boundary of a face, and a contradiction occurs again. So f_6 has two vertices in $V(G) - V(A_0) \cup V(f_4)$. Further, f_5 has two vertices in $V(G) - V(A_0) \cup V(f_4) \cup V(f_6)$. Then G has patch A_1. So $\{e_3, e', a\}$ is a trivial edge-cut and $e_3 \in \{u_4w, u_4u_3, u_2u_1, u_2u_3, u_7u_8, u_7u_6, u_5u_6, u_5w_5\}$. For each case of e_3, we can show that $G - S$ has at least two perfect matchings, a contradiction. Here, we take $e_3 = u_1u_2$ as an example. Then $\{v_3v_4, e, u_3u_2, u_4w\} \subset M$, $G - \{v_3, v_4, v_1, v_2, u_3, u_2, u_4, w\}$ is 2-edge-connected since each face is bounded by a cycle and has more than one perfect matchings by Theorem 2.6, a contradiction.

If f_1 and f_2 share one edge, then G has patch A_2^1. So $G \cong (r, 4, t)$, $r \geq 2$. If $r = 2$, then $G \cong (2, 4, 2)$. For $r \geq 3$, G has patch A_2^2. Now, $e_3 \in \{ww_1, wu_4, u_4u_3, u_5w_2, u_5u_6, u_7u_6, u_7u_8, u_2u_1, u_2u_3\}$. For each case of e_3, we can confirm that $G-S$ has more than one perfect matchings , a contradiction. For the case that f_1 and f_3 share one edge, the analogous arguments as above can be implemented.

If f_2 and f_3 share one edge, G has patch A_3^1. So $G \cong (r, 4, t)$, $r \geq 2$. If $r = 2$, then $G \cong (2, 4, 2)$. For $r \geq 3$, G has patch A_3^2, and $e_3 \in \{u_5w_1, u_5u_6, u_7u_6, u_7u_8, u_2u_1, u_2u_3, u_4u_3, u_4w\}$. We can conclude a similar contradiction for each case of e_3. ■

5 Proof of Lemma 4.5

Suppose that $\{e_1, e_2, e\}$ is a trivial edge-cut of G. Then G_1 is a single vertex v_1. If exactly one of two faces sharing edge e is a triangle, then G has patch B as shown in Fig. 8. Since S and $\{v_2u_1, e_2, e_3\}$ anti-force the same perfect matching M, we can demonstrate that G has subgraph T or $G \cong (2, 4, 2)$ by Lemma 4.4.

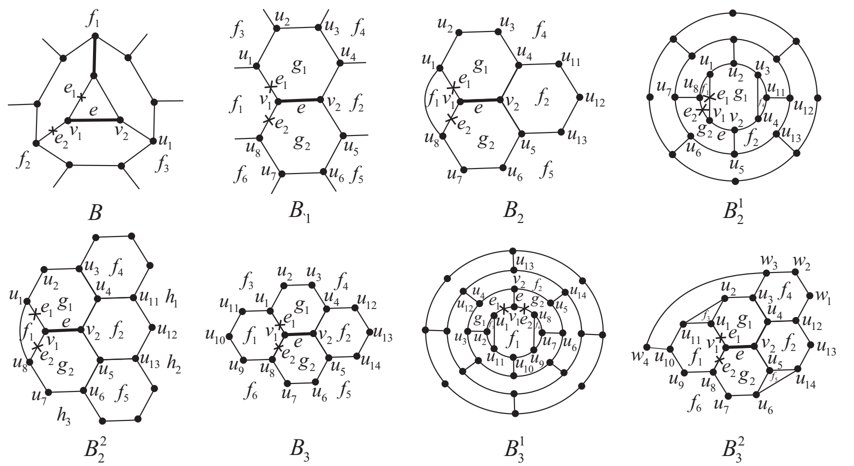

Figure 8: Illustration for the proof of Lemma 4.5 (I).

Now we suppose that the two faces sharing edge e are hexagons. Then G has patch B_1 as shown in Fig. 8. Since $G-\{v_1,v_2\}$ is 2-edge-connected by Lemma 3.4, $e_3 \in G-\{v_1,v_2\}$ and $G' := G-\{v_1,v_2\}-e_3$ is connected and has a unique perfect matching $M\backslash\{e\}$. By Theorem 2.6 G' has a cut-edge e' in $M\backslash\{e\}$. So $\{e_3,e'\}$ is an edge-cut of $G-\{v_1,v_2\}$. By analogous argument as in the proof of Lemma 4.4, $\{e_3,e',b\}$ is a trivial edge-cut or a trivial cyclic edge-cut of G, where $b \in \{e_1,e_2,v_2u_4,v_2u_5\}$. We need a larger patch for considering e_3. In the following we suppose that G does not contain subgraph T. Then at most one of f_1 and f_2 is a triangle.

