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A New Lower Bound for the Multiplicative Degree–Kirchhoff Index

José Luis Palacios

Electrical and Computer Engineering Department,
The University of New Mexico, Albuquerque, NM 87131, USA
jpalacios@unm.edu

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Abstract

For $G = (V, E)$ an arbitrary simple undirected connected graph, its multiplicative degree-Kirchhoff index is defined by $R^*(G) = \sum_{i<j} d_i d_j R_{ij}$. We show that

$$R^*(G) \geq n - 1 + 2|E|(n - 2),$$

and the equality is attained by the complete graph K_n and the star graph S_n.

1 Introduction

For a simple connected undirected graph $G = (V, E)$ with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E, the multiplicative degree-Kirchhoff index was proposed by Chen and Zhang in [3], and defined as

$$R^*(G) = \sum_{i<j} d_i d_j R_{ij},$$

where R_{ij} is the effective resistance between the vertices i and j, and d_i is the degree of the vertex i. (For all graph theoretical terms the reader is referred to [10])

References [1], [4], [6], [7], [8] [9] and [11] are a sample of articles where the index $R^*(G)$ has been studied. In [8], through an expression in terms of eigenvalues of the transition probability matrix, and Lagrange multipliers, we gave the general lower bound

$$R^*(G) \geq 2|E| \left(n - 2 + \frac{1}{n} \right) \tag{1}$$

for an arbitrary G.

In [9] we improved (1) to

$$R^*(G) \geq 2|E|\left(n-2+\frac{1}{\Delta+1}\right), \tag{2}$$

where Δ is the largest degree of the graph. This bound was used to prove that the star graph S_n attains the minimum of $R^*(G)$ for all G.

In [1], using majorization, we found the bound

$$R^*(G) \geq 2|E|\left(\frac{1}{1+\frac{\sigma}{\sqrt{n-1}}}+\frac{(n-2)^2}{n-1-\frac{\sigma}{\sqrt{n-1}}}\right), \tag{3}$$

where $\sigma^2 = \frac{2}{n}\sum_{(i,j)\in E}\frac{1}{d_i d_j} = \frac{2}{n}R_{-1}(G)$, and $R_{-1}(G)$ is the generalized Randić index with $\alpha = -1$. The equalities in (1), (2) and (3) are attained by the complete graph K_n.

In this article we want to give a new lower bound that improves our previous ones, and is derived using a refinement of the electric arguments in [9].

2 The bound

Lemma 1 [9] *Let i and j be any two vertices in G.*

If $(i,j) \in E$ then

$$R_{ij} \geq \frac{d_i+d_j-2}{d_i d_j-1}. \tag{4}$$

If $(i,j) \notin E$ then

$$R_{ij} \geq \frac{1}{d_i}+\frac{1}{d_j}; \tag{5}$$

Now we can prove

Proposition 1 *For any n-vertex graph $G=(V,E)$ we have*

$$R^*(G) \geq n-1+2|E|(n-2). \tag{6}$$

The equality is attained by the complete graph K_n and the star graph S_n.

Proof. Using (4) we see that if $(i,j) \in E$ then

$$d_i d_j R_{ij} \geq R_{ij}+d_i+d_j-2,$$

and therefore

$$\sum_{\substack{i<j\\(i,j)\in E}} d_i d_j R_{ij} \geq \sum_{\substack{i<j\\(i,j)\in E}} (R_{ij}+d_i+d_j-2) = n-1-2|E|+\sum_{\substack{i<j\\(i,j)\in E}}(d_i+d_j), \tag{7}$$

where in the equality we have used Foster's first formula (see [5]). Now using (5) we can write, for $(i,j) \notin E$

$$d_i d_j R_{ij} \geq d_i + d_j,$$

and therefore

$$\sum_{\substack{i<j \\ (i,j)\notin E}} d_i d_j R_{ij} \geq \sum_{\substack{i<j \\ (i,j)\notin E}} (d_i + d_j). \tag{8}$$

Putting together (7) and (8) we obtain

$$R^*(G) = \sum_{\substack{i<j \\ (i,j)\in E}} d_i d_j R_{ij} + \sum_{\substack{i<j \\ (i,j)\notin E}} d_i d_j R_{ij}$$

$$\geq n - 1 - 2|E| + \sum_{\substack{i<j \\ (i,j)\in E}} (d_i + d_j) + \sum_{\substack{i<j \\ (i,j)\notin E}} (d_i + d_j)$$

$$= n - 1 - 2|E| + \sum_{i<j} (d_i + d_j) = n - 1 - 2|E| + (n-1)\sum_{i=1}^{n} d_i$$

$$= n - 1 + 2|E|(n-2) \qquad \bullet$$

It is also clear that for K_n we have $2|E| = n(n-1)$ and the bound gives the precise value $R^*(G) = (n-1)^3$.

Also, for any G we have $|E| \geq n-1$, and therefore applying directly (6) we have that for any graph G

$$R^*(G) \geq n - 1 + 2(n-1)(n-2) = (n-1)(2n-3) = R^*(S_n),$$

proving in a more direct way than in [9] that the minimal multiplicative degree-Kirchhoff index is attained by the star graph S_n.

Since $\Delta \leq n-1$ it is clear that

$$2|E| \leq \Delta n \leq \Delta n - \Delta + n - 1 = (\Delta + 1)(n-1),$$

and therefore (6) is always better than (2).

To study the behavior of the bound (3) in order to compare it to (6) not a trivial task. As shown in [1], the factor $F(n,\sigma) = \dfrac{1}{1 + \frac{\sigma}{\sqrt{n-1}}} + \dfrac{(n-2)^2}{n - 1 - \frac{\sigma}{\sqrt{n-1}}}$ is an increasing function of σ for $\dfrac{1}{\sqrt{n-1}} \leq \sigma < 1$, and the minimum occurs when $\sigma = \frac{1}{\sqrt{n-1}}$ for the complete graph K_n, but as we decrease the number of edges and hence $F(n,\sigma)$ increases, the factor $2|E|$ works in the opposite direction. At least we can say that since $\frac{1}{\sqrt{n-1}} \leq \sigma < 1$, then $\dfrac{\sigma}{\sqrt{n-1}} \to 0$, as $n \to \infty$, and therefore the bound (3) is roughly similar to the bound

$$R^*(G) \geq 2|E|\left(1 + \frac{(n-2)^2}{n-1}\right) = \frac{2|E|}{n-1} + 2|E|(n-2). \tag{9}$$

It is plain that (6) is better than (9) if and only if $2|E| \leq (n-1)^2$, that is, the bounds (6) and (9) are not comparable, so the bounds (3) and (6) would seem to be also not comparable, and if there were examples where (3) performs better than (6) (we have not been able to find any), they should be found among very dense graphs with a large value of $|E|$. However, it must be reminded that (9) is a crude estimate of (3), and we conjecture that (6) is always better than (3).

As a final remark, notice that the lower bound (6) for a unicyclic graph U becomes

$$R^*(U) \geq n - 1 + 2n(n-2) = 2n^2 - 3n - 1.$$

One may ask: how good is this bound? The answer is provided in [4]: the minimal degree-Kirchhoff index among unicyclic graphs has a value of $2n^2 - \frac{5}{3}n - 5$, so our lower bound does not miss this extremal graph by much; in fact, both expressions coincide for $n = 3$.

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