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Note on the Bounds on Wiener Number of a Graph

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Abstract

Theorem 7 in [H. B. Walikar, V. S. Shigehalli, H. S. Ramane, Bounds on the Wiener number of a graph, MATCH Commun. Math. Comput. Chem., 50 (2004), 117–132] related to the Wiener number of a graph interms of the chromatic number is not correct. This note provides the correct version of the theorem.

Introduction

Let G be a connected graph on n vertices and m edges. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The *distance* between the vertices v_i and v_j in G is equal to the length of a shortest path joining them and is denoted by $d(v_i, v_j)$.

The *distance number* of a vertex u, denoted by $d(u|G)$ is defined as

$$d(u|G) = \sum_{v \in V(G)} d(u, v) \ .$$

The *Wiener number* $W(G)$ of a graph G is defined as the sum of the distances between all unordered pairs of vertices of G [3], that is,

$$W(G) = \sum_{1 \le i < j \le n} d(v_i, v_j) = \frac{1}{2} \sum_{u \in V(G)} d(u|G) \ .$$

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors required to color the vertices of G such that no two adjacent vertices have same color. The set of all vertices with same color in a coloring of G is independent and is called the *color class* of G. The *complete k-partite graph* denoted by $K_{n_1,n_2,\ldots,n_k}$, is a graph whsose vertex set can be partitioned into k subsets $V_1, V_2, \ldots, V_k$, where $|V_i| = n_i$ such that no two vertices in any V_i, $1 \leq i \leq k$ are adjacent and each vertex of V_i is adjecent to all vertices of V_j, $i \neq j$. For graph theoretic terminology we refer [1].

In [2] the following theorem was given.

Theorem 1 [2]: *Let G be any connected graph of order n with chromatic number $\chi(G) = t$. Then*

$$W(G) \geq \frac{n(t+1) - 2t}{2} \ . \tag{1}$$

Further the equality holds if and only if $G \cong K_{n_1,n_2,\ldots,n_t}$.

Unfortunately some mistakes are found in the proof of the Theorem 1 and hence this theorem is not correct. Among several examples, one of the counterexample to the equality of Eq. (1) is the complete multipartite graph $K_{1,1,2}$. For which $W(K_{1,1,2}) = 7$ and $(n(t+1) - 2t)/2 = 5$.

Correct Statement

Theorem 2: *Let G be a connected graph of order n with chromatic number $\chi(G) = t$. Let $C_1, C_2, \ldots, C_t$ be the color classes of G, where $|C_i| = n_i$, $i = 1, 2, \ldots, t$. Then*

$$W(G) \geq \frac{n(n-2)}{2} + \frac{1}{2}\sum_{i=1}^{t} n_i^2 \,, \tag{2}$$

with equality holds if and only if $G \cong K_{n_1,n_2,\ldots,n_t}$.

Proof. Vertex set $V(G)$ can be partitioned into t color classes $C_1, C_2, \ldots, C_t$ where $|C_i| = n_i$, $i = 1, 2, \ldots t$ and $n = \sum_{i=1}^{t} n_i$. If $u, v \in C_i$, then $d(u,v) \geq 2$ and if $u \in C_i$ and $v \in V(G) - C_i$, then $d(u,v) \geq 1$.

Let $u \in C_i$, $i = 1, 2, \ldots, t$. Then

$$\begin{aligned} d(u|G) &= \sum_{v \in V(G)} d(u,v) = \sum_{v \in C_i} d(u,v) + \sum_{v \in V(G) - C_i} d(u,v) \\ &\geq 2(n_i - 1) + (n - n_i) = n + n_i - 2 \ . \end{aligned}$$

Therefore

$$\begin{aligned} W(G) &= \frac{1}{2}\sum_{u\in V(G)} d(u|G) = \frac{1}{2}\sum_{i=1}^{t}\sum_{u\in C_i} d(u|G) \geq \frac{1}{2}\sum_{i=1}^{t}\sum_{u\in C_i}(n+n_i-2) \\ &= \frac{1}{2}\sum_{i=1}^{t} n_i(n+n_i-2) = \frac{1}{2}\left[(n-2)\sum_{i=1}^{t} n_i + \sum_{i=1}^{t} n_i^2\right] = \frac{n(n-2)}{2} + \frac{1}{2}\sum_{i=1}^{t} n_i^2 \ . \end{aligned}$$

For equality:

If $G = K_{n_1,n_2,\ldots,n_k}$, then $W(G) = \frac{n(n-2)}{2} + \frac{1}{2}\sum\limits_{i=1}^{t} n_i^2$.

On the other hand, if $W(G) = \frac{n(n-2)}{2} + \frac{1}{2}\sum\limits_{i=1}^{t} n_i^2$, then the vertex set $V(G)$ can be partitioned into t color classes $C_1, C_2, \ldots, C_t$, where $|C_i| = n_i$, $i = 1, 2, \ldots, t$. We claim that if $u \in C_i$ and $v \in C_j$, $i \neq j$ then u and v are adjacent. For, if u and v are not adjacent, where $u \in C_i$ and $v \in C_j$, $i \neq j$, then $\sum\limits_{v\in V(G)-C_i} d(u,v) > n - n_i$. This implies that

$$\begin{aligned} d(u|G) &= \sum_{v\in V(G)} d(u,v) = \sum_{v\in C_i} d(u,v) + \sum_{v\in V(G)-C_i} d(u,v) \\ &> 2(n_i-1) + (n-n_i) = n + n_i - 2 \ . \end{aligned}$$

Therefore

$$W(G) > \frac{n(n-2)}{2} + \frac{1}{2}\sum_{i=1}^{t} n_i^2 \ ,$$

a contradiction.

Again, if u and v belongs to the same color class, then $d(u,v) = 2$. For, if u and v are adjacent in C_i, then $\sum\limits_{v\in C_i} d(u,v) = n_i - 1$. Therefore

$$\begin{aligned} d(u|G) &= \sum_{v\in V(G)} d(u,v) = \sum_{v\in C_i} d(u,v) + \sum_{v\in V(G)-C_i} d(u,v) \\ &> (n_i-1) + (n-n_i) = n - 1 \ . \end{aligned}$$

Therefore

$$W(G) = \frac{1}{2}\sum_{u\in V(G)} d(u|G) > \frac{n(n-1)}{2},$$

again a contradiction.

Hence $G \cong K_{n_1,n_2,\ldots,n_t}$. ■

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