Firstly, we presume that exactly one of f_1 and f_2 is a triangle, say f_1. Then G has patch B_2. By Lemma 3.4 $G-\{v_1,v_2,u_1,u_8\}$ is 2-edge-connected, so $e_3 \notin \{u_1u_2,u_7u_8\}$. By Lemma 4.1, $e_3 \neq u_1u_8$. Hence $e_3 \in G-\{v_1,v_2,u_1,u_8\}$. If f_4 or f_5 is a triangle, then $G \cong (r,4,t)$, $r \geq 2$. For $r=2$, $G \cong (2,4,2)$. For $r \geq 3$, G has patch B_2^1. So $e_3 \in \{u_2u_3,u_3u_4,u_4u_{11},u_5u_6,u_5u_{13},u_{11}u_{12}\}$. For each case of e_3, $G-S$ has at least two perfect matchings, a contradiction. So we may suppose that both f_4 and f_5 are hexagons. Then G has patch B_2^2 and $e_3 \in \{u_3u_4,u_4u_{11},u_5u_6,u_5u_{13}\}$. We can verify that $G-S$ has at least two perfect matchings for each case of e_3, a contradiction.

Next, we suppose that both f_1 and f_2 are hexagons. Then G has patch B_3 as shown in Fig. 8. Note that f_3, f_4, f_5 and f_6 are distinct faces, and at most one of f_3 and f_4 (resp. f_5 and f_6) is a triangle. Since S and $\{v_2u_4,v_2u_5,e_3\}$ anti-force the same perfect matching M, it is sufficient to consider the following four cases.

Case 1: Both f_3 and f_6 are triangles. Then $e_3 \in \{u_1u_{11},u_1u_2,u_2u_3,u_3u_4,u_4u_{12},u_5u_{14}, u_5u_6,u_6u_7,u_8u_7,u_8u_9,u_9u_{10},u_{10}u_{11}\}$ and $G \cong (r,4,t)$, $r \geq 3$. For $r=3$, $G \cong (3,4,1)$. For $r \geq 4$, G has patch B_3^1. Then $G-S$ has at least two perfect matchings for each case of e_3, a contradiction.

Case 2: Both f_3 and f_5 are triangles. Then $e_3 \in \{u_{10}u_{11},u_1u_{11},u_1u_2,u_2u_3,u_3u_4,u_4u_{12}, u_{13}u_{14},u_5u_{14},u_5u_6,u_6u_7,u_8u_7,u_8u_9\}$ and G has patch B_3^2. Since G does not contain subgraph T, $w_4u_9 \notin E(G)$. So f_4 and f_6 are disjoint, and G has patch $B_3^{2,1}$ (see Fig. 9). If f_7 and f_6 share an edge, i.e. w_5w_6, then $w_6w_8 \in E(G)$, $w_2w_4 \in E(G)$ and $G \cong (3,4,1)$. If f_7 and f_6 are disjoint, then G has patch $B_3^{2,2}$. The similar contradiction occurs for each case of e_3.

Case 3: f_3 is a triangle, and f_4, f_5 and f_6 are hexagons. Then $e_3 \in \{u_1u_2,u_2u_3,u_3u_4,u_4 u_{12},u_5u_{14},u_5u_6,u_7u_8,u_8u_9,u_1u_{11},u_{10}u_{11}\}$. f_4 and f_5 are disjoint, so G has patch B_3^3. Since

G does not contain subgraph T, $w_4u_9 \notin E(G)$. So f_4 and f_6 are disjoint.

If f_5 and f_6 share an edge, then G has patch $B_3^{3,1}$. We assume that h_3 and f_4 are disjoint, otherwise, w_2w_4, w_1w_8 and w_5u_{13} are edges of G and $G \cong (3,4,1)$. Further h_3 and f_2 are disjoint since h_3, h_1 and h_2 are distinct faces of G. So each face of $G - \{v_1, v_2, u_8, u_9\}$ is bounded by a cycle, and thus $G - \{v_1, v_2, u_8, u_9\}$ is 2-edge-connected. For $e_3 = u_7u_8$, $\{e, u_8u_9\} \subset M$, so $G - \{v_1, v_2, u_8, u_9\}$ and then $G - S$ has at least two perfect matchings, a contradiction. The similar contradiction would occur for $e_3 \in \{u_5u_6, u_3u_4, u_4u_{12}, u_1u_2, u_{10}u_{11}, u_1u_{11}, u_2u_3, u_8u_9, u_5u_{14}\}$.

If f_5 and f_6 are disjoint, then G has patch $B_3^{3,2}$. If h_3 and f_4 are disjoint, then $G - S$ has at least two perfect matchings for each case of e_3. If h_3 and f_4 share an edge, then $w_2w_4 \in E(G)$ and $G \cong (r,4,t)$, $r \geq 4$. We always obtain similar contradiction for each case of e_3.

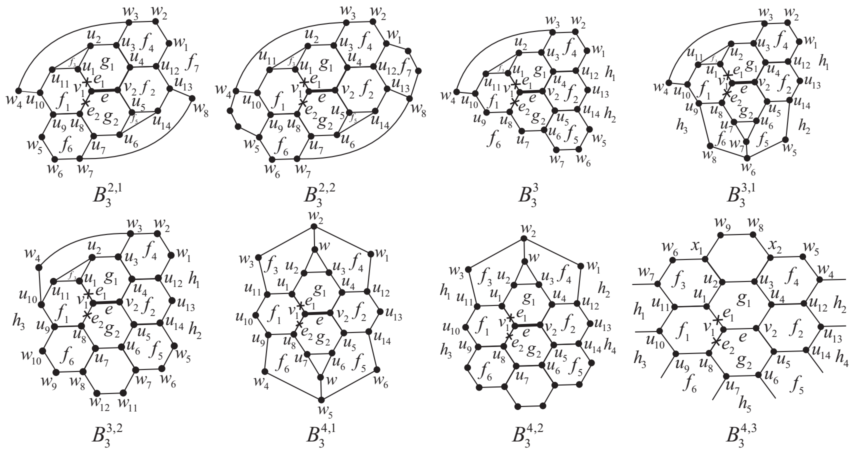

Figure 9: Illustration for the proof of Lemma 4.5 (II).

Case 4: f_3, f_4, f_5 and f_6 are hexagons. Then $e_3 \in \{u_1u_2, u_1u_{11}, u_4u_3, u_4u_{12}, u_5u_{14}, u_5u_6, u_8u_7, u_8u_9\}$.

If f_3 and f_4 share an edge, and f_5 and f_6 share an edge, then G has patch $B_3^{4,1}$ and $G \cong (r,6,t)$, $r \geq 2$. For $r = 2$, $G \cong (2,6,2)$ or $(2,6,0)$. In fact $(2,6,2) \cong (3,4,1)$ and $(2,6,0)$ has subgraph T. For $r \geq 3$, $G - S$ has more than one perfect matchings for each case of e_3, a contradiction.

If f_3 and f_4 share an edge, and f_5 and f_6 are disjoint, then G has patch $B_3^{4,2}$. Since G

does not contain subgraph T, h_1 and h_2 are distinct faces of G. So $u_{10}u_{13} \notin E(G)$, and h_3 and h_4 are distinct faces. Further h_1 is different from h_4, and h_2 is different from h_3 since G has no 5-cycles by Lemma 2.4. For each case of e_3, we can show that S does not anti-force M.

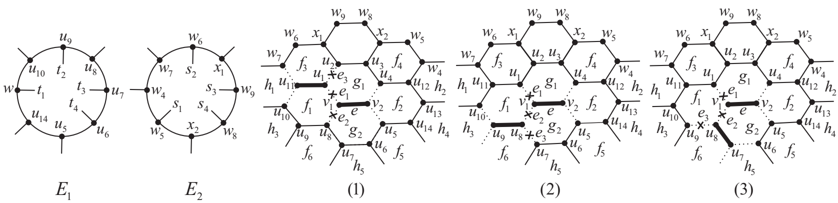

Figure 10: Edge-cuts E_1 and E_2, and some cases of e_3 in patch $B_3^{4,3}$.

Now, we consider the remaining case: f_3 and f_4 are disjoint, and f_5 and f_6 are disjoint. Then G has patch $B_3^{4,3}$ as shown in Fig. 9. We claim that h_1 is not h_4 or h_2, and h_3 is not h_4. Suppose that $h_1 = h_4$. Then $w_7u_{13} \in E(G)$ and u_{10} and u_{14} are connected by a path $u_{10}wu_{14}$ of length 2. Now G has a 4-edge-cut $E_1 = \{t_1, t_2, t_3, t_4\}$ as shown in Fig. 10. In an inductive way we have that G has infinite many vertices, a contradiction. If $h_1 = h_2$ or $h_3 = h_4$, then $u_{10}u_{13}$ and w_7w_4 are edges of G and G has a 4-edge-cut $E_2 = \{s_1, s_2, s_3, s_4\}$ as shown in Fig. 10. So G either has subgraph T or has infinite many vertices, a contradiction.

We show four cases of e_3 as follows, the other four cases could be proved similarly. For $e_3 = u_1u_{11}$, we can easily obtain that $G - S$ has more than one perfect matchings. For $e_3 = u_1u_2$ as shown in Fig. 10(1), h_1 and f_2 are disjoint, and h_1 and g_2 are disjoint since h_1 is different from h_5, f_5, h_4 and h_2. Then $\{e, u_1u_{11}\} \subset M$ and each face of $G - \{v_1, v_2, u_1, u_{11}\}$ is bounded by a cycle. By Theorem 2.6 $G - S$ has at least two perfect matchings, a contradiction. For $e_3 = u_7u_8$ as shown in Fig. 10(2), h_3 is disjoint with g_2 and f_2 since h_3 is different from h_5, f_5, h_4 and h_2, and f_6 and f_2 are disjoint since f_6 is different from h_4 and h_2. So a similar contradiction occurs. For $e_3 = u_8u_9$ as shown in Fig. 10(3), since h_5 is not one of h_3, h_1, h_4 and h_2, h_5 is disjoint with f_1 and f_2. We also know that f_6 and f_2 are disjoint. So we have a similar contradiction.

